PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 10, Fasc. 2 (1989), p. 283-295

DECOUPLING AND STOCHASTIC INTEGRATION IN UMD BANACH SPACES*

BY

TERRY R. McCONNELL (SYRACUSE)

Abstract. Rosiński and Suchanecki have characterized the class of deterministic E-valued functions integrable with respect to Brownian motion, where E is a given Banach space. We extend their result to random predictable integrands in case E belongs to the class UMD. The proof is based upon some new decoupling inequalities for E-valued martingale difference sequences.

Introduction. In this paper we construct the stochastic integral with respect to Brownian motion of a predictable process taking values in a UMD Banach space. The criterion for integrability is that the integrand process should have sample paths which belong almost surely to the space of deterministic integrands for which the integral exists. This parallels the real-valued case, in which a predictable process is integrable in Itô's sense iff it has square integrable sample paths.

Since their introduction by Burkholder in [5], the UMD Banach spaces have proved the natural setting for vector-valued generalizations of real-valued results which are based upon orthogonality (see, for example, [6], [3], and [12]). Since the usual construction of the Itô integral depends upon the fact that multiplication by a predictable process preserves orthogonality of increments, it is quite natural to study the Itô integral of UMD-valued integrands.

Let W and W^* be independent Brownian motions defined on a common probability space Ω, E a UMD Banach space over $\boldsymbol{R}$ or $\boldsymbol{C}$, and $e: \Omega \times [0, T] \to E$ a process which is predictable relative to W. (See section 3 for precise definitions.) Our method is to use decoupling inequalities. Roughly speaking, these reduce the problem of defining the integral $\int e(\omega, t)\, dW(t)$ to that of defining $\int e(\omega, t)\, dW^*(t)$, $0 \leqslant t \leqslant T$. The latter is much easier to analyze since the integrand is independent of W^*. The same method has been used by Kwapień and Woyczyński [11] in the real case, but their proof does not extend to the vector-valued case. Our first task (section 2) is then to obtain suitable

* Supported in part by the Alfred P. Sloan Foundation.

decoupling inequalities for E-valued martingale difference sequences. The resulting inequalities, which should be of independent interest, do not hold in any wider class of Banach spaces.

The class of deterministic Wiener-integrable functions $f\colon [0, T] \to E$, where E is a given Banach space, not necessarily UMD, has been characterized in [18]. This characterization is not very explicit in general, but, as pointed out in [18], it is possible to give easily verifiable necessary and sufficient conditions for integrability in some special cases of interest. For example, let $E = l^p$, $1 < p < \infty$ (this space is UMD). Let $f\colon \Omega \times [0, T] \to E$ be predictable and write

$$f(\omega, t) = \sum_{j=1}^{\infty} f_j(\omega, t) e_j$$

in terms of the standard Schauder basis of l^p. Then the integral $\int_0^T f(\omega, t)\, dW(t)$ exists, in the sense explained in section 3, iff

$$\sum_{j=1}^{\infty} \Big(\int_0^T f_j^2(\omega, t)\, dt\Big)^{p/2} < \infty \quad \text{a.s.} \tag{1.1}$$

(See the discussion at the end of section 3.)

Previous work on Itô stochastic integration of vector-valued integrands has focused on the Hilbert-valued case or, more generally, on the case of integrands with values in a 2-uniformly smooth space ([16]. Also see [8] and [19]. It follows from the results of [16] that a UMD space is 2-smooth iff it has type 2.) In the latter case the stochastic integral has been defined under the condition

$$\mathrm{E} \int_0^T \|f(\omega, t)\|_E^2\, dt < \infty. \tag{1.2}$$

We should point out that, in the case of the 2-smooth spaces l^p, $2 < p < \infty$, condition (1.2) is much more restrictive than (1.1), even without the expectation sign.

There is also a well-developed theory of stochastic integration of operator-valued integrands with respect to Hilbert-space valued square integrable martingales. See [15], [2], and [14].

Throughout this paper λ will denote normalized Lebesgue measure on $[0, T]$. Two positive variable quantities A and B are related by $A \approx B$ if their ratio is bounded below and above by constants whenever either is finite. The symbol $\|\ \|$ denotes the norm of the Banach space E.

2. Decoupling inequalities for martingale differences. Recall that the Banach space E is UMD (for Unconditional Martingale Differences) if there exists a p, $1 < p < \infty$, and a constant β_p such that, for every n, every E-valued martingale difference sequence $d_1, d_2, \ldots, d_n$, and every sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ of numbers in $\{+1, -1\}$ we have

$$\mathrm{E}\|\sum_{j=1}^{n}\varepsilon_j d_j\|^p \leqslant \beta_p^p \mathrm{E}\|\sum_{j=1}^{n} d_j\|^p. \tag{2.1}$$

As soon as (2.1) holds for one such p, it holds for all p in the range $1<p<\infty$. Examples of UMD spaces are l^p, $L^p[0, 1]$, and the Schatten C^p spaces. (See [5] or [7] for further information on UMD spaces.) The spaces l^1, $L^1[0, 1]$, and C^1 are not UMD.

We begin with a very simple decoupling result. Our reason for including it is that it shows very clearly the connection between decoupling and the defining property (2.1) of UMD spaces.

PROPOSITION 2.1. *Let $\eta_1, \eta_2, \ldots$ be a sequence of independent real-valued centered Gaussian random variables. Let $\eta_1^*, \eta_2^*, \ldots$ be an independent copy of this sequence defined on the same probability space. Let E be a UMD Banach space and $e_1, e_2, \ldots$ a sequence of strongly measurable E-valued random variables such that e_n is a function of $\eta_1, \ldots \eta_{n-1}$. Then*

$$\mathrm{E}\|\sum_{j=1}^{n} e_j\eta_j\|^p \approx \mathrm{E}\|\sum_{j=1}^{n} e_j\eta_j^*\|^p \tag{2.2}$$

for each $1<p<\infty$.

Proof. Write $\eta_j = x_j + y_j$ with the pair x_j, y_j i.i.d. (hence also centered Gaussian.) We may assume that either the right or left-hand side of (2.2) is finite. Then the sequence $e_1x_1, e_1y_1, e_2x_2, e_2y_2, e_3x_3, \ldots$ is a martingale difference sequence. Applying (2.1) and the reverse inequality to this sequence with $\varepsilon_j = (-1)^{j+1}$ we obtain

$$\mathrm{E}\|\sum_{j=1}^{n} e_j(x_j+y_j)\|^p \approx \mathrm{E}\|\sum_{j=1}^{n} e_j(x_j-y_j)\|^p.$$

The desired result follows since the sequences $\{e_j(x_j-y_j)\}$ and $\{e_j\eta_j^*\}$ have the same distribution.

Two $\mathscr{F}_n$-adapted martingale difference sequences $\{d_j\}$ and $\{d_j^*\}$ are said to be *tangent relative to* $\mathscr{F}_n$ (terminology essentially due to Jacod [10]) if for each n the pair (d_n, d_n^*) is conditionally i.i.d. almost surely, given $\mathscr{F}_{n-1}$. The main result of this section may now be stated.

THEOREM 2.2. *Let E be a UMD Banach space. If p satisfies $1<p<\infty$, there is a constant γ_p such that, for every pair $\{d_j\}$ and $\{d_j^*\}$ of E-valued tangent martingale difference sequences such that $\mathrm{E}\|d_j\|^p$ and $\mathrm{E}\|d_j^*\|^p$ are finite for each j, we have*

$$\mathrm{E}\|\sum_{j=1}^{n} d_j^*\|^p \leqslant \gamma_p^p \mathrm{E}\|\sum_{j=1}^{n} d_j\|^p, \quad n = 1, 2, \ldots \tag{2.3}$$

We also have the weak-type inequality

$$\lambda \mathrm{P}\big(\sup_n \|\sum_{j=1}^{n} d_j^*\| > \lambda\big) \leqslant c \sup_n \mathrm{E}\|\sum_{j=1}^{n} d_j\|, \quad \lambda > 0, \tag{2.4}$$

for some constant c. The constants in these inequalities depend only on E and p.

In the case $E = \mathbf{R}$ or, more generally, for E finite dimensional, inequalities (2.3) have been obtained by Zinn [21]. We believe that inequality (2.4) is new even in the real case.

Example 2.3. Let $X_1, X_2, \ldots$ be a sequence of independent, mean zero real-valued random variables and $X_1^*, X_2^*, \ldots$ an independent copy of this sequence. Let $e_1, e_2, \ldots$ be E-valued with e_j measurable $\sigma(\{X_1, \ldots, X_{j-1}, X_1^*, \ldots, X_{j-1}^*\})$. Suppose $\mathrm{E}\|e_j X_j\| < \infty$ for each j. Then $\{e_j X_j\}$ and $\{e_j X_j^*\}$ are tangent sequences. (The same holds if the e_j are real-valued and the X_j are E-valued.) Note that Proposition 2.1 corresponds to the special case in which the X_j are Gaussian and the e_j are independent of the starred variables.

Lemma 2.4. *Let E be a Banach space and X and Y i.i.d. E-valued Bochner-integrable random variables defined on a common probability space. Suppose Φ is biconcave on $E \times E$, i.e., concave in each variable separately, and such that $\Phi(X+Y, X-Y)$ is an integrable random variable. Also assume that Φ is lower semi-continuous in its first variable. Then*

$$\mathrm{E}\,\Phi(X+Y, X-Y) \leqslant \Phi(2EX, 0). \tag{2.5}$$

Proof. Since X and Y are i.i.d.,

$$\begin{aligned}
\mathrm{E}\,\Phi(X+Y, X-Y) &= \frac{1}{2}\mathrm{E}\,\Phi(X+Y, X-Y) + \frac{1}{2}\mathrm{E}\,\Phi(X+Y, Y-X) \\
&\leqslant \mathrm{E}\,\Phi(X+Y, 0) \quad \text{(concavity in second variable)} \\
&\leqslant \Phi(2EX, 0) \quad \text{(concavity in first variable).}
\end{aligned}$$

Proof of Theorem 2.2. We apply the methods and results of D. L. Burkholder. It is shown in [7] that there is a biconcave function Φ on $E \times E$ such that

$$\left\|\frac{x+y}{2}\right\|^p - \beta_p^p \left\|\frac{x-y}{2}\right\|^p \leqslant \Phi(x, y), \qquad x, y \in E. \tag{2.6}$$

Here β_p is as in (2.1). Furthemore, for some $0 < C < \infty$,

$$|\Phi(x, y)| \leqslant C(1 + \|x\|^p + \|y\|^p), \tag{2.7}$$

and

$$\Phi(0, 0) = 0. \tag{2.8}$$

Indeed, the existence of such a function Φ is equivalent to (2.1) (see [7]). Put

$$F_n = \sum_{i=1}^{n} d_i \quad \text{and} \quad G_n = \sum_{i=1}^{n} d_i^*.$$

Then, by (2.6) and Lemma 2.1,

$$\mathrm{E}\|G_n\|^p - \beta_p^p \mathrm{E}\|F_n\|^p \leqslant \mathrm{E}\Phi(G_n+F_n, G_n-F_n) = \mathrm{E}\mathrm{E}\big(\Phi(G_n+F_n, G_n-F_n)|\mathscr{F}_{n-1}\big)$$
$$\leqslant \mathrm{E}\Phi(G_{n-1}+F_{n-1}, G_{n-1}-F_{n-1}),$$

where we used (2.5) in the last step. Repeating the argument $n-1$ times and applying (2.8), we have $\mathrm{E}\|G_n\|^p - \beta_p^p \mathrm{E}\|F_n\|^p \leqslant \Phi(0, 0) = 0$, which proves (2.3) with $\gamma_p = \beta_p$.

To prove (2.4) we again use the results and methods of Burkholder from [7]. It is shown in section 8 of [7] that there is a biconvex function u on $\mathrm{E} \times \mathrm{E}$ such that

(2.9) $$u(x, y) \leqslant \|x+y\| \quad \text{if } \max\{\|x\|, \|y\|\} \geqslant 1,$$

(2.10) $$u(x, y) \leqslant u(0, 0)+\|x+y\|,$$

and

(2.11) $$u(0, 0) > 0.$$

Let

$$\tau = \inf\{n: \|\sum_{j=1}^{n} d_j^*\| > 1\},$$

with $\inf \emptyset = +\infty$. Note that $\{d_j^* 1_{\{\tau > j\}}\}$ and $\{d_j 1_{\{\tau > j\}}\}$ are again tangent sequences. Put

$$F_n = \sum_{j=1}^{n} 1_{\{\tau > j\}} d_j \quad \text{and} \quad G_n = \sum_{j=1}^{n} 1_{\{\tau > j\}} d_j^*.$$

We shall prove that, for each n,

(2.12) $$\mathrm{P}\big(\sup_{1 \leqslant l \leqslant n} \|\sum_{j=1}^{l} d_j^*\| > 1\big) \leqslant \frac{2}{u(0, 0)} \mathrm{E}\|\sum_{j=1}^{n} d_j\|.$$

Inequality (2.4) then follows by homogeneity and passage to the limit as n tends to infinity. Now

$$\mathrm{P}\big(\sup_{1 \leqslant l \leqslant n} \|\sum_{j=1}^{l} d_j^*\| > 1\big) \leqslant \mathrm{P}(\|G_n\| > 1) \leqslant \mathrm{P}\big(\max\{\|F_n+G_n\|, \|F_n-G_n\|\} > 1\big).$$

By (2.9),

$$u(0, 0)\,\mathrm{P}\big(\max\{\|F_n+G_n\|, \|F_n-G_n\|\} > 1\big)$$
$$\leqslant \mathrm{P}\big(2\|F_n\| - u(F_n+G_n, F_n-G_n)+u(0, 0) \geqslant u(0, 0)\big)u(0, 0).$$

By (2.10) the expression $2\|F_n\| - u(F_n+G_n, F_n-G_n)+u(0, 0)$ is nonnegative, so Chebyshev's inequality gives the upper bound $2\mathrm{E}\|F_n\| - \mathrm{E}u(F_n+G_n, F_n-G_n)$

$+u(0, 0)$. Using Lemma 2.4 exactly as in the proof of (2.3) above we have $-\mathrm{E}u(F_n+G_n, F_n-G_n) \leqslant -u(0, 0)$, and the desired result follows.

In the case $E = \boldsymbol{R}$ there exists [7] a function u satyisfying (2.9) and (2.10) with $u(0, 0) = 1$. Thus, in this case (2.4) holds with $c = 2$.

We shall conclude this section by showing that if inequalities (2.3) hold for any $1 < p < \infty$, then E must be UMD.

Definition 2.5. Let $r_1, r_2, \ldots$ be the Rademacher sequence. An E-valued martingale difference sequence d_n of the form $d_n = e_n r_n$, where e_n is an E-valued function of $r_1, r_2, \ldots, r_{n-1}$, will be termed a *Walsh-Paley martingale difference sequence.*

It can be shown that E is UMD if and only if (2.1) holds for all Walsh-Paley martingale difference sequences [13].

Now let $r_1^*, r_2^*, \ldots,$ be an independent copy of the Rademacher sequence and $e_j = e_j(r_1, r_2, \ldots, r_{j-1}, r_1^*, \ldots, r_{j-1}^*)$ be E-valued functions. Then by (2.3) (see Example 2.3) we have

$$\mathrm{E}\|\sum_{j=1}^{n} e_j r_j\|^p \approx \mathrm{E}\|\sum_{j=1}^{n} e_j r_j^*\|^p. \tag{2.13}$$

Equivalently, we have the two-sided inequalities

$$\mathrm{E}\|\sum_{j=1}^{n} r_j^* d_j\|^p \approx \mathrm{E}\|\sum_{j=1}^{n} d_j\|^p, \tag{2.14}$$

whenever d_j is a Walsh-Paley martingale difference sequence and r_j^* are independent of the d_j. Now inequality (2.1) for the d_j follows at once from (2.14) since the r_j^* are symmetric random variables (one may take $\beta_p = \gamma_p^2$).

Remark 2.6. Garling [9] has raised the following interesting open questions: In which Banach spaces do the right-hand (resp. left-hand) inequalities in (2.13) hold when the e_j are functions of only one of the Rademacher sequences?

Remark 2.7. We may take $\gamma_p = \beta_p$ in (2.3). Are these the best constants? The size of these constants is of some interest. For example, if α_p denote the best constants in the E-valued Riesz inequalities (see [6]), then it can be shown that $\alpha_p \leqslant \gamma_p^2$.

3. Stochastic integration with respect to Brownian motion. Let E be a Banach space and $W(t)$ standard real-valued Brownian motion. Following [18] we shall declare a *non-random* strongly Lebesgue measurable function $f: [0, T] \to E$ to be W-integrable if there exists a sequence f_n of E-valued simple functions such that $\|f_n - f\|$ tends to zero in measure and

$$f_n \circ W(t) \equiv \int_0^t f_n(s)\, dW(s)$$

converges in probability at $t = T$ (hence uniformly in $0 \leqslant t \leqslant T$ in probability by Lévy's inequality). One defines a random E-valued continuous function $f \circ W(\bullet)$ as the limit of the random functions $f_n \circ W(\bullet)$ and checks easily that this definition does not depend on the chosen sequence f_n. The class of such W-integrable f, denoted by $\mathscr{L}([0, T], E, W)$, becomes an F-space in any of the (equivalent) metrics

$$\|f\|_{\mathscr{L}} \equiv \|f\|_0 + (E\|f \circ W(T)\|^p)^{1/p^*},$$

where $p^* = \max\{p, 1\}$, $0 < p < \infty$, and

$$\|f\|_0 = \int_0^T \min\{\|f(s)\|, 1\}\, ds.$$

See [12] for proofs and further discussion.

In particular, it is shown there that the class $\mathscr{L}([0, T], E, W)$ (hereafter shortened to $\mathscr{L}$) is highly dependent on E and cannot in general be characterized by any moment condition. Indeed, there are Banach spaces E and *uniformly bounded* E-valued functions f which fail to be W-integrable. We need below the fact that, given f in $\mathscr{L}$ and φ^* in E^*, we have $\varphi^* \circ f$ in $L^2[0, T]$ so that the usual Wiener integral

$$\int_0^T \varphi^* \circ f(s)\, dW(s)$$

is well-defined. Also, if $f \in \mathscr{L}$ and g is a real-valued Lebesgue measurable function such that $|g| \leqslant 1$, then $gf \in \mathscr{L}$ (see [18]).

Let $\mathscr{F}_t$ be the completed filtration of W. Define the family of *elementary integrands* on $[0, T]$, $\mathscr{E}(E, T)$, to be the linear space of functions on $\Omega \times [0, T]$ spanned by functions of the form

$$x\, 1_A(\omega)\, 1_{[0]}(t), \quad A \text{ in } \mathscr{F}_0,\ x \text{ in } E,$$

and by functions of the form

$$x\, 1_A(\omega)\, 1_{(s,t]}, \quad A \text{ in } \mathscr{F}_s,\ x \text{ in } E.$$

Thus, a typical function e in $\mathscr{E}(E, T)$ has the form

$$e(\omega, t) = x_0\, 1_{A_0}(\omega)\, 1_{[0]}(t) + \sum_{j=1}^{n} x_j\, 1_{A_j}(\omega)\, 1_{(t_{j-1}, t_j]}(t), \tag{3.1}$$

where $0 \leqslant t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant T$, $x_j \in E$, $A_0 \in \mathscr{F}_0$, and $A_j \in \mathscr{F}_{t_{j-1}}$, $j = 1, 2, \ldots, n$. We define the process $e \circ W(t)$ for $0 \leqslant t \leqslant T$ by

$$e \circ W(t) = \int_0^t e(\omega, s)\, dW(s) \equiv \sum_{j=1}^{n} x_j\, 1_{A_j}(\omega)\big[W(t_j \wedge t) - W(t_{j-1} \wedge t)\big]. \tag{3.2}$$

Finally, let $\mathscr{L}_P$ denote the class of all strongly $P \otimes \lambda$-measurable E-valued functions e such that

$$\varphi^* \circ e \text{ is } \mathscr{F}_t\text{-predictable for each } \varphi^* \text{ in } E^*, \tag{3.3}$$

and

(3.4) $$t \to e(\omega, t) \text{ belongs to } \mathscr{L} \text{ for } P\text{-a.e. } \omega.$$

The main result of this section may now be stated.

THEOREM 3.1. *Suppose e belongs to $\mathscr{L}_P$. Then there exists a sequence of elementary integrands such that, for each φ^* in E^*,*

(3.5) $$\varphi^* \circ e_n \to \varphi^* \circ e \text{ in } L^2(0, T], \text{ a.s.,}$$

and

(3.6) $$e_n \circ W(t) \text{ converges uniformly in } t \text{ in probability.}$$

The limiting process, denoted by $e \circ W(t)$, is independent (up to indistinguishability) of the choice of e_n satisfying (3.5) *and* (3.6).

A natural question is whether we can have strong rather than weak convergence in (3.5). We believe that this is generally not possible, even in the non-random case, unless E has cotype 2.

We may assume that there is an independent copy W^* of W defined on the same sample space as W. Thus W^* is independent of $\mathscr{F}_\infty$ and is itself a standard Brownian motion. It then follows from Fubini's theorem and the results of [18] that for each e in $\mathscr{L}_P$ there is a random variable $e \circ W^*$ taking values in the space of continuous E-valued functions such that, for every φ^* in E^*,

(3.7) $$\varphi^*(e \circ W^*) = \varphi^*(e) \circ W^* \quad \text{a.s.,}$$

the latter being defined as an ordinary Wiener integral. Also, we get that $\mathrm{E}(\|e \circ W^*\|^p | \mathscr{F}_\infty) < \infty$ a.s. for every $0 < p < \infty$, by Fernique's theorem.

LEMMA 3.2. *Given e in $\mathscr{L}_P$, there exists a sequence e_n of elementary integrands such that for each φ^* in E^* we have*

(3.8) $$\varphi^* \circ e_n \to \varphi^* \circ e \text{ in } L^2[0, T] \text{ a.s.}$$

and

(3.9) $$e_n \circ W^*(t) \to e \circ W^*(t) \text{ uniformly in } t \text{ in probability.}$$

Proof. We assume $T = 1$ for convenience. Note that (3.8) follows from (3.9.), (3.7) and well-known properties of the Wiener integral. For $\delta > 0$ define T_δ on $\mathscr{L}$ by $T_\delta e(s) = e(s - \delta)$ if $\delta \leqslant s \leqslant 1$ and $T_\delta e(s) = 0$ otherwise. We shall show that T_δ maps $\mathscr{L}$ into itself and, moreover,

(3.10) $$T_\delta e \circ W^*(t) \to e \circ W^*(t)$$

uniformly in t in probability as $\delta \to 0$.

To prove this we first show that if $A(n)$ are Borel sets with $A(n) \downarrow A(\infty)$ and $\lambda(A(\infty)) = 0$, then $\mathrm{E}\|e 1_{A(n)} \circ W^*(1)\|^p \to 0$ for any $0 < p < \infty$. If we let

$X_n = e\,1_{A(n)\setminus A(n-1)} \circ W^*(1)$, then $\sum_{n=1}^{\infty} X_n$ converges a.s. by the Itô-Nisio Theorem. Thus

$$e\,1_{A(n)} \circ W^*(1) = \sum_{j=n}^{\infty} X_j \to 0 \text{ a.s.},$$

and the desired result follows from [18], Lemma 2.1.

Now take any $0 < p \leqslant 1$ and $\varepsilon > 0$ and choose Δ so small that for any $0 < \delta \leqslant \Delta$ we have

$$\mathrm{E}\|e\,1_{(1-\delta, 1]} \circ W^*(1)\|^p < \varepsilon/4.$$

Let $e_\delta = e\,1_{[0, 1-\delta]}$. We may find a simple function e_s, supported in $[0, 1-\Delta]$, such that $\mathrm{E}\|e_s \circ W^*(1) - e_\Delta \circ W^*(1)\|^p < \varepsilon/4$ and, hence,

$$\mathrm{E}\|e_s \circ W^*(1) - e_\delta \circ W^*(1)\|^p < \varepsilon/2 \quad \text{for each } 0 \leqslant \delta \leqslant \Delta.$$

By translation invariance of W^* we also have

$$\mathrm{E}\| T_\delta(e_s) \circ W^*(1) - T_\delta(e_\delta) \circ W^*(1)\|^p < \varepsilon/2.$$

Now $T_\delta(e_\delta) = T_\delta(e)$, so combining these estimates yields

$$\mathrm{E}\|T_\delta(e) \circ W^*(1) - e \circ W^*(1)\|^p \leqslant \mathrm{E}\| T_\delta(e_s) \circ W^*(1) - e_s \circ W^*(1)\|^p + \varepsilon.$$

It is easy to show that the first term on the right-hand side tends to 0 as δ tends to 0, and (3.10) follows since ε was arbitrary (uniform convergence in t follows from convergence at $t = 1$ via Lévy's inequality.)

We will now show how to approximate functions in $\mathscr{L}$ by dyadic step functions. Unfortunately, the natural approximation method destroys predictability in the case of random integrands. The purpose of the "shift" operators T_δ introduced above is to restore that predictability.

Let $\mathscr{G}_n$ denote the nth dyadic σ-field of $[0, 1]$ and $\mathscr{Q}_n$ the σ-field generated by the increments $W^*(j2^{-n}) - W^*((j-1)2^{-n})$ for $j = 1, 2, \ldots, 2^n$. If e belongs to $\mathscr{L}$, then $\varphi^* \circ e$ belongs to $L^2[0, 1]$ for each φ^* in E^*. Thus $E(e|\mathscr{G}_n)$ is well-defined in the Pettis sense and it is easy to check that

$$\int_0^1 \mathrm{E}(e|\mathscr{G}_n)(s)\, dW^*(s) = \mathrm{E}\Big(\int_0^1 e(s)\, dW^*(s) | \mathscr{Q}_n\Big) \quad \text{a.s.}$$

We conclude from the convergence theorem for right-closed martingales, valid in an arbitrary Banach space, that

$$\int_0^1 \mathrm{E}(e|\mathscr{G}_n)(s)\, dW^*(s) \to e \circ W^*(1) \quad \text{a.s.}$$

As above, this may be strengthened to

(3.11) $\quad \mathrm{E}(e|\mathscr{G}_n) \circ W^*(t) \to e \circ W^*(t)$ uniformly in t in probability.

Finally, if $e(\omega,t)$ belongs to $\mathscr{L}_p$, then $\mathrm{E}(T_\delta e(\omega,\cdot)|\mathscr{G}_n)\in\mathscr{E}(E,1)$ for $2^{-n}<\delta$. Thus, by combining (3.10) and (3.11), it is easy to construct a sequence e_n with the desired properties.

The next result provides the link between the distributions of $e\circ W$ and $e\circ W^*$ for predictable e. Kwapień and Woyczyński have proved a similar result in the case $E=\mathbf{R}$. (Their proof does not extend to the vector-valued case.) We shall obtain our result directly from the L^2 case ($p=2$ in (2.3)) by means of the extrapolation method of Burkholder and Gundy (the method of proof is modeled on that of [4] Theorem 6.2).

Lemma 3.3. *There is a constant C such that for any $0<a$, b and any $e\in\mathscr{E}(E,T)$ the estimate*

$$\mathrm{P}(\sup_{0\leqslant t\leqslant T}\|e\circ W(t)\|>b)\leqslant C(a/b)^2+2\mathrm{P}(\|e\circ W^*(T)\|\geqslant a) \tag{3.12}$$

holds. We may take $c=\gamma_2^2$ in (2.3).

Proof. We shall first show that if τ is any stopping time relative to the joint σ-fields $\mathscr{F}_t$ of W and W^* *satisfying* $0\leqslant\tau\leqslant T$ a.s., then

$$\mathrm{E}\|e\circ W(\tau)\|^2\leqslant C\mathrm{E}\|e\circ W^*(\tau)\|^2 \tag{3.13}$$

with $C=\gamma_2^2$ in (2.3). By a standard approximation argument involving the quasi-left continuity of Brownian paths we may assume that τ takes only finitely many values. Thus, taking into account the form of e (see (3.1) above), we have

$$e(\omega,t)\,1_{\{\tau(\omega)>t\}}=x_0\,1_{A_0}(\omega)\,1_{[0]}(t)+\sum_{j=1}^{N}x_j\,1_{A_j}(\omega)\,1_{(t_{j-1},t_j]}(t)$$

for suitable $0=t_0\leqslant t_1\leqslant\ldots\leqslant t_N\leqslant T$, $x_j\in E$, $A_0\in\mathscr{F}_0$, and $A_j\in\mathscr{F}_{t_{j-1}}$, $j=1,2,\ldots,N$. Now (3.13) follows readily from (2.3).

Define stopping times μ and ν by $\mu=\inf\{t:\|e\circ W(t)\|\geqslant b\}$ and $\nu=\inf\{t:\|e\circ W^*(t)\|\geqslant a\}$. Then

$$\mathrm{P}(\sup_{0\leqslant t\leqslant 1}\|e\circ W(t)\|>b,\ \sup_{0\leqslant t\leqslant 1}\|e\circ W^*(t)\|<a)\leqslant\mathrm{P}(\|e\circ W(\mu\wedge\nu\wedge T)\|\geqslant b)$$

$$\leqslant b^{-2}\mathrm{E}\|e\circ W(\mu\wedge\nu\wedge T)\|^2\leqslant Cb^{-2}\mathrm{E}\|e\circ W^*(\mu\wedge\nu\wedge T)\|^2\leqslant C(a/b)^2,$$

where we used (3.13) with $\tau=\mu\wedge\nu\wedge T$. Now (3.12) follows since

$$\mathrm{P}(\sup_{0\leqslant t\leqslant T}\|e\circ W(t)\|>b)$$

$$\leqslant\mathrm{P}(\sup_{0\leqslant t\leqslant T}\|e\circ W(t)\|>b,\ \sup_{0\leqslant t\leqslant T}\|e\circ W^*(t)\|<a)+\mathrm{P}(\sup_{0\leqslant t\leqslant T}\|e\circ W^*(t)\|\geqslant a).$$

By exactly the same method of proof we obtain the reverse inequality

(3.14) $$\mathrm{P}\big(\sup_{0\leqslant t\leqslant T}\|e\circ W^*(t)\|>b\big)\leqslant C(a/b)^2+\mathrm{P}\big(\sup_{0\leqslant t\leqslant T}\|e\circ W(t)\|\geqslant a\big)$$

for all $a, b>0$ with $c=\gamma_2^2$ in (2.3).

Remark 3.4. Note that the proof of Lemma 3.3 used stopping times defined in terms of *both* processes W and W^*. Therefore, the full strength of Theorem 2.2 is needed — Proposition 2.1 alone is not sufficient.

Proof of Theorem 3.1. To define the process $e\circ W$ for e in $\mathscr{L}_p$ choose any sequence e_n of elementary integrands satisfying (3.8) and (3.9) of Lemma 3.2. Combining (3.9) with (3.12) we may define the random continuous function $e\circ W(t)$ as the uniform limit in probability of the $e_n\circ W$. To check that $e\circ W$ is independent of the chosen sequence e_n, suppose f_n is any other sequence in $\mathscr{E}(E, T)$ satisfying (3.8) and (3.9). Let $H(\omega, t)$ denote the uniform limit in probability of $f_n\circ W$. Then, for any t in $[0, T]$ and any φ^* in E^*, we have

$$\varphi^*\big(H(\omega, t)\big)=\int_0^t\varphi^*\circ e(\omega, s)\,dW(s)=\varphi^*\big(e\circ W(\omega, t)\big)\qquad\text{a.s.}$$

by (3.7). We conclude that $H(\omega, t)=e\circ W(\omega, t)$ a.s. for each fixed t. But since both $H(\omega, t)$ and $e\circ W(\omega, t)$ have continuous sample paths, the two processes are indistinguishable.

As noted in [18] the class $\mathscr{L}$ can be described explicitly in certain cases. For example, let $(\Sigma, \mathscr{Q}, \mu)$ be a measure space and take $E=L^p(d\mu)$ for some $1\leqslant p<\infty$. Then a strongly measurable E-valued function f may be viewed as a jointly $\mu\otimes\lambda$-measurable function, and we have f in $\mathscr{L}$ iff

$$\int_\Sigma\Big(\int_0^T|f(t, s)|^2dt\Big)^{p/2}\mu(ds)<\infty.$$

To see this recall [18] that f belongs to $\mathscr{L}$ iff f, considered as an E-valued random variable on the sample space $[0, T]$, is pre-Gaussian. The stated criterion thus follows from the well-known condition for an L^p-valued random variable to be pre-Gaussian (see [20] or [1] exercise 13, p. 205). In particular, if f: $\Omega\times[0, T]\to l^p$ is predictable, then the stochastic integral $f\circ W$ exists iff

$$\sum_{j=1}^{\infty}\Big(\int_0^T f_j^2(\omega, t)\,dt\Big)^{p/2}<\infty\qquad\text{a.s.,}$$

where $f=\sum_{j=1}^{\infty}f_je_j$ in terms of the standard basis of l^p.

Remark 3.5. The only properties of Brownian motion used in this section are its integrability, quasi-left continuity, and the independence of increments. Therefore, the methods and results of this section carry over with minimal changes to any process having these properties. (The class of integrable deterministic processes will vary from process to process.) Also, it seems very likely that integrability may be dropped for symmetric processes because of the possibility of truncation.

Added in proof. The author has learned that (2.3) and Lemma 2.4 had been obtained independently by P. Hitczenko.

REFERENCES

[1] A. Araujo and E. Giné, *The Central Limit Theorem for Real and Banach-valued random variables*, Wiley, New York 1980.

[2] C. Asperti, *Sur l'intégration stochastique par rapport à une martingale hilbertienne de carré intégrable*, C. R. Acad. Sci. Paris, Ser. I, 304 (1987), p. 241–244.

[3] E. Berkson, T. A. Gillespie, and P. S. Muhly, *Abstract spectral decompositions quaranteed by the Hilbert transform*, Proc. London Math. Soc. 53 (1986), p. 489–517.

[4] D. L. Burkholder, *Distribution function inequalities for martingales*, Ann. Probab. 1 (1973), p. 19–42.

[5] – *A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional*, ibidem 9 (1981), p. 997–1011.

[6] – *A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions*, Conference on Harmonic Analysis in Honor of Antoni Zygmund, University of Chicago, 1981, Wadsworth International Group, Belmont, California, 1 (1983), p. 270–286.

[7] – *Martingales and Fourier Analysis in Banach Spaces*, C.I.M.E. Lectures, Varenna, Italy, 1985, Lecture notes in mathematics 1206 (1986), Springer, New York.

[8] E. Dettweiler, *Banach space valued processes with independent increments and stochastic integration*, Probability in Banach Spaces IV, Oberwohlfach 1982, Lecture notes in mathematics 990, Springer, Berlin 1983, p. 54–83.

[9] D. J. H. Garling, *Brownian motion and UMD spaces*, Probability and Banach spaces (Preceedings 1985), ibidem 1221, Springer, New York 1987.

[10] J. Jacod, *Une généralization des semimartingales: les processus admettent un processus à accroissement indépendents tangent*, ibidem 1059, Springer, Berlin 91–118.

[11] S. Kwapień and W. A. Woyczyński, *Semimartingale integrals via decoupling inequalities and tangent processes*, Ann. Probab. (to appear).

[12] T. R. McConnell, *On Fourier multiplier transformations of Banach-valued functions*, Trans. Amer. Math. Soc. 285 (1984), p. 739–757.

[13] – *A Skorohod-like representation in infinite dimensions*, Probability in Banach Spaces V, Tufts University, 1984, Lecture Notes in Mathematics 1153, Springer, Berlin 1985, p. 359–368.

[14] M. Metivier and J. Pellaumail, *Stochastic Integration*, Academic Press, New York 1980.

[15] M. Metivier and G. Pistone, *Une formule d'isométrie pour l'intégrale stochastique hilbertienne et équations d'évolution linéaires stochastiques*, Z. für Wahrscheinlichkeitsth. verw. Geb. 33 (1975), p. 1–18.

[16] G. Pisier, *Martingales with values in uniformly convex spaces*, Israel J. Math. 20 (1975), p. 326–350.

[17] J. Rosiński, *Bilinear random integrals*, Dissertationes Mathematicae 259 (1987), p. 1–71.

[18] J. Rosiński and Z. Suchanecki, *On the space of vector-valued functions integrable with respect to the white noise*, Coll. Math. 431 (1980), p. 183–201.

[19] W. Zh. Yang and T. M. Tsai, *Itô's integration on 2-smooth Banach space*, Chinese J. Math. 11 (1983), p. 193–206.

[20] N. Vakhania, *Sur une propertié des repartitions normales de probabilités dans les espaces* l_p $(1 < p < \infty)$, C. R. Acad. Sci. Paris 260 (1965), p. 1334–1336.

[21] J. Zinn, *Comparison of martingale difference sequences*, Probability in Banach Spaces V, Tufts University, 1984, Lecture Notes in Mathematics 1153, Springer, Berlin 1985, p. 453–457.

Department of Mathematics
Syracuse University
215 Carnegie
Syracuse, N.Y. 13244–1150
USA

Received on 25. 5. 1987;
revised version on 2. 2. 1988