On the asymptotic homological dimension of hyperbolic groups

Jacek Świątkowski

Abstract. We introduce the notion of asymptotic homological dimension $asdim_h$ of a metric space (invariant under quasiisometry), and show that $\dim \partial_\infty \Gamma + 1 \le asdim_h \Gamma \le asdim_+ \Gamma$ for a (word-) hyperbolic group Γ ($asdim_+$ is the large scale dimension defined by M. Gromov). We show also that $asdim_h \Gamma \le 2$ for a certain class of hyperbolic groups (introduced by M. Gromov) that we call strongly isoperimetric groups.

0. Introduction

It has proved to be useful to look at infinite groups as geometric objects, especially to study their asymptotic (or large scale) properties, i.e. those which are invariant with respect to quasiisometries. In his recent paper [**6**] M. Gromov introduces several such properties and invariants, among them the large scale dimension $asdim_+$ (see definition A.2 of Appendix). Another such invariant for hyperbolic groups (in this paper by hyperbolic we shall mean word-hyperbolic in the sense of Gromov, see [**7**]) is the topological dimension $\dim \partial_\infty \Gamma$ of ideal boundary of a group Γ. In [**6**] Gromov conjectured that the equality $asdim_+ \Gamma = \dim \partial_\infty \Gamma + 1$ holds for any hyperbolic group Γ. In this paper we shall establish (in remark A.7 of the Appendix) the inequality $asdim_+ \Gamma \ge \dim \partial_\infty \Gamma + 1$.

In section 9 of the same paper [**6**] M. Gromov describes an interesting class of hyperbolic groups, that we call here *strongly isoperimetric groups* (see definition 5.1). Gromov states without proof that $\dim \partial_\infty \Gamma \le 1$ for those groups. We prove it here rigorously (see remark 5.8).

In this paper, we use a method, which follows the idea of the homological dimension due to P. Alexandrov (see [**1**]). In section 1 we introduce the notion of *asymptotic homological dimension* $asdim_h$ of a metric space. We prove that it is a quasiisometry invariant (proposition 1.6), and show the inequality $asdim_h \le asdim_+$ (Appendix).

The main results of this paper are the following.

Theorem 4.1. *For any hyperbolic group* Γ $asdim_h\Gamma \geq \dim \partial_\infty\Gamma + 1$.

Theorem 5.7. *If* Γ *is a strongly isoperimetric (hyperbolic) group, then* $asdim_h\Gamma \leq 2$.

To obtain these results, we use the properties of *horospheres* and *parabolic-gradient-geodesic projections* in hyperbolic groups (section 2), and relate the asymptotic homological dimension of a horosphere with that of a group itself (section 3).

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1. Asymptotic homological dimension

To define asymptotic homological dimension $asdim_h$, we need the (notion of) ϵ-homology H_*^ϵ for a metric space X. If not stated explicitely otherwise, through all the paper we shall use as coefficients for homology the group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Definition 1.1. If $\epsilon > 0$ is a real number, a q-dimensional ϵ-*simplex* in a metric space X is a $(q+1)$-tuple $(x_0, x_1, \ldots, x_q)$ of elements of X satisfying $d(x_i, x_j) \leq \epsilon$ for $0 \leq i \neq j \leq q$. Then we define ϵ-chains with coefficients in $\mathbb{T}$, a boundary operator ∂, and ϵ-homology H_*^ϵ in an obvious manner.

We define the support $supp\, u$ of an ϵ-chain u, to be the set of all vertices of its simplices (with nonzero coefficients).

We will say that an ϵ-cycle is ϵ-*homologically trivial* (respectively *nontrivial*) in X if its ϵ-homology class in $H_*^\epsilon X$ is zero (respectively nonzero). By ϵ-*filling* of an ϵ-cycle z we will mean an ϵ-chain w such that $\partial w = z$.

Definition 1.2. Let X be a metric space and p be a positive integer. We say that $asdim_h X \leq p$ if for each $r > 0$ there exists $\alpha > 0$, depending only on X and r, such that for $q \geq p$ any q-dimensional r-cycle ϕ which is r-homologically trivial in X, is also α-homologically trivial in its support $supp\,\phi$.

We say that $asdim_h\, X = p$, if p is the minimal number such that $asdim_h\, X \leq p$.

Recall the definition of quasiisometry:

Definition 1.4. Let (X, d_X) and (Y, d_Y) be metric spaces, and $F : X \to Y$ a mapping (not necessarily continuous). F is called a (C, R, L)-*quasiisometry*, for some constants $C \geq 1$, $R \geq 0$, $L \geq 0$, if the following conditions are satisfied:

(i) $\frac{1}{C} \cdot d_X(x_1, x_2) - R \leq d_Y(F(x_1), F(x_2)) \leq C \cdot d_X(x_1, x_2) + R$ for all $x_1, x_2 \in X$;

(ii) for each $y \in Y$ there exists $x \in X$ such that $d_Y(y, F(x)) \leq L$.

F is a *quasiisometry*, if it is a (C, R, L)-quasiisometry for some C, R and L.

Remark 1.5.

(a) Note that if $F : X \to Y$ is a (C, R, L)-quasiisometry, then there exists a $(C, RC + 2LC + 2L, RC)$-quasiisometry $E : Y \to X$;

(b) if $F : X \to Y$ and $E : Y \to Z$ are both (C, R, L)-quasiisometries, then the composition $E \circ F : X \to Z$ is a $(C^2, CR + R, CL + L + R)$-quasiisometry;

(c) from (a) and (b) it follows that the relation of quasiisometry is the equivalence relation between metric spaces.

PROPOSITION 1.6. *The asymptotic homological dimension $asdim_h$ is a quasiisometry invariant of a metric space.*

Before giving a proof of this proposition, we need one more definition and an easy technical lemma, which we state without proof.

Definition 1.7. Let z be an ϵ-chain in a metric space X and let $f : supp\, z \to X$ be a function mapping vertices of z into points of X so that $d(v, f(v)) < \epsilon'$ for some $\epsilon' > 0$ and for all vertices $v \in supp\, z$. Then we call f an *ϵ'-deformation* of ϵ-chain z. We denote by $f_* z$ the image chain of z by the mapping f.

LEMMA 1.8. *(cf. [**1**] p. 118, lemma 4). The image $f_* z$ of an ϵ-chain z by an ϵ'-deformation $f : supp\, z \to X$, is an $(\epsilon + 2\epsilon')$-chain. Moreover, if z is an ϵ-cycle, then $f_* z$ is $(\epsilon + 2\epsilon')$-homologous to z; in fact $f_* z = z + \partial u$, where the vertices of u are only those of z or $f_* z$.*

Proof of proposition 1.6.

Assume that $F : X \to Y$ is a (C, R, L)-quasiisometry, and that $asdim_h\, X = p$. We shall prove that $asdim_h\, Y \leq p$.

Choose any $q \geq p$, and any q-dimensional r-cycle ϕ, which is r-homologically trivial in Y, with r-filling ψ. We shall find a constant α (independent on q and ϕ) such that ϕ can be α-filled in its support $supp\, \phi$.

Construct a mapping $E : Y \to X$ as follows. If $y \in Im\, F$, put $E(y) = x$ for some x such that $F(x) = y$. If $y \notin Im\, F$, choose $y_1 \in Im\, F$ such that $d_Y(y, y_1) \leq L$, and put $E(y) = E(y_1)$. Then E is a $(C, 2LC + RC + 2L, RC)$-quasiisometry referred to in remark 1.5.(a). It follows that $E_*\phi$ is an r'-cycle in X, and $E_*\psi$ is its r'-filling, where $r' = Cr + 2LC + RC + 2L$. Thus, there is $\alpha' > 0$ and an α'-cycle u with $supp\, u \subset supp\, E_*\phi$, which is an α'-filling of $E_*\phi$.

Returning to Y by the mapping F, we get that $F_* u$ is a $(C\alpha' + R)$-filling of $(Cr' + R)$-cycle $(F \circ E)_*\phi$ in its support $supp\, (F \circ E)_*\phi$. Note, that since $(F \circ E)|_{Im\, F} = id$, we have $d_Y(F \circ E(y), y) \leq B$ for each $y \in Y$, where $B = 3C^2L + C^2R + 2CL + R + L$, and thus $(F \circ E)|_{supp\, \phi}$ is a B-deformation. By lemma 1.8, this enables to construct, using also $F_* u$, a B'-filling w of ϕ, with $supp\, w \subset supp\, \phi \cup supp\, (F \circ E)_*\phi$, where $B' = \max(B, C\alpha' + R)$.

Now, since $(F \circ E)|_{supp\, \phi}$ is a B-deformation, we can choose another B-deformation

$f : supp\,(F \circ E)_* \phi \cup supp\,\phi \to supp\,\phi$ so that $f|_{supp\,\phi} = id$. Then $f_* w$ is a $(B' + 2B)$-filling of ϕ and $supp\, f_* w \subset supp\,\phi$. Thus the proposition follows by puttig $\alpha = B' + 2B$.

Remark 1.9. Asymptotic homological dimension $asdim_h$ is an invariant of a hyperbolic group Γ, since the word metrics on Γ for different finite generating sets S are quasiisometric.

2. Horospheres and parabolic–gradient–geodesic projections in hyperbolic groups.

The object of our study is a δ-hyperbolic group Γ with a word metric d defined with respect to a fixed finite generating set S, and a Cayley graph $G(\Gamma, S)$ with respect to this metric; we fix this notation. All paths p that we consider, including geodesics, are polygonal curves in $G(\Gamma, S)$ parametrized by arc length so that $p(k) \in \Gamma \subset G(\Gamma, S)$ for $k \in \mathbb{Z}$.

Consider the ideal boundary $\partial_\infty \Gamma$ of a group Γ, as it is defined in section 7 of [**5**]. Note that since the Cayley graph $G(\Gamma, S)$ is a proper geodesic metric space, then according to [**5**] 5.25, any point of ideal boundary can be joined by a geodesic in $G(\Gamma, S)$ to any element of Γ, as well as to any other point of $\partial_\infty \Gamma$ (compare also [**2**] Chapitre 2, Proposition 2.1).

Lemma 2.1. *Let γ be a geodesic joining two points a, b of ideal boundary $\partial_\infty \Gamma$, and $g \in \Gamma$ any element. Then there exists $N \in \mathbb{Z}$ such that for any $n > N$ we have $d(g, \gamma(n)) + 1 = d(g, \gamma(n+1))$.*

Proof: Note that the triangle inequality $d(g, \gamma(n)) + 1 \geq d(g, \gamma(n+1))$ holds for each $n \in \mathbb{Z}$.

Suppose that the sharp inequality $d(g, \gamma(n)) + 1 > d(g, \gamma(n+1))$ holds for infinitely many n with $n \to +\infty$. Then it follows that $\lim_{n \to +\infty} n - d(g, \gamma(n)) = +\infty$, which violates the triangle inequality $d(\gamma(n), \gamma(0)) - d(g, \gamma(n)) \leq d(g, \gamma(0))$, giving a contradiction.

Definition 2.2 (compare [**3**] 3.3). Let γ be a geodesic in Γ. We define a family of sets $\{H_\gamma(k) : k \in \mathbb{Z}\}$ by

$$H_\gamma(k) = \{g \in \Gamma : \lim_{n \to +\infty} [n - d(g, \gamma(n))] = k\}.$$

These are well defined by lemma 2.1, and will be called *horospheres* in Γ with respect to a geodesic γ. If $\lim_{n \to +\infty} \gamma(n) = a \in \partial_\infty \Gamma$, we say that horospheres $\{H_\gamma(k) : k \in \mathbb{Z}\}$ are centered at a.

Remark 2.3. The following properties of horospheres are clear according to lemma 2.1 and its proof:

(a) $H_\gamma(k_1) \cap H_\gamma(k_2) = \emptyset$ for $k_1 \neq k_2$;

(b) $\Gamma = \bigcup_{k \in \mathbb{Z}} H_\gamma(k)$;

(c) any curve joining some points of $H_\gamma(k_1)$ and $H_\gamma(k_2)$ intersects $H_\gamma(k)$ for any $k_1 < k < k_2$.

Definition 2.4 (compare [**3**] 3.4.1). We say that a path p in Γ is a *gradient path* with respect to a horosphere system $\{H_\gamma(k) : k \in \mathbb{Z}\}$, if there is a constant $m \in \mathbb{Z}$ such that $n - k = m$ whenever $p(n) \in H_\gamma(k)$.

Remark 2.5. Note that, by remark 2.3.(c), each gradient path is a geodesic segment (compare also [**3**] 4.4.2).

Remark 2.6. Note that, by lemma 2.1, for any $g \in \Gamma$ there is $N \in \mathbb{Z}$ such that the curve $\phi = [g, \gamma(N)] \cup \gamma|_{[N, +\infty)}$, for any geodesic segment $[g, \gamma(N)]$, is a gradient geodesic joining $g \in \Gamma$ with the center a of the horosphere system.

Proposition 2.7. *Let $x \in \partial_\infty \Gamma \setminus \{a\}$, where a is a center of a horosphere system $\{H_\gamma(k) : k \in \mathbb{Z}\}$. Then there exists a gradient (with respect to this system) geodesic ϕ_x joining x to a.*

Proof: Let η be any geodesic joining x to a, parametrized so that $\lim_{n\to+\infty} \eta(n) = a$.

A short and rough argument is as follows:

Consider a metric ρ on a Cayley graph $G(\Gamma, S)$, equivalent to the original word metric (i.e. inducing the same topology on the graph), such that $\partial_\infty \Gamma \cup G(\Gamma, S)$ is a compact completion of $(G(\Gamma, S), \rho)$ (cf. [**2**], chapter 11). Then the family $\{\phi_n : n \in \mathbb{N}\}$ of gradient rays joining $\eta(-n)$ to a (see remark 2.5) is a relatively compact family of continuous mappings from the closed unit interval I to $\partial_\infty \Gamma \cup G(\Gamma, S)$. Thus there is a subsequence n_k of natural numbers such that ϕ_{n_k} is pointwise convergent to a limit curve ϕ_x, which is easily verified to be a gradient geodesic joining x to a.

We give also a more elementary and detailed argument:

As previously, consider a family $\{\phi_n : n \in \mathbb{N}\}$ of gradient rays from $\eta(-n)$ to a, and parametrize them in such a way that $\phi_n(k) \in H_\gamma(k)$. By [**3**] 1.3.4, each point $\phi_n(k)$ remains at distance bounded by 4δ from η, and thus an easy argument shows that $d(\phi_n(k), \phi_m(k)) \leq 16\delta$ for any k in the domain of ϕ_n and ϕ_m.

We start the inductive construction of ϕ_x:

Step 1. Note that, since the set $\{\phi_n(0) : n \in \mathbb{Z}$ such that 0 is in the domain of $\phi_n\} \subset H_\gamma(0)$ is bounded and thus finite, there is an element v_0 in it, and an infinite subsequence $(i^0_n)_{n\in\mathbb{N}}$ of negative integers such that $v_0 = \phi_{i^0_n}(0)$ for $n \in \mathbb{N}$.

Step 2 (general inductive step). Having constructed points v_j for $|j| \leq m$, and a subsequence $(i^m_n)_{n\in\mathbb{N}}$ of negative integers, we choose points $v_{m+1} \in H_\gamma(m+1)$ and $v_{-m-1} \in H_\gamma(-m-1)$, and a subsequence $(i^{m+1}_n)_{n\in\mathbb{N}}$ of i^m_n, such that $v_j = \phi_{i^{m+1}_n}(j)$ for all $|j| \leq m+1$ and all $n \in \mathbb{N}$.

By above inductive procedure, we get a family $\{v_j : j \in \mathbb{Z}\}$ of points. We finish the proof by putting $\phi_x = \bigcup_{j\in\mathbb{Z}}[v_j, v_{j+1}]$ and noting that ϕ_x is a gradient geodesic joining x to a, becouse any its subsegment $[v_{-m}, v_m]$ is gradient by construction, and $\lim_{n\to+\infty} v_n = a$ and $\lim_{n\to-\infty} v_n = x$.

Remark 2.8. Note that the behaviour of gradient geodesics with respect to a horosphere system is closely analogous to the behaviour of geodesic rays starting at some fixed point $g_0 \in \Gamma$ with respect to sphere system $\{S(g_0, n) : n \in \mathbb{N}\}$.

Definition 2.9. Let $\{H_\gamma(n) : n \in \mathbb{Z}\}$ be a system of horospheres centered at point $a \in \partial_\infty\Gamma$. For each $x \in \partial_\infty\Gamma \setminus \{a\}$ choose a gradient geodesic ϕ_x joining x to a, and parametrize it so that $\phi_x(n) \in H_\gamma(n)$ for $n \in \mathbb{Z}$. Then define a *parabolic–gradient–geodesic projection* $P_n : \partial_\infty\Gamma \setminus \{a\} \to H_\gamma(n)$ by $P_n(x) = \phi_x(n)$.

Note that a projection P_n depends on a choice of gradient geodesics $\{\phi_x : x \in \partial_\infty\Gamma\setminus\{a\}\}$.

Let us state a proposition, which is a consequence of corollary 1.3.5 of [**3**], and which expresses the intuitive fact that the bigger n is, the more contracting P_n is:

PROPOSITION 2.10. *If $n > m$ are integers, δ is a hyperbolicity constant for Γ, and $x, y \in \partial_\infty\Gamma \setminus \{a\}$, then*

$$d(P_n(x), P_n(y)) \le d(P_m(x), P_m(y)) + 16\delta.$$

A somewhat deeper approach to the same intuition is included in the following:

PROPOSITION 2.11. *(cf. [**5**] chapter 8). There is a metric d_a on $\partial_\infty\Gamma \setminus \{a\}$ (which restricts to the natural topology on the ideal boundary) satisfying the following properties.*

Let $C \ge 16\delta$ be any constant; then if $x, y \in \partial_\infty\Gamma$, we have

(1) $\forall\epsilon > 0\, \exists N(\epsilon) \in \mathbb{Z}\, \forall n \ge N(\epsilon)\; d_a(x, y) < \epsilon \Rightarrow d(P_n(y), P_n(x)) \le C$, *and the function* $N(\epsilon)$ *satisfies* $\lim_{\epsilon\to 0} N(\epsilon) = -\infty$;

(2) $\forall N \in \mathbb{Z}\ \exists \epsilon(N) > 0\ \forall n \leq N\ d(P_n(x), P_n(y)) \leq C \Rightarrow d(x,y) < \epsilon(N)$ *and moreover* $\lim_{N \to -\infty} \epsilon(N) = 0$.

*The metrics of this kind constructed in chapter 8 of [**5**] are called Busemann's metrics.*

We define also a version of parabolic-gradient-geodesic projection between two horospheres of the same system, and state its key property.

Definition 2.12. Let $\{H_\gamma(i) : i \in \mathbb{Z}\}$ be a system of horospheres centered at $a \in \partial_\infty \Gamma$. Given $n \in \mathbb{Z}$, for each $g \in H_\gamma(n)$ choose a gradient geodesic ray ϕ_g joining g to a, and parametrize it so that $\phi_g(i) \in H_\gamma(i)$ for $i \geq n$. If $k \geq n$, define $P_{n,k} : H_\gamma(n) \to H_\gamma(k)$ by $P_{n,k}(g) = \phi_g(k)$.

Note that a projection $P_{n,k}$ depends on a choice of gradient geodesics $\{\phi_g : g \in H_\gamma(n)\}$.

The direct analogue of proposition 2.10 is the following:

PROPOSITION 2.13. *If $m \geq k \geq n$ and $x, y \in H_\gamma(n)$, then*

$$d(P_{n,m}(x), P_{n,m}(y)) \leq d(P_{n,k}(x), P_{n,k}(y)) + 16\delta.$$

3. DIMENSION $asdim_h$ OF HOROSPHERES IN A HYPERBOLIC GROUP.

In this section we give a proof of the following:

THEOREM 3.1. *If H is a horosphere in a hyperbolic group Γ, then*

$$asdim_h \Gamma \geq asdim_h H + 1.$$

PROOF: Let $p = asdim_h H$. Then there is a constant $r > 0$ such that for any $\alpha > 0$ there is a $(p-1)$-dimensional r-cycle ϕ_α with an r-filling ψ_α in H, which cannot be α-filled in its support $supp\, \phi_\alpha$.

We may assume that $H = H_\gamma(0)$ for some geodesic γ. Using a family of parabolic-gradient-geodesic projections $P_{0,k} : H_\gamma(0) \to H_\gamma(k)$ for $k > 0$ (see definition 2.12), we shall construct for any $\alpha > 0$, out of ϕ_α and ψ_α, a p-dimensional $(r + 16\delta + 2)$-cycle c_α and its $(r + 16\delta + 2)$-filling u_α in Γ, and then prove that $c_{5\alpha}$ is α-homologically nontrivial in its support $supp\, c_{5\alpha}$. This will finish the proof by implying that $asdim_h \Gamma \geq p + 1$.

Construction of the cycle c_α and its filling u_α.

Given $k \geq 0$ let $\phi_{\alpha,k} = (P_{0,k})_* \phi_\alpha$ and let $\psi_{\alpha,k} = (P_{0,k})_* \psi_\alpha$, and note that, by proposition 2.13, $\phi_{\alpha,k}$ is an $(r + 16\delta)$-cycle and $\psi_{\alpha,k}$ is an $(r + 16\delta)$-filling of it, for any $k \geq 0$. Since for each $k \geq 0$ chains $\phi_{\alpha,k+1}$ and $\psi_{\alpha,k+1}$ are the images of $\phi_{\alpha,k}$ and $\psi_{\alpha,k}$ by 1-deformations, let $c_{\alpha,k}$ and $u_{\alpha,k}$ be the $(r + 16\delta + 2)$-chains of lemma 1.8, such that $\partial c_{\alpha,k} = \phi_{\alpha,k+1} - \phi_{\alpha,k}$, and $\partial u_{\alpha,k} = \psi_{\alpha,k+1} - \psi_{\alpha,k} + c_{\alpha,k}$. We finish the construction by choosing integer $l > \alpha$ and putting

$c_\alpha = \psi_{\alpha,l} - \psi_{\alpha,0} + \sum_{i=1}^{l} c_{\alpha,i}$ and $u_\alpha = \sum_{i=1}^{l} u_{\alpha,i}$.

Cycle $c_{5\alpha}$ is α-homologically nontrivial in its support.

We shall prove the existence of the following homomorphisms of homology groups (see definition 1.1):

excision homomorphism

$e : H_n^\alpha(supp\, c_{5\alpha}, \mathbb{T}) \to H_n^\alpha(supp\, \psi_{5\alpha}, supp\, \psi_{5\alpha} \cap \bar{N}_{2\alpha}(supp\, \phi_{5\alpha}); \mathbb{T})$;

boundary homomorphism

$\Delta : H_n^\alpha(supp\, \psi_{5\alpha}, supp\, \psi_{5\alpha} \cap \bar{N}_{2\alpha}(supp\, \phi_{5\alpha}); \mathbb{T}) \to H_{n-1}^\alpha(\bar{N}_{2\alpha}(supp\, \phi_{5\alpha}), \mathbb{T})$;

2α-deformation homomorphism

$f_* : H_{n-1}^\alpha(\bar{N}_{2\alpha}(supp\, \phi_{5\alpha}), \mathbb{T}) \to H_{n-1}^{5\alpha}(supp\, \phi_{5\alpha}, \mathbb{T})$;

where $\bar{N}_{2\alpha}(supp\, \phi_{5\alpha}) = \{g \in H_\gamma(0) : d(g, supp\, \phi_{5\alpha}) \leq 2\alpha\}$, and $f : \bar{N}_{2\alpha}(supp\, \phi_{5\alpha}) \to$

$supp\,\phi_{5\alpha}$ is any 2α-deformation such that $f|_{supp\,\phi_{5\alpha}} = id$.

Define an excision homomorphism e as follows:

for any α-cycle z with $supp\,z \subset supp\,c_{5\alpha}$, let z_e be an α-chain consisting of those simplices of z (with coefficients in $\mathbb{T}$) which have some (at least one) vertices in $supp\,\psi_{5\alpha} \setminus \bar{N}_{2\alpha}(supp\,\phi_{5\alpha})$.

Claim. The other vertices of such simplices (if there are any) are contained in $supp\,\psi_{5\alpha} \cap \bar{N}_{2\alpha}(supp\,\phi_{5\alpha})$.

Proof of claim:

Suppose not. Then there is an α-simplex with vertices $v_1 \in supp\,\psi_{5\alpha} \setminus \bar{N}_{2\alpha}(supp\,\phi_{5\alpha})$ and $v_2 \in supp\,c_{5\alpha} \setminus supp\,\psi_{5\alpha}$. Then, by construction of $c_{5\alpha}$, there is a point $g \in supp\,\phi_{5\alpha}$ such that $v_2 = P_{0,k}(g)$. Since $d(v_1, v_2) \leq \alpha$ and $d(v_1, g) > 2\alpha$, by the triangle inequality we get $k = d(v_2, g) > \alpha$. But since $v_1 \in H_\gamma(0)$ and $v_2 \in H_\gamma(k)$, this implies by remark 2.3.(c) that $d(v_1, v_2) > \alpha$, giving a contradiction. Thus the claim follows.

Note that from the claim it follows that $supp\,\partial z_e \subset supp\,\psi_{5\alpha} \cap \bar{N}_{2\alpha}(supp\,\phi_{5\alpha})$, and that z_e is a relative α-cycle in $(supp\,\psi_{5\alpha}, supp\,\psi_{5\alpha} \cap \bar{N}_{2\alpha}(supp\,\phi_{5\alpha}))$. We then define $e([z]) = [z_e]$, where $[w]$ denotes the homology class of cycle w.

Since by relative α-cycle z in (X, Y) we mean an α-chain in X, for which the boundary ∂z has support contained in Y, we define the boundary homomorphism Δ as induced by the mapping $z \mapsto \partial z$ on the chain level.

The 2α-deformation homomorphism f_* is well defined according to lemma 1.8.

Having constructed well defined homomorphisms e, Δ and f_*, notice that $e([c_{5\alpha}]) = [\psi_{5\alpha}]$, and therefore $f_* \circ \Delta \circ e([c_{5\alpha}]) = [\phi_{5\alpha}]$. Since, by assumption, $[\phi_{5\alpha}] \neq 0$ in $H^{5\alpha}_{p-1}(supp\,\phi_{5\alpha}, \mathbb{T})$, it follows that $[c_{5\alpha}] \neq 0$ in $H^{\alpha}_{p}(supp\,c_{5\alpha}, \mathbb{T})$, which finishes the proof of theorem 3.1.

4. Relation with topological dimension of ideal boundary

In this section we proof the following:

Theorem 4.1. *For any hyperbolic group* Γ $\dim \partial_\infty \Gamma + 1 \leq asdim_h \Gamma$.

The structural relationship between dimensions dim and $asdim_h$ stems from the following result. Here, as always, we consider homology with coefficients $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Theorem 4.2. *(Alexandrov, [**1**] 4.2.2).*

Let X *be a compact metric space with topological dimension* $\dim X = p$*. Then there exists a closed subset* $\Phi \subset X$*, and an* $\epsilon_0 > 0$ *such that for each* $\eta > 0$ *there exists an* η*-cycle* z *of dimension* $p - 1$*, which is* ϵ_0*-homologically nontrivial in* Φ*, although it is* η*-homologically trivial in* X*.*

Remark 4.3. It follows from theorem 4.2 that if X is a compact metric space and $\dim X = p$, then there exists a closed subset $\Phi \subset X$, and $\epsilon_0 > 0$ such that for each $\eta > 0$ there exists η-chain w of dimension p, being a relative η-cycle ϵ_0-homologically nontrivial in (X, Φ), with $z = \partial w$ being an η-cycle ϵ_0-homologically nontrivial in Φ.

Let us recall a technical result from the book of Alexandrov:

Lemma 4.4. *([**1**] theorem 8, p. 124).*

Given a compact metric space X *and its closed subset* Φ*, for each* $\epsilon_0 > 0$ *there exists* $\sigma > 0$ *such that for any* $\rho > 0$ *each relative* σ*-cycle* w *in* (X, Φ) *is* ϵ_0*-homologous with some relative* ρ*-cycle* w_ρ*.*

The last preparatory result is the following

Lemma 4.5. *Given a metric space Φ and $\epsilon' = \frac{1}{3}\epsilon_0 > 0$, let z be an ϵ'-cycle ϵ_0-homologically nontrivial in Φ, and z' the image of its any ϵ'-deformation in Φ. Then z' is an ϵ_0-cycle which is ϵ_0-homologically nontrivial in Φ.*

Proof: Suppose on the contrary that z' is ϵ_0-homologically trivial in Φ. Since, by lemma 1.8, z' is $3\epsilon'$-homologous to z and $3\epsilon' = \epsilon_0$, it follows that z is ϵ_0-homologically trivial in Φ, giving a contradiction.

Proof of theorem 4.1: Let $\dim \partial_\infty \Gamma = p$. Let $X \subset \partial_\infty \Gamma$ be a closed proper subset such that $\dim X = \dim \partial_\infty \Gamma = p$, and let $a \in \partial_\infty \Gamma \setminus X$. Let $\{H_\gamma(n) : n \in \mathbb{Z}\}$ be a system of horospheres centered at a.

For any $\alpha > 0$ we shall find $n_\alpha \in \mathbb{Z}$, and an r-cycle ϕ_α with $r = 16\delta$, which is r-homologically trivial in $H_\gamma(n_\alpha)$, but α-homologically nontrivial in its support $supp\, \phi_\alpha$. Since, by the proof of theorem 3.1, it is possible to construct out of the cycle $\phi_{5\alpha}$ an r'-cycle c_α which is r'-homologically trivial in Γ, but α-homologically nontrivial in its support $supp\, c_\alpha$, where $r' = r + 16\delta + 2$, this will finish the proof of theorem.

Let d_a be the Busemann's metric of proposition 2.11 restricted to the set $X \subset \partial_\infty \Gamma \setminus \{a\}$. According to remark 4.3, since $\dim X = p$, there exists a closed subset $\Phi \subset X$ and an $\epsilon_0 > 0$ such that for each $\eta > 0$ there exists an η-chain w_η of dimension p, being a relative η-cycle ϵ_0-homologically nontrivial in (X, Φ), with $z = \partial w_\eta$ being ϵ_0-homologically nontrivial in Φ. Fix σ as in lemma 4.4 applied to $\Phi \subset X$ and ϵ_0, take $\eta = \sigma$ and fix an η-chain w_η as above.

Construction of r-cycle ϕ_α and its r-filling ψ_α.

Fix any $\alpha > 0$. Let ϵ with $0 < \epsilon < \epsilon_0$ be so small, that the number $N(\epsilon)$ of proposition 2.11.(1) for $C = r$ is so small, that putting $C = \alpha$ in proposition 2.11.(2), we get $\epsilon(N(\epsilon)) < \epsilon'$, where $\epsilon' = \epsilon_0/3$. Then put $n_\alpha = N(\epsilon)$.

By lemma 4.4, for $\rho = \min(\epsilon, \epsilon')$, there exists a relative ρ-cycle w_ρ ϵ_0-homologous to w_η in (X, Φ). Note that then w_ρ is ϵ_0-homologically nontrivial in (X, Φ), and that $z_\rho = \partial w_\rho$

is a ρ-cycle ϵ_0-homologically nontrivial in Φ.

Now put $\phi_\alpha = (P_{n_\alpha})_* z_\rho$ and $\psi_\alpha = (P_{n_\alpha})_* w_\rho$, where $P_{n_\alpha} : \partial_\infty \Gamma \setminus \{a\} \to H_\gamma(n_\alpha)$ is the parabolic-gradient-geodesic projection of section 2. By the choice of n_α and ρ, recalling that $r = 16\delta$, it follows that ϕ_α is an r-cycle and ψ_α its r-filling in $H_\gamma(n_\alpha)$.

Claim. *Cycle ϕ_α is α-homologically nontrivial in its support.*

Suppose on the contrary that w_α is an α-filling of ϕ_α in its support. Consider a function $f : supp\, \psi_\alpha \to \Phi$ such that $P_{n_\alpha} \circ f(v) = v$. Then, since n_α is small enough, $f_* \phi_\alpha$ is an ϵ'-cycle with ϵ'-filling $f_* \psi_\alpha$ in Φ. But according to lemma 4.5, since $f_* \phi_\alpha$ is also an ϵ'-deformation image of z_ρ, it is ϵ_0-homologically nontrivial in Φ, which contradicts the existence of the ϵ'-filling $f_* \psi_\alpha$, and the claim follows.

This finishes the proof of theorem.

5. Dimension of strongly isoperimetric groups.

In chapter 9 of [**6**] M. Gromov shows that most (in a precise statistical sense) of groups with presentations of a certain type are hyperbolic, and moreover strongly isoperimetric in the following sense:

Definition 5.1. A group Γ is called *strongly isoperimetric*, if it admits a finite presentation $\Gamma =< S|R >$, for which there is a constant $C > 0$ such that for any reduced word w_0 on the alphabet $S \cup S^{-1}$ representing the unit element $e \in \Gamma$, and for any reduced Dehn diagram P of w_0, one has

$$|P| \leq C \cdot |w_0|$$

where $|w_0|$ denotes the length of w_0 and $|P|$ denotes the number of cells in P (corresponding to conjugates of cyclically reduced relators of R).

See [**10**] for the definition of Dehn diagram (compare also [**8**] III.9 where it is called simple diagram).

Example 5.2. Groups satisfying several types of small cancellation conditions are strongly isoperimetric (see [**10**] 2.6 for more details).

Remark 5.3. Note that strongly isoperimetric groups are hyperbolic (cf. [**7**] 1.43 or [**9**] Theorem 2.5).

The main result of this section will be based on the properties of the following 2-dimensional piecewise Euclidean complex:

Definition 5.4. Given a group with presentation $\Gamma =< S|R >$, a *Dehn complex* $C(\Gamma; S, R)$ is a cellular complex that arises in the following way. Let p be a polygonal closed path in a Cayley graph $G(\Gamma, S)$ of one of the following forms:

(1) p corresponds to a cyclically reduced relator of R;

(2) p is the image of some path of form (1) by automorphism of the Cayley graph determined by some element of Γ (Γ acts by automorphisms on $G(\Gamma, S)$).

For each such path we attach to the Cayley graph a Euclidean 2-cell so that its polygonal boundary is glued to $G(\Gamma, S)$ according to the parametrization of a path (paths that differ by cyclic reparametrizations are identified).

LEMMA 5.5. *If Γ is a strongly isoperimetric group with respect to a finite presentation $\Gamma =< S|R >$, then the homology group $H_2[C(\Gamma; S, R)]$ of its Dehn complex vanishes for any coefficients.*

PROOF: Since $C(\Gamma; S, R)$ is a 2-dimensional simply connected complex (or 1-dimensional if R is empty, but this case is obvious), then according to universal coefficients theorem it is enough to show that $H_2[C(\Gamma; S, R)]$ vanishes for integer coefficients.

Suppose on the contrary that $H_2(C(\Gamma; S, R), \mathbb{Z}) \neq 0$. Realize a certain homology nonzero element as a geometric cycle, that is an image of a cellular pseudomanifold M

by a cellular nondegenerate mapping $f : M \to C(\Gamma; S, R)$ (compare [**4**] 1.3.7). Since we consider the dimension 2, we may assume M to be an orientable manifold; since $C(\Gamma; S, R)$ is simply connected, M may be assumed to be a 2-sphere. Furthermore, we may assume that the realization f is reduced on M in the sense that no two distinct 2-cells of M having a common edge are mapped onto the same cell of $C(\Gamma; S, R)$; otherwise we could reduce the realization f in a finite number of steps, possibly changing the nonzero homology element (we omit the elementary details).

We can then find two distinct vertices of M, at which not less than three cells meet, and take arbitrarily big covering M' of M branched over these two points, being again a 2-sphere. Removing any 2-cell from M' we get a reduced Dehn diagram P violating the inequality $|P| \leq C \cdot |w_0|$, which gives a contradiction.

Remark 5.6. Note that for small cancellation groups the nonexistence of realization $f : M \to C(\Gamma; S, R)$ modelled on a 2-sphere is a consequence of the Gauss-Bonnet theorem (compare [**10**] 2.4 or [**8**] p. 246).

THEOREM 5.7. *If Γ is a strongly isoperimetric group, then $asdim_h \Gamma \leq 2$.*

PROOF: We shall show that for each $r > 0$ there is $\alpha > 0$ such that each q-dimensional r-cycle in Γ with $q \geq 2$ is α-homologically trivial in its support.

Case $q = 2$. Let ϕ be a 2-dimensional r-cycle in Γ. Triangulate the complex $C(\Gamma; S, R)$ by subdividing each 2-cell into triangles, without introducing new vertices. We can subdivide the cycle ϕ, to make it a simplicial cycle in $C(\Gamma; S, R)$, as follows:

for any 1-simplex appearing in ϕ choose a geodesic segment joining its vertices; then to each 2-simplex there corresponds a geodesic triangle in the Cayley graph $G(\Gamma, S)$. Since it is δ-thin, we can subdivide it into geodesic triangles with vertices in Γ and with lengths of edges bounded by $\delta + 1$. By the isoperimetric inequality, there is $m > 0$ such that each such triangle of the subdivision can be filled (after cyclic reduction) by no more than m cells of $C(\Gamma; S, R)$. After dividing the cells into triangles we get a subdivision $\tilde{\phi}$ of ϕ, which is a simplicial cycle in $C(\Gamma; S, R)$.

Note that there is a constant $C > 0$, depending on δ, m and a maximal diameter of cells in $C(\Gamma; S, R)$ (assumed to be regular Euclidean n-gons), such that

(1) vertices of $\tilde{\phi}$ are at distance bounded by C from vertices of ϕ;

(2) $\tilde{\phi}$ is C-homologous to ϕ by a C-chain with support contained in $supp\, \phi \cup supp\, \tilde{\phi}$.

From lemma 5.5 and the fact that $\dim C(\Gamma; S, R) = 2$ it follows that $\tilde{\phi} = 0$ as a $(\delta + 1)$-chain, and thus ϕ can be C-filled in the C-neighbourhood of its support. Applying the appropriate C-deformation we get a $3C$-filling of ϕ in its support, and we finish the proof of this case by putting $\alpha = 3C$.

Case $q > 2$**.** Let ϕ be a q-dimensional r-cycle. We can subdivide a top dimensional simplex σ of it in the following way: subdivide all 2-faces of σ as in the previous case; then for a 3-face τ choose one its vertex, say v, and make a simplicial cone over the already subdivided boundary $\partial\tau$ with v as a cone vertex; use the same procedure recursivly for all faces of dimensions up to q. Denote by $\tilde{\phi}$ a new cycle gotten from ϕ by subdivision of all its simplices as above. Observe that if C is the constant of the previous case, then

(1) vertices of $\tilde{\phi}$ are at distance bounded by C from those of ϕ;

(2) $\tilde{\phi}$ is $(r+C)$-homologous with ϕ by a chain with support contained in $supp\, \phi \cup supp\, \tilde{\phi}$.

Note also, that according to the previous case, if τ is a 3-face of ϕ and $\rho = \partial\tau$, then the subdivision $\tilde{\rho}$ of ρ gives a simplicial 2-cycle in $C(\Gamma; S, R)$ which is equal to zero as an $(r+C)$-cycle. This means that $\tilde{\phi} = 0$ as an $(r+C)$-cycle, so that ϕ is $(r+C)$-homologically trivial in a C-neighbourhood of its support. After applying an appropriate C-deformation we get that ϕ is $(3C+r)$-homologically trivial in its support, which finishes the proof after putting $\alpha = 3C + r$.

Remark 5.8. In section 9.B of [**6**] M. Gromov states without proof the following property of strongly isoperimetric groups:

If Γ *is strongly isoperimetric then* $\dim \partial_\infty \Gamma \leq 1$.

Note that this fact is a direct consequence of theorems 4.1 and 5.1.

Appendix: Comparison with Gromov's large scale dimension.

In [**6**] M. Gromov defines the large scale dimension $asdim_+$ of a metric space (see definition A.2 below). In this Appendix we prove the following:

Theorem A. *Let* X *be a metric space. Then* $asdim_h X \leq asdim_+ X$.

Definition A.1. Let $\{A_j : j \in J\}$ be a family of subsets of a metric space X. Given $r > 0$ we say that this family is *r-neighbouring*, if to each $j \in J$ we can assign an element $v_j \in A_j$ so that $d(v_{j_1}, v_{j_2}) \leq r$ for any $j_1, j_2 \in J$.

Definition A.2. Given a metric space X, we say that $asdim_+ X \leq p$ if for any $r > 0$ there exist $\alpha > 0$ and a covering $\{A_i : i \in I\}$ of X by sets satisfying $diam\, A_i \leq \alpha$, with r-multiplicity bounded by $p+1$ (*r-multiplicity* is defined as $\max\{card\, J : J \subset I$ such that a family $\{A_j : j \in J\}$ is r-neighbouring$\}$).

Definition A.3. An *abstract r-nerve* of a covering $\{A_i : i \in I\}$ is a simplicial complex K with the vertex set I defined as follows: a finite subset $J \subset I$ spans an abstract simplex of K if and only if the family $\{A_j : j \in J\}$ is r-neighbouring.

Remark A.4. If a covering $\{A_i : i \in I\}$ has r-multiplicity $p+1$, its r-nerve K is a p-dimensional simplicial complex.

Definition A.5. Let $\{A_i : i \in I\}$ be a covering of a metric space X, and $f : I \to X$ be a function such that $f(i) \in A_i$. A *geometric r-nerve* of this covering with respect to f is a simplicial complex $K_f = f_* K$, where K is the abstract nerve of this covering. We equip K_f with a piecewise Euclidean metric induced by distances of vertices $\{f(i) : i \in I\}$ with respect to the metric in X.

Remark A.6. (a) The simplicial function $f : K \to K_f$ needn't be an isomorphism, since f needn't be injective.

(b) If the covering $\{A_i : i \in I\}$ has r-multiplicity $p+1$ then $\dim K_f \leq p$.

(c) If $diam\, A_i \leq \alpha$ for $i \in I$, then K_f consists of $(r+2\alpha)$-simplices.

The theorem A is an evident consequence of the following:

PROPOSITION. *Assume that $asdim_+ X = p$. Then for each $r > 0$ there exists $\beta > 0$ such that:*

(1) *each p-dimensional r-cycle which is r-homologically trivial in X is β-homologically trivial in its support;*

(2) *for $q > p$, each q-dimensional r-cycle in X is β-homologically trivial in its support.*

PROOF OF (2): Since $asdim_+ X = p$, for given $r > 0$ there exists a covering $\{A_i : i \in I\}$ of X, with $diam\, A_i \leq \alpha$ and r-multiplicity $p+1$. For $q > p$, let ϕ be any q-dimensional r-cycle in X. Let $g : supp\, \phi \to I$ be a function such that $v \in A_{g(v)}$ for every $v \in supp\, \phi$. Then $(f \circ g)_* \phi$ is a simplicial cycle in K_f. Since $\dim K_f \leq p$ and $q > p$, we have $H_q(K_f, \mathbb{T}) = 0$, and so in fact $(f \circ g)_* \phi = 0$ as an $(r+2\alpha)$-cycle. But since $f \circ g$ is an α-deformation, we get by lemma 1.8 that ϕ is $(r+2\alpha)$-homologically trivial in α-neighbourhood of its support, and thus $(r+4\alpha)$-homologically trivial in $supp\, \phi$. We complete the proof by putting $\beta = r + 4\alpha$.

PROOF OF (1): Given $r > 0$, consider a covering $\{A_i : i \in I\}$ of X, with $diam\, A_i \leq \alpha$ and r-multiplicity $p+1$. Let ϕ be any p-dimensional r-cycle, and ψ its r-filling in X.

Let $I_0 = \{i \in I : A_i \cap supp\,\phi \neq \emptyset\}$ and $X_0 = \bigcup_{i\in I_0} A_i$. Let ψ_0 be a r-chain consisting of those simplices of ψ (with coefficients in $\mathbb{T}$) which have at least one vertex in the set $\bar{N}_r(X_0) = \{x \in X : d(x, X_0) \leq r\}$. Note that the boundary $\partial\psi_0$ can be decomposed into the sum $\partial\psi_0 = \phi + \lambda$ of r-cycles having disjoint support.

Let K_0 be a subcomplex of K consisting of those simplices $(v_1, \ldots, v_m)$ with $v_i \in I \setminus I_0$ for $1 \leq i \leq m$, for which there is $v_0 \in I_0$ such that $(v_0, v_1, \ldots, v_m) \in K$. Note that, by definition, $\dim K_0 = p - 1$. Let $g : supp\,\lambda \to I$ be a function such that $v \in A_{g(v)}$ for $v \in supp\,\lambda$. Realize that $(f \circ g)_*\lambda$ is a simplicial cycle in $f(K_0)$, but since $\dim f(K_0) \leq p - 1$, it is equal to 0 as an $(r + 2\alpha)$-cycle. Thus λ is $(r + 2\alpha)$-homologically trivial in α-neighbourhood of its support, which implies that ϕ is $(r + 2\alpha)$-homologically trivial in $(r + 2\alpha)$-neighbourhood of its support, and so finally it is $(3r + 6\alpha)$-homologically trivial in $supp\,\phi$.

The proof is completed by putting $\beta = 3r + 6\alpha$.

Remark A.7. Note that theorem A together with theorem 4.1 give an inequality

$$\dim \partial_\infty \Gamma + 1 \leq asdim_+ \Gamma.$$

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