Geometric constructions and elements of Galois' theory
List 7
Algebraic equations and radicals

0. Solve the following quadratic equation (with coefficients being complex numbers)

$$x^2 + ix - 1 + i = 0.$$

HINT: verify that $\sqrt{3-4i} = \pm(2-i)$, and make use of this fact in your solution.

1. Describe any radical extension of the field Q containing the number

$$\text{(a)} \quad \sqrt[5]{(\sqrt{2}+1)/(\sqrt[3]{\sqrt{7}-2}-\sqrt{3})}, \quad \text{(b)} \quad \sqrt{3/2} + i \cdot (2 + \sqrt[4]{1+\sqrt{12}}).$$

Estimate from above the degree of each of the above two numbers.

2. Prove, by referring to Cardano formulas, that roots of any polynomial of degree 3, with constructible numbers as coefficients, can be expressed in terms of rational numbers and radicals.
3. Describe the general form of an element of the following extension
   (a) $Q(\sqrt{3})(1+i)$,
   (b) $Q(\sqrt{3})(1+\sqrt{3}i)$,
   (c) $Q(\sqrt[3]{2})(\sqrt{1+\sqrt[3]{2}})$,
   expressing these elements in terms of rational numbers and radicals.
4. Show that solutions of any equation of any of the forms $ax^6 + bx^3 + c = 0$ and $ax^6 + bx^4 + cx^2 + d = 0$ can be expressed in terms of coefficients and radicals. Find other forms of eqations of degree greater than 4, and distinct from $x^n - a = 0$, whose all solutions can be expressed by radicals. Find equations as above of arbitrarily large degree.
5. Give an example of a polynomial of degree 6, with all coefficients distinct from 0 (for any degree of a variable), whose roots can be expressed by radicals.
6. Can one express in terms of rational numbers and radicals the number equal to the radius of the circle, whose area equals $\sqrt[3]{3}$? Justify your answer with an argument.