Foundation of geometry and non-euclidean geometry
List 3. Klein's model of non-euclidean geometry.

Basic geometric objects in Klein's model

1. Indicate a point in the interior of any convex angle, through which there is no line intersecting both arms of the angle. For a given angle as above, describe the set of all points with this property. (Confront this with the incorrect Legendre's proof, presented at one of the lectures, of the fact that the sum of angles of any triangle equals $180°$.)
2. Present an example of three pairwise disjoint halfplanes in Klein's model of non-euclidean plane. Is such a situation possible in euclidean plane?
3. Given two asymptotic lines k and m, and a point A in the area between them, decide how many lines pass through A and do not intersect k and m. Describe as many positions for A as possible (with respect to a fixed pair of lines k and m), which result with distinct solutions of the above problem.

Perpendicular lines, acute and obtuse angles

4. Show that any two asymptotic lines have no common perpendicular.
5. Prove that any two distinct lines perpendicular to a given line are divergent.
6. In the Klein's model, find an example of a right angle and an acute angle such that the area of the former is contained in the interior of the latter. BEWARE: an angle is acute if it is contained in some right angle having the same angle vertex and common one of the arms.
7. Indicate in Klein's model an example of Lambert quadrilateral, and justify that, apart from its three consecutive right angles, the fourth angle in this quadrilateral is acute.
8. Give an example (together with a justification) of a quadrilateral having all four interior angles acute.

Non-euclidean distances and measures of segments

9. Show that the logarithmic coordinate

$$x_A = \frac{1}{2}\left|\ln \frac{|PA|/|AQ|}{|PB|/|BQ|}\right|$$

   of a point A on a chord PQ:
   (a) is 0 when A is a euclidean center of the chord PQ;
   (b) increases when A moves in the direction from P to Q;
   (c) changes into an opposite number after transposing the roles of the enpoints P and Q in the chord.
10. Given point A lying at the center of the unit circle forming the Klein's model, and a point B at euclidean distance $1/2$ from A, calculate the non-euclidean distance $|AB|_{\mathrm{NE}}$, and find the non-euclidean center of the segment AB.
11. Indicate in Klein's model a segment of non-euclidean measure 1000.
12. Construct in Klein's model an example of Saccheri quadrilateral. Check that the interior angles of this quadrilateral not adjacent to its base side (i.e. the side with right angles at its endpoints) are acute.

13. Prove that the hypotenuse in any right triangle in Klein's model has strictly greater measure than any of the legs, in the following two steps.
    (a) First, consider the case of a triangle in which the common vertex of the hypotenuse and the leg under consideration is the (euclidean) center of the model.
    (b) Next, by referring to (a) and to the congruence axiom K3, prove the same for any non-euclidean right triangle.
14. By referring to the previous exercise, show that the orthogonal projection A' onto a given line p of any point A is the closest point to A among all points lying on p.
15. Lines p and q are asymptotic, $A \in q$, and A' is the orthogonal projection of A onto p. Show that the non-euclidean distance $|AA'|$ converges to zero when A converges to the ideal point at infinity represented by the common point of p and q on the boundary of the model. Assume in your argument that p is represented by a diameter in the model (i.e. a chord which passes through the center of the model).

Non-euclidean circles and isometries

16. Check that the intervals in Klein's model having common endpoint at the center of the model have the same non-euclidean measure if and only if their euclidean lengths are the same. Deduce that non-euclidean circles centered at the center S of the model (i.e. sets of all points lying at a fixed non-euclidean distance from S) coincide with ordinary euclidean circles centered at S.
17. Let P be a point in Klein's model distinct from the center S of the model. Prove that no euclidean circle centered at P is a non-euclidean circle centered at P.
18. Using Exercise 16, give an example of a circle in the non-euclidean plane which is not inscribed in any triangle (one cannot circumscribe any triangle around it). Calculate the minimal non-euclidean radius of such a circle.
19. For non-euclidean circles centered at the center of the model, show that any inscribed angle based at a diameter in any such circle is not the right angle. Is it acute or obtuse?
20. Show that each ordinary (euclidean) rotation around the center S of Klein's model is a non-euclidean isometry of the non-eucliedean plane. This means that each such rotation preserves non-euclidean distances (i.e. distance between any two points is the same as distance between their images).

Other exercises

21. A triangle with one ideal vertex has at the remaining vertices the angles of measure $90°$ and $30°$. Calculate the (non-euclidean) length of the side of this triangle which connects its two ordinary vertices. HINT: put this triangle in such a position in the model that the vertex with angle measure $30°$ is placed in the center of the model.
22. For any α with $0 < \alpha < 90°$ construct in Klein's model an example of Lambert quadrilateral whose angle measure at the fourth vertex equals α.