Exercises - Combinatorial Group Theory
List 2.
Ping-pong Lemma, Banach–Tarski paradox, and Magnus representation.

Applications of ping-pong lemma

1. State and prove a version of ping-pong lemma for three elements of a group.
2. Show that, if parameter d is large enough, the following two maps of R^2, y $a, b \in GL(2, R)$, genrate a free group: $a = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$, $b = tat^{-1}$, where t is the rotation through the angle $\pi/4$ around the origin of the coordinate system. Hint: Use the geometric form of the maps a and b and argue geometrically
3. **(Schottky groups)** A group G acts by bijections on a set X, and Y_1, Y_2, Y_3, Y_4 are some pairwise disjoint subsets of X. Suppose that a, b are such elements of G that:
   (1) a maps the complement of Y_1 into Y_2,
   (2) b maps the complement of Y_3 into Y_4.
   Show that the subgroup of G generated by $\{a, b\}$ is free with respect to this set. Hint: show first that a^{-1} maps the complement of Y_2 into Y_1, and then apply appropriately the ping-pong lemma.
4. **(Axial automorphisms of trees)** An automorphism γ of a tree T is *axial* if there is a bi-infinite polygonal path $A \subset T$ (called *the axis* of γ) preserved by γ and such that γ acts on A "translating" it by certain non-zero amount of segments (this number of segments is then called the *translation number* of γ, and we denote it $d(\gamma)$).
   (a) Show that any two axial automorphisms of a tree T having disjoint axes generate a free group.
   (b) Let γ, δ be axial automorphisms of a tree T, and suppose that the intersection of their axes is a bounded polygonal path consisting of l edges of T. Prove that if the translation numbers of γ and δ are greater than l then these elements generate a free subgroup of the automorphism group of T.

Approaching Banach–Tarski paradox

5. **(Embedding of F_2 into $SO(3)$)** Let $\theta = \arccos(1/3)$. Prove that the rotations A and B through the angle θ around the axes Oz i Ox, respectively, generate a free subgroup of rank 2 in the group $SO(3)$.
   Hints: (1) every nontrivial element of the free group $F_{\{a,b\}}$, up to conjugation, is re[presented by a reduced word terminating with the letter a. (2) By induction with respect to the word length show that, if we express the rotations A, B in terms of appropriate 3×3 matrices, then for any reduced word over the alphabet $\{A, B, A^{-1}, B^{-1}\}$ terminating with A and having the length n, the corresponding matrix from $SO(3)$ has the first column of form $\frac{1}{3^n} \cdot (a, b\sqrt{2}, c)^T$, where a, b, c are some integers, and b is not divisible by 3 (in particular, $b \neq 0$).
6. **(Paradoxical decomposition of a free group)** Find a decomposition of the pree group f rank 2 into four subsets, $F_2 = A \sqcup B \sqcup C \sqcup D$, such that some left translations of A and B (and similarly some left translations of C and D) also give a decomposition of F_2. More precisely, there are elments $g, h \in F_2$ such that the following two disjoint unions are decompositions of F_2: $F_2 = A \sqcup gB = C \sqcup hD$.

Few preparations not related to free groups.

Sets X, Y are *piecewise congruent* if we can decompose them into finitely many pieces, $X = X_1 \sqcup \ldots \sqcup X_n$ oraz $Y = Y_1 \sqcup \ldots \sqcup Y_n$, such that for each i the pieces X_i and Y_i are congruent.

Exercise A. Piecewise congruence is an equivalence relation.
Exercise B. The circle S^1 on the plane is piecewise congruent to the circle $S^1 \setminus \{p\}$ with one point removed.
Exercise C. [can be resolved using Exercise 2] The sphere S^2 in the 3-space is piecewise congruent with the complement $S^2 \setminus C$ of its any countable subset C.

7. **(Paradoxical decomposition of the sphere S^2)** Show that the sphere S^2 and the disjoint union $S^2 \sqcup S^2$ of its two copies are piecewise congruent.
   Hints: (1) Let C be the set of intersections of the sphere S^2 (viewed as srtandardly embedded in R^3) with axes of all rotations corresponding to nontrivial elements of the group $F_{\{A,B\}}$, under its embedding into $SO(3)$ as in Exercise 5; then $F_{\{A,B\}}$ acts on the complement $S^2 \setminus C$ freely, i.e. all orbits of this action are in 1–1 correspondence with the group. (2) Paradoxical decomposition of the group F_2 as in Exercise 6 induces then piecewise congruence of $S^2 \setminus C$ with the disjoint union of two copies of $S^2 \setminus C$.

Magnus representation and its consequences

Let Q_S be the ring of formal power series with respect to noncommuting variables $\xi_s : s \in S$, with integer coefficients, and let U_S be the multiplicative group of units (i.e. invertible elements) in this ring. Copnsider the map $\psi : S \to U_S$ given by $s \mapsto 1 + \xi_s$ for all $s \in S$.

8. Prove that the homomorphism $\bar{\psi} : F_S \to U_S$ which extends ψ is a monomorphism (equivalently, the set $1 + \xi_s : s \in S$ forms a basisi of a free subgroup in the group U_S).

Consider the ideal $\Delta := (\xi_s : s \in S) \subset Q_S$, i.e. the ideal generated by all monomials ξ_s. For each natural n consider the power Δ^n of the ideal Δ, i.e. the ideal consisting of all power series in which the coefficients of all monomials of degree less than n vanish. Consider also the lower central series of the group F_S, i.e. the sequence of normal subgroups $F_S^{(k)} \triangleleft F_S$ given recursively by: $F_S^{(1)} = F_S$, $F_S^{(k+1)} = [F_S^{(k)}, F_S]$.

9. Prove that for the commutator subgroup $F_S^{(2)}$ of the group F_S we have $\bar{\psi}(F_S^{(2)}) = \bar{\psi}(F_S) \cap [1 + \Delta^2]$.
10. Prove that for each natural n we have $\bar{\psi}(F_S^{(n)}) \subset 1 + \Delta^n$. Deduce that $\bigcap_n F_S^{(n)} = \{1\}$.
11. **(Residual nilpotency of free groups)** Prove that a nonabelian free group F_S is not nilpotent (i.e. for each n the subgroup $F_S^{(n)}$ is nontrivial). Show also that F_S is *residually nilpotent*, i.e. for each $g \in F_S \setminus \{1\}$ there is a nilpotent group P and a homomorphism $h : F_S \to P$ which maps g to a nontrivial element $h(g) \neq 1$ in P.