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EXPLICIT SOLUTIONS OF MOMENT PROBLEMS, I*

BY

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Abstract. The moment problems minimizing (maximizing) $\{\mathrm{E}\|X_1-X_2\|^p\colon \mathrm{E}\|X_i\|^{q_j}=a_{ij};\ i=1,2,\ j=1,\ldots,n\}$, where X_i takes values in a separable norm space, are investigated. Explicit solutions when $n=1$ are given for all $p\geqslant 0$ and $q_j\geqslant 0$ ($0^0 := 0$). If $n=2$, the minimization problem is solved for $1\leqslant q_1\leqslant p<q_2$ and the maximization problem for $0<p\leqslant q_1$, $1\leqslant q_1<q_2$ or $0<q_1<q_2<p$. Possible generalizations and open problems are presented.

1. Introduction. In probability theory the following two measure theoretic problems are well known (see e.g. [6], [10], [12]-[14], [18] and references there):

A. Marginal problem. *For fixed probability measures (laws) P_1 and P_2 on a measurable space U and a measurable function c on the product space $U^2=U\times U$*

$$\text{(1.1)}\qquad \textit{minimize (maximize)} \int_{U^2} c(x,y)\,P(dx,dy),$$

where the laws P on U^2 have marginals P_1 and P_2, i.e.

$$\text{(1.2)}\qquad \pi_i P = P_i,\quad i=1,2,.$$

B. Moment problem. *For fixed real numbers a_{ij} and real-valued continuous functions f_{ij} $(i=1,2;\ j=1,\ldots,n)$*

$$\text{(1.3)}\qquad \textit{minimize (maximize)} \int_{U^2} c(x,y)\,P(dx,dy),$$

where the law P on U^2 satisfies the marginal moment conditions

$$\text{(1.4)}\qquad \int_U f_{ij}\,dP_i = a_{ij},\qquad i=1,2,\ j=1,\ldots,n.$$

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Dual relationships and explicit solutions of the marginal problem for different spaces U and criterion functions c are given in [4–6], [10], [13–15], and [18].

In citing some results concerning duality and explicit solutions of A, we shall use the following notation:

(1.5) (U, d) is a separable metric space with metric d;

(1.6) $\mathscr{P}(U^k)$ is the space of all Borel probability measures on the Cartesian product U^k;

(1.7) $\mathscr{H}$ is the class of all convex functions

$$H\colon [0, \infty) \to [0, \infty), \quad H(0) = 0, \quad K_H := \sup_{t \geqslant 0} H(2t)/H(t) < \infty;$$

(1.8) $$\mathscr{P}_H := \{P \in P(U)\colon \int_U H(d(x, a))\, P(dx) < \infty\}, \quad H \in \mathscr{H}(^1);$$

(1.9) $$D(x, y) := H(d(x, y));$$

(1.10) $$\mathrm{Lip}(U) := \{f\colon U \to \boldsymbol{R},\ \sup_{x \in U} |f(x)| < \infty,\ \exists \alpha(f) > 0\colon |f(x) - f(y)| \leqslant \alpha(f)\, d(x, y) \quad \text{for all } x, y \in U\};$$

(1.11) $$\mathscr{G}_H := \{(f, g)\colon f, g \in \mathrm{Lip}(U),\ f(x) + g(y) \leqslant D(x, y),\ x, y \in U\};$$

(1.12) $$\overline{\mathscr{G}}_H := \{(f, g)\colon f, g) \in \mathrm{Lip}(U),\ f(x) \geqslant 0,\ g(y) \geqslant 0,\ f(x) + g(y) \geqslant D(x, y),\ x, y \in U\}.$$

THEOREM A. (i) (Duality solutions of A). *Let (U, d) be a separable metric space $H \in \mathscr{H}$, $P_1, P_2 \in \mathscr{P}_H$. Then*

(1.13) $$\hat{\mathscr{L}}_H(P_1, P_2) := \inf\{\int_{U^2} D(x, y)\, P(\mathrm{d}x, dy)\colon P \in \mathscr{P}(U^2),\ \pi_i P = P_i,\ i = 1, 2\}$$
$$= \sup\{\int_U f\, dP_1 + \int_U g\, dP_2\colon (f, g) \in \underline{\mathscr{G}}_H\}$$

and

(1.14) $$\check{\mathscr{L}}_H(P_1, P_2) := \sup\{\int_{U^2} D(x, y)\, P(dx, dy)\colon P \in \mathscr{P}(U^2),\ \pi_i P = P_i,\ i = 1, 2\}$$
$$= \inf\{\int_U f\, dP_1 + \int_U g\, dP_2\colon (f, g) \in \overline{\mathscr{G}}_H\}.$$

(ii) (Explicit solutions of A). *If $(U, d) = (\boldsymbol{R}, |\cdot|)$ and H is a convex function, then*

(1.15) $$\hat{\mathscr{L}}_H(P_1, P_2) = \int_0^1 D(F_1^{-1}(t), F_2^{-1}(t))\, dt,$$

(¹) $\mathscr{P}_H$ does not depend on $a \in U$ for any $H \in \mathscr{H}$.

$$\mathscr{L}_H(P_1, P_2) = \int_0^1 D\left(F_1^{-1}(t), F_2^{-1}(1-t)\right) dt, \tag{1.16}$$

where F_j *is the distribution function corresponding to* P_j *and* F_j^{-1} *is its inverse,* $j = 1, 2$.

Indication. (i) see Rachev [15]. (ii) see Cambanis, Simons and Stout [5].

Kellerer [10] provides duality solutions of A for general c; however, in case $c = D$, his dual solutions are not as sharp as these in (i).

The possible solutions of the marginal problems are related to the dual and explicit expressions for the so-called *minimal metrics* (see [21, 22]) and *maximal distances* (see [15]) that are fruitful in the development of a considerable range of stability problems for stochastic models (see [22] and [14]).

Owing to the number of its important applications (see [1, 2], [6–9], [11, 12], and [19] the moment problem can also be treated as an approximation of the marginal problem. Indeed, if the laws P_1 and P_2 (see (1.2)) are not determined completely and if only some functionals of P_1 and P_2 are given (see (1.4)), then one has to solve the problem B instead of A. The significance of the moment problem for the theory of probability metrics was also stressed by Sholpo [20] and Rachev [15].

General dual representations of moment problems on a compact space U are given in [6, 11, 12]. In a more general case of a completely regular topological space U, dual expressions are given in [12] under a "tightness" condition on the pairs (f_{ij}, a_{ij}), $i = 1, 2$, $j = 1, \ldots, n$.

The present paper is devoted to the explicit solutions of some moment problems on separable metric space U with metric d. In this case, considering a "rich enough" probability space $(\Omega, \mathscr{A}, \Pr)$ without atoms and the space $\mathscr{X} = \mathscr{X}(U)$ of all U-valued random variables (rv's) X on $(\Omega, \mathscr{A}, \Pr)$ one can rewrite the moment problem B as follows:

$$\text{minimize (maximize)} \quad \{\boldsymbol{E}\, c(X_1, X_2) \colon \boldsymbol{E} f_{ij}(X_i) = a_{ij},\ i = 1, 2,\ j = 1, 2, \ldots, n\}. \tag{1.17}$$

In fact, the above assumptions guarantee that the set of all Borel probability measures on U^2 coincides with the set of all joint distributions $\Pr_{X,Y}$ of pairs of rv's X, Y (see [16]). The main reason for considering not arbitrary but separable space (U, d) is that we need the measurability of d. For example, if $c(x, y) = d(x, y)$ in (1.5) and (U, d) is a metric space of cardinality $> \mathfrak{C}$, then the metric $d\colon U \times U \to \boldsymbol{R}$ is not measurable with respect to the product σ-algebra $\mathscr{B}(U) \times \mathscr{B}(U)$ ($\mathscr{B}(U)$ is the Borel σ-algebra on (U, d)), see [3].

In Section 2 explicit solutions are given for the problem (1.17) in the case where $n = 1$, U is a separable norm space with norm $\|\cdot\|$ and

$$c(x, y) = h(\|x-y\|), \quad f_{i1}(x) := g(\|x\|),\ i = 1, 2. \tag{1.18}$$

In particular, for any $p \geqslant 0$ and $q \geqslant 0$ we solve (1.17) with

$$c(x, y) = \|x-y\|^p, \quad f_{i1}(x) = \|x\|^q. \tag{1.19}$$

(Here and in the sequel 0^0 means 0).

In Section 3 we assume that U is again separable norm space but $n \geqslant 2$. In this case, among other results, explicit solutions are given for the moment problem (1.17) with

$$c(x, y) = \|x-y\|^p, \quad f_{ij}(x) = \|x\|^{q_j}, \quad i = 1, 2, j = 1, 2, \tag{1.20}$$

for $1 \leqslant q_1 \leqslant p \leqslant q_2$ when minimizing in [1.17], and for $0 < p \leqslant q_1$, $1 \leqslant q_1 < q_2$ or $0 < q_1 < q_2 < p$ when maximizing in (1.17). Here, we also give the explicit solution of the well known moment problem:

$$\text{minimize (maximize) } \{E\|X\|^p : E\|X\|^{q_i} = a_i, \quad i = 1, 2,\} \tag{1.21}$$

for all nonnegative p, q_1 and q_2.

In Section 4 we apply the results of Section 3 to obtain precise bounds for $\hat{\mathscr{L}}_H(P_1, P_2)$ and $\check{\mathscr{L}}_H(P_1, P_2)$ when the moments $\int \|x\|^q P_i(dx)$ $(i = 1, 2)$ are fixed. Some open problems are offered.

2. Moment problems with one fixed pair of marginal moments. Let U be a separable norm space with norm $\|\cdot\|$ and $\mathscr{M}$ be the class of all strictly increasing continuous functions $f: [0, \infty] \to [0, \infty]$, $f(0) = 0$, $f(\infty) = \infty$. In the present section we treat the explicit representations of the following extremal functionals:

$$I(h, g; a, b) := \inf\{Eh(\|X-Y\|): X, Y \in \mathscr{X}(U), Eg(\|X\|) = a, Eg(\|Y\|) = b\}, \tag{2.1}$$

$$S(h, g; a, b) := \sup\{Eh(\|X-Y\|): X, Y \in \mathscr{X}(U), Eg(\|X\|) = a, Eg(\|Y\|) = b\}, \tag{2.2}$$

where $a > 0$, $b > 0$, $h \in \mathscr{M}$, $g \in \mathscr{M}$. In particular, for all $p \geqslant 0$, $q \geqslant 0$ the values

$$I(p, q; a, b) := I(h, g; a, b)(h(t) = t^p, g(t) = t^q), \tag{2.23}$$

$$S(p, q; a, b) := S(h, g; a, b)(h(t) = t^p, g(t) = t^q) \tag{2.4}$$

are calculated. Note that here and in the sequal $E\|X-Y\|^0$ means $\Pr(X \neq Y)$.

The scheme of the proofs of all statements here is as follows: first we prove the necessary inequalities that give us the required bounds and then we construct pairs of random variables which achieve the bounds or approximate them with arbitrary precision.

Let $f, f_1, f_2 \in \mathscr{M}$ and consider the following conditions (here and in the sequel f^{-1} is the inverse function of $f \in \mathscr{M}$):

$A(f_1, f_2)$: $f_1 \circ f_2^{-1}(t)$ $(t \geqslant 0)$ is convex;

$B(f)$: $f^{-1}(Ef(\|X+Y\|)) \leqslant f^{-1}(Ef(\|X\|)) + f^{-1}(Ef(\|Y\|))$ for any $X, Y \in \mathscr{X}$;

$C(f)$: $Ef(\|X+Y\|) \leqslant Ef(\|X\|) + Ef(\|Y\|)$ for any $X, Y \in \mathscr{X}$;

$D(f_1, f_2)$: $\lim\limits_{t\to\infty} \left(f_1(t)/f_2(t)\right) = 0$;

$E(f_1, f_2)$: $f_1 \circ f_2(t)(t \geqslant 0)$ is concave;

$F(f_1, f_2)$: f_1 is concave and f_2 is convex;

$G(f_1, f_2)$: $\lim\limits_{t\to\infty} \left(f_1(t)/f_2(t)\right) = \infty$.

Obviously, if $h(t) = t^p$, $g(t) = t^q$ $(p > 0, q > 0)$, then $A(h,g) \Leftrightarrow p \geqslant q$, $B(g) \Leftrightarrow q \geqslant 1$, $C(g) \Leftrightarrow q \leqslant 1$, $D(h,g) \Leftrightarrow q > p$, $E(h,g) \Leftrightarrow q \geqslant p$, $F(h,g) \Leftrightarrow p \leqslant 1 \leqslant q$, $G(h,g) \Leftrightarrow p > q$, and hence conditions A-G cover all possible values of the pairs (p, q).

In the next theorem we establish explicit solutions of the moment problem (1.17) under conditions (1.18) and combinations of requirements A-G on h and g.

THEOREM 1. *Let $a \geqslant 0$ and $b \geqslant 0$, $a+b > 0$.*

(i) *We have*

$$(2.5)\qquad I(h,g;\, a,\, b) = \begin{cases} h(|g^{-1}(a) - g^{-1}(b)|) & \text{if } A(h,g) \text{ and } B(g) \text{ hold,} \\ h \circ g^{-1}(|\alpha - \beta|) & \text{if } A(h,g) \text{ and } C(g) \text{ hold,} \\ 0 & \text{if } D(h,g) \text{ holds.} \end{cases}$$

(ii) *For any $u \in U$, $h \in \mathcal{M}$ and $g \in \mathcal{M}$ we have*

$$(2.6)\qquad \inf\{\Pr\{X \neq Y\} : Eg(\|X\|) = a,\ Eg(\|Y\|) = b\} = 0,$$

$$(2.7)\qquad \inf\{E\,h(\|X - Y\|) : \Pr\{X \neq u\} = a,\ \Pr\{Y \neq u\} = b\} = 0 \qquad (a,\, b \in [0, 1]).$$

(iii) *We have*

$$(2.8)\qquad S(h,g;\, a,b) = \begin{cases} h(g^{-1}(a) + g^{-1}(b)) & \text{if } F(h,g) \text{ holds or if } B(g) \text{ and } E(h,g) \text{ hold,} \\ h \circ g^{-1}(\alpha + \beta) & \text{if } C(g) \text{ and } E(h,g) \text{ hold,} \\ \infty & \text{if } G(h,g) \text{ holds.} \end{cases}$$

(iv) *For any $u \in U$, $h \in \mathcal{M}$, $g \in \mathcal{M}$.*

$$(2.9)\qquad \sup\{\Pr\{X \neq Y\} : Eg(\|X\|) = a,\ Eg(\|Y\|) = b\} = 1,$$

$$(2.10)\qquad \sup\{\Pr\{X \neq Y\} : \Pr\{X \neq u\} = a,\ \Pr\{Y \neq u\} = b\} = \min(a+b, 1) \qquad (a, b \in [0, 1]),$$

$$(2.11)\qquad \sup\{Eh(\|X - Y\|) : \Pr(X \neq u) = a,\ \Pr\{X \neq u\} = b\} = \infty.$$

Proof. (i) *Case* 1. Let $A(h,g)$ and $B(g)$ be fulfilled. Write

$$\varphi(a,b) := h(|g^{-1}(a) - g^{-1}(b)|),\ a \geqslant 0,\ b \geqslant 0.$$

Claim 1. $I(h,g;a,b) \geqslant \varphi(a,b)$ By the Jensen's inequality and $A(h,g)$

$$(2.12)\qquad h \circ g^{-1}(EZ) \leqslant E\,h \circ g^{-1}(Z).$$

Taking $Z = g(\|X-Y\|)$ and using $B(g)$ we obtain

$$h^{-1}(Eh(\|X-Y\|)) \geqslant g^{-1}(Eg(\|X-Y\|)) \geqslant |g^{-1}(E(\|X\|))-g^{-1}(E(\|Y\|))|$$

for any $X, Y \in \mathscr{X}$, which proves the claim.

Claim 2. There exists an "optimal" pair (X^*, Y^*) of rv's such that $Eg(\|X^*\|) = a$, $Eg(\|Y^*\|) = b$, $E(\|X^*-Y^*\|) = \varphi(a, b)$. Let $\bar{e}$ here and in the sequel be a fixed point of U with $\|\bar{e}\| = 1$. Then the required pair (X^*, Y^*) is given by

$$X^* = g^{-1}(a)\bar{e}, \qquad Y^* = g^{-1}(b)\bar{e}, \tag{2.13}$$

which proves the claim.

Case 2. Let $A(h,g)$ and $C(g)$ be fulfilled. Write $\varphi_1(t) := h \circ g^{-1}(t)$, $t \geqslant 0$. As in Claim 1 we get $I(h,g;a,b) \geqslant \varphi_1(|a-b|)$. Suppose that $a > b$ and for each $\varepsilon > 0$ define a pair $(X_\varepsilon, Y_\varepsilon)$ of rv's as follows:

$$\Pr\{X_\varepsilon = c_\varepsilon \bar{e},\ Y_\varepsilon = \bar{0}\} = p_\varepsilon, \qquad \Pr\{X_\varepsilon = d_\varepsilon \bar{e},\ Y_\varepsilon = d_\varepsilon \bar{e}\} = 1-p_\varepsilon;$$

here

$$\bar{0} := 0\bar{e},\ p_\varepsilon := \frac{a-b}{a-b+\varepsilon},\ c_\varepsilon := g^{-1}(a-b+\varepsilon),\ d_\varepsilon := g^{-1}\left(\frac{b}{1-p_\varepsilon}\right). \tag{2.14}$$

Then $(X_\varepsilon, Y_\varepsilon)$ enjoys the side conditions in (2.1) and

$$Eh(\|X_\varepsilon - Y_\varepsilon\|) = \varphi_1(a-b+\varepsilon)\frac{a-b}{a-b+\varepsilon}.$$

Letting $\varepsilon \to 0$ we claim (2.5).

Case 3. (i) Let $D(h,g)$ be fulfilled. In order to obtain (2.5) it is sufficient to define a sequence (X_n, Y_n), $n \geqslant N$, such that

$$\lim_{n\to\infty} Eh(\|X_n - Y_n\|) = 0, \qquad Eg(\|X_n\|) = a, \qquad Eg(\|Y_n\|) = b.$$

An example of such a sequence is the following one:

$$\Pr\{X_n = \bar{0},\ Y_n = \bar{0}\} = 1-c_n-d_n,$$

$$\Pr\{X_n = na\bar{e},\ Y_n = \bar{0}\} = c_n, \qquad \Pr\{X_n = \bar{0},\ Y_n = nb\bar{e}\} = d_n,$$

where $c_n = a/g(na)$, $d_n = b/g(nb)$ and N satisfies $c_N + d_N < 1$.

(ii) Define the sequence (X_n, Y_n), $n = 2, 3, \ldots$, such that

$$\Pr\{X_n = g^{-1}(na)\bar{e},\ Y_n = g^{-1}(nb)\} = \frac{1}{n}, \qquad \Pr\{X_n = \bar{0}, Y_n = \bar{0}\} = \frac{n-1}{n}.$$

Hence, $Eg(\|X_n\|) = a$, $Eg(\|Y_n\|) = b$ and $\Pr(X_n \neq Y_n) = 1/n$ which shows (2.6).

Further suppose $a \geqslant b$. Without loss of generality we may assume that $u = \bar{0}$. Then consider the random pair $(\tilde{X}_n, \tilde{Y}_n)$ with the following joint distribution:

$$\Pr\{\tilde{X}_n = \bar{0},\ \tilde{Y}_n = \bar{0}\} = 1-a, \qquad \Pr\left\{\tilde{X}_n = \frac{1}{n}\bar{e},\ \tilde{Y}_n = \bar{0}\right\} = a-b,$$

$$\Pr\left\{\tilde{X}_n = \frac{1}{n}\bar{e},\ \tilde{Y}_n = \frac{1}{n}\bar{e}\right\} = b.$$

Obviously, $(\tilde{X}_n, \tilde{Y}_n)$ satisfies the side conditions

$$\Pr(\tilde{X} \neq 0) = a,\ \Pr(\tilde{Y}_n \neq \bar{0}) = b, \quad \text{and} \quad \lim_{n\to\infty} Eh(\|X_n - Y_n\|) = 0,$$

which proves (2.7).

The proofs of (iii) and (iv) are quite analogous to those of (i) and (ii) respectively, q.e.d.

Note that if $A(h,g)$ and $B(g)$ hold we have constructed an *optimal* pair (X^*, Y^*) (see (2.13)), i.e. (X^*, Y^*) realizes the infimum in (2.1). However, if $D(h,g)$ holds and $a \neq b$, then optimal pairs do not exist, because $Eh(\|X-Y\|) = 0$ implies $a = b$.

COROLLARY 1. *For any* $a \geqslant 0,\ b \geqslant 0,\ a+b \geqslant 0,\ p \geqslant 0,\ q \geqslant 0,$

$$(2.15) \qquad I(p,q;\,a,b) = \begin{cases} |a^{1/q} - b^{1/q}|^p & \text{if } p \geqslant q \geqslant 1, \\ |a-b|^{p/q} & \text{if } p \geqslant q,\ 0 < q < 1, \\ 0 & \text{if } 0 \leqslant p < q \text{ or } q = 0,\ p > 0, \\ |a-b| & \text{if } p = q = 0. \end{cases}$$

$$(2.16) \qquad S(p,q;\,a,b) = \begin{cases} (a^{1/q} + b^{1/q})^p & \text{if } 0 \leqslant p \leqslant q,\ q \geqslant 1, \\ (a+b)^{p/q} & \text{if } 0 \leqslant p \leqslant q < 1,\ q \neq 0, \\ \infty & \text{if } p > q \geqslant 0, \\ \min(a+b,\ 1) & \text{if } p = q = 0. \end{cases}$$

Remark. Obviously, if $q = 0$ in (2.15) or (2.16), the values of I and S make sense for $a, b \in [0,1]$.

3. Moment problems with two fixed pairs of marginal moments. The main part of this section is devoted to the explicit description of the bounds

$$(3.1) \qquad I(h, g_1, g_2;\ a_1, b_1, a_2, b_2) := \inf Eh(\|X-Y\|),$$

$$(3.2) \qquad S(h, g_1, g_2;\ a_1, b_1, a_2, b_2) := \sup Eh(\|X-Y\|),$$

where $h, g_1, g_2 \in \mathcal{M}$, and the infimum in (3.1) and the supremum in (3.2) are taken over the set of all pairs of rv's $X, Y \in \mathcal{X}(U)$ satisfying the moment conditions

(3.3) $$E g_i(\|X\|) = a_i, \quad E g_i(\|Y\|) = b_i, \quad i = 1, 2.$$

In particular, if $h(t) = t^p$, $g_i(t) = t^{q_i}$, $i = 1, 2$ ($p \geqslant 0$, $q_2 > q_1 \geqslant 0$), we write

(3.4) $$I(p, q_1, q_2; a_1, b_1, a_2, b_2) := I(h, g_1, g_2; a_1, b_1, a_2, b_2),$$

(3.5) $$S(p, q_1, q_2; a_1, b_1, a_2, b_2) := S(h, g_1, g_2; a_1, b_1, a_2, b_2).$$

The moment problem with two pairs of marginal conditions in considerably more complicated and, in the present section, our results are not as complete as in the previous one.

THEOREM 2. *Let the conditions* $A(g_2, g_1)$ *and* $G(g_2, g_1)$ *hold. Let* $a_i \geqslant 0$, $b_i \geqslant 0$, $i = 1, 2$, $a_1 + a_2 > 0$, $b_1 + b_2 > 0$ *and*

(3.6) $$g_1^{-1}(a_1) \leqslant g_2^{-1}(a_2), \quad g_1^{-1}(b_1) \leqslant g_2^{-1}(b_2).$$

(i) *If* $A(h, g_1)$, $B(g_1)$ *and* $D(h, g_2)$ *are fulfilled, then*

(3.7) $$I(h, g_1, g_2; a_1, b_1, a_2, b_2) = I(h, g_1; a_1, b_1) = h(|g_1^{-1}(a_1) - g_1^{-1}(b_1)|).$$

(ii) *Let* $D(h, g_2)$ *be fulfilled. If* $F(h, g)$ *holds or if* $B(g)$ *and* $E(h, g)$ *hold, then*

(3.8) $$S(h, g_1, g_2; a_1, b_1, a_2, b_2) = S(h, g_1; a_1, b_1) = h(g_1^{-1}(a_1) + g_1^{-1}(b_1)).$$

(iii) *If* $G(h, g_2)$ *is fulfilled and* $g_1^{-1}(a_1) \neq g_2^{-1}(a_2)$ *or* $g_1^{-1}(b_1) \neq g_2^{-1}(b_2)$, *then*

(3.9) $$S(h, g_1, g_2; a_1, b_1, a_2, b_2) = S(h, g_1; a_1, b_1) = \infty.$$

Proof. By Theorem 1 (i) we have

(3.10) $$I(h, g_1, g_2; a_1, b_1, a_2, b_2) \geqslant I(h, g_1, a_1, b_1) = \varphi(a_1, b_1).$$

Further we shall define an appropriate sequence of rv's (X_t, Y_t) that satisfy the side conditions (3.3) and

$$\lim_{t \to \infty} E h(\|X_t - Y_t\|) = \varphi(a_1, b_1).$$

Let $f(x) = g_2 \circ g_1^{-1}(x)$. Then, by the Jensen's inequality and $A(g_2, g_1)$,

(3.11) $$f(a_1) = f(E g_1(\|X\|)) \leqslant E f \circ g_1(\|X\|) = a_2$$

as well as $f(b_1) \leqslant b_2$. Moreover, $\lim_{t \to \infty} (f(t)/t) = \infty$ by $G(g_1, g_2)$.

Case 1. Suppose that $f(a_1) < a_2$, $f(b_1) < b_2$.

CLAIM. If the functions $f \in \mathcal{M}$ and the reals c_1, c_2 possess the properties

(3.12) $$f(c_1) < c_2, \quad \lim_{t \to \infty} (f(t)/t) = \infty,$$

then there exist a positive t_0 and a function $k(t)$, $t \geqslant t_0$, such that, for any $t \geqslant t_0$,

(3.13) $$0 < k(t) < c_1,$$

$$tf\big(c_1-k(t)\big)+k(t)\,f(c_1+t) = c_2\big(k(t)+t\big), \tag{3.14}$$

$$\frac{k(t)}{k(t)+t} \leqslant \frac{c_2}{f(c_1+t)}, \tag{3.15}$$

and

$$\lim_{t\to\infty} k(t) = 0. \tag{3.16}$$

Indeed, let us take such t_0 that $f(c_1+t)/(c_1+t) > c_2/c_1$, $t \geqslant t_0$, and consider the equation $F(t,x) = c_2$, where

$$F(t,x) := f(c_1-x)t/(x+t)+f(c_1+t)x/(x+t).$$

For each $t \geqslant t_0$ we have $F(t,c_1) > c_2$, $F(t,0) = f(c_1) < c_2$. Hence, for each $t \geqslant t_0$ there exists such an $x = k(t)$ that $k(t)\in(0,c_1)$ and $F\big(t,k(t)\big) = c_2$, which provides (3.13) and (3.14). Further, (3.14) implies (3.15) as well as (3.13) and (3.15) imply (3.16). The claim is established.

From the claim we see that there exist $t_0 > 0$ and functions $l(t)$ and $m(t)$ $t \geqslant t_0$, such that for all $t > t_0$ we have

$$0 < l(t) < a_1, \quad 0 < m(t) < b_1, \tag{3.17}$$

$$tf\big(a_1-l(t)\big)+l(t)\,f(a_1+t) = a_2\big(l(t)+t\big), \tag{3.18}$$

$$tf\big(b_1-m(t)\big)+m(t)\,f(b_1+t) = b_2\big(m(t)+t\big), \tag{3.19}$$

$$\lim_{t\to\infty} l(t) = 0,\ \lim_{t\to\infty} m(t) = 0. \tag{3.20}$$

By (3.17)–(3.20) and the conditions $A(h,g_1)$, $D(h,g_2)$ and $G(g_2,g_1)$ one can easily check that the rv's (X_t, Y_t), $t > t_0$, determined by the equalities

$$\Pr\{X_t = x_i(t),\ Y_t = y_j(t)\} = p_{ij}(t), \quad i,j = 1,2,$$

where

$$x_1(t) := g_1^{-1}\big(a_1-l(t)\big)\bar{e},\ x_2(t) := g_1^{-1}(a_1+t)\bar{e},$$

$$y_1(t) := g_1^{-1}\big(b_1-m(t)\big)\bar{e},\ y_2(t) := g_1^{-1}(b_1+t)\bar{e},$$

$$p_{11}(t) := \min\{t/\big(l(t)+t\big),\ t/\big(m(t)+t\big)\},$$

$$p_{12}(t) := t/\big(l(t)+t\big)-p_{11}(t),$$

$$p_{21}(t) := t/\big(m(t)+t\big)-p_{11}(t),$$

$$p_{22}(t) := \min\{l(t)/\big(l(t)+t\big),\ m(t)/\big(m(t)+t\big)\}$$

have all the desired properties.

Case 2. Suppose $f(a_1) = a_2$ (i.e. $g_1^{-1}(a_1) = g_2^{-1}(a_2)$), $f(b_1) < b_2$. Then we determine (X_t, Y_t) by the equalities

$$\Pr\{X_t = g_1^{-1}(a_1),\ Y_t = y_1(t)\} = t/(m(t)+t),$$
$$\Pr\{X_t = g_1^{-1}(a_1),\ Y_t = y_2(t)\} = m(t)/(m(t)+t).$$

Case 3. The cases $(f(a_1) < a_2, f(b_1) = b_2)$, $(f(a_1) = a_2, f(b_1) = b_2)$ are considered in the same way as in Case 2.

(ii) and (iii) are proved by the analogous arguments, q.e.d.

COROLLARY 2. *Let* $a_i \geqslant 0,\ b_i \geqslant 0,\ a_1+a_2 > 0,\ b_1+b_2 > 0,\ a_1^{1/q_1} \leqslant a_2^{1/q_2}$ $b_1^{1/q_1} \leqslant b_2^{1/q_2}$.

(i) *If* $1 \leqslant q_1 \leqslant p < q_2$, *then*

$$\text{(3.21)}\qquad I(p, q_1, q_2;\ a_1, b_1, a_2, b_2) = I(p, q_1;\ a_1, b_1) = (a_1^{1/q_1} - b_1^{1/q_1})^p.$$

(ii) *If* $0 < p \leqslant q_1,\ 1 \leqslant q_1 < q_2$, *then*

$$\text{(3.22)}\qquad S(p, q_1, q_2;\ a_1, b_1, a_2, b_2) = S(p, q_1;\ a_1, b_1) = (a_1^{1/q_1} + b_1^{1/q_1})^p.$$

(iii) *If* $0 < q_1 < q_2 < p$ *and* $a_1^{1/q_1} = a_2^{1/q_2}$ *or* $b_1^{1/q_1} = b_2^{1/q_2}$, *then*

$$S(p, q_1, q_2;\ a_1, b_1, a_2, b_2) = S(p, q_1;\ a_1, b_1) = \infty.$$

Corollary 2 describes situations in which the "additional moment information" $a_2 = E\|X\|^{q_2}$, $b_2 = E\|Y\|^{q_2}$ does not affect the bounds

$$I(p, q_1, q_2;\ a_1, b_1, a_2, b_2) = I(p, q_1, a_1, a_2),$$
$$S(p, q_1, q_2;\ a_1, b_1, a_2, b_2) = S(p, q_1, a_1, a_2)$$

(and likewise Theorem 2).

We conclude this section by giving the explicit solution of the following moment problem: *determine the extremal values*

$$\text{(3.23)}\quad \tilde{I} := \tilde{I}(p, q_1, q_2, a_1, a_2) := \inf\{E\|X\|^p : E\|X\|^{q_1} = a_1,\ E\|X\|^{q_2} = a_2\},$$

$$\text{(3.24)}\quad \tilde{S} := \tilde{S}(p, q_1, q_2, a_1, a_2) := \sup\{E\|X\|^p : E\|X\|^{q_1} = a_1,\ E\|X\|^{q_2} = a_2\}$$

for all $p \geqslant 0,\ 0 \leqslant q_1 \leqslant q_2$.

THEOREM 3. *Let* $p \geqslant 0,\ 0 \leqslant q_1 \leqslant q_2$. *Then*

$$\tilde{I} = \begin{cases} \dfrac{a_1^{(q_2-p)/(q_2-q_1)}}{a_2^{(q_1-p)/(q_2-q_1)}} & \text{if } 0 \leqslant p \leqslant q_1 < q_2, \\[2ex] \dfrac{a_2^{(p-q_1)/(q_2-q_1)}}{a_1^{(p-q_2)/(q_2-q_1)}} & \text{if } 0 \leqslant q_1 < q_2 \leqslant p, \\[2ex] a_1^{p/q_1} & \text{if } 0 < q_1 \leqslant p < q_2 \text{ or } a_1^{1/q_1} = a_2^{1/q_2},\ 0 < q_1 < q_2, \\ a_i & \text{if } p = q_i,\ i = 1 \text{ or } 2, \\ 0 & \text{if } 0 = q_1 < p < q_2 \end{cases}$$

and

$$\tilde{S} = \begin{cases} \dfrac{a_1^{(q_2-p)/(q_2-q_1)}}{a_2^{(q_1-p)/(q_2-q_1)}} & \text{if } 0 \leqslant q_1 \leqslant p \leqslant q_2,\ q_1 < q_2, \\ a_1^{p/q_1} & \text{if } 0 < p \leqslant q_1 < q_2 \text{ or } 0 = p < q_1 < q_2 \\ & \text{or } a_1^{1/q_1} = a_2^{1/q_2},\ 0 < q_1 < q_2, \\ a_i & \text{if } p = q_i,\ i = 1 \text{ or } 2, \\ \infty & \text{if } 0 \leqslant q_1 < q_2 < p,\ a_1^{1/q_1} \neq a_2^{1/q_2}, \\ 1 & \text{if } p = 0,\ 0 < q_1 < q_2 \text{ or } a_1^{1/q_1} = a_2^{1/q_2}. \end{cases}$$

Proof. If $a_1^{1/q_1} = a_2^{1/q_2}$, then it is easy to see that $\|X\| = a_1^{1/q_1}$ with probability 1. Hence,

$$\tilde{I} = \tilde{S} = a_1^{p/q_1} \quad \text{if } p < 0,\ 0 < q_1 < q_2,\ a_1^{1/q_1} = a_2^{1/q_2}. \tag{3.25}$$

Further, suppose that $a_1^{1/q_1} \neq a_2^{1/q_2}$. For any $r > 1$ and any real-valued ξ and η

$$E|\xi\eta| \leqslant (E|\xi|^r)^{1/r}(E|\eta|^{r/(r-1)})^{(r-1)/r} \tag{3.26}$$

by the Hölder's inequality.

Let $0 < r_1 < r_2 < r_3$, $r = (r_3 - r_1)/(r_3 - r_2)$, $\xi = \|X\|^{r_1(r_3-r_2)/(r_3-r_1)}$, $\eta = \|X\|^{r_3(r_2-r_1)/(r_3-r_2)}$. Then, by (3.26),

$$E\|X\|^{r_2} \leqslant (E\|X\|^{r_1})^{(r_3-r_2)/(r_3-r_1)}(E\|X\|)^{r_3})^{(r_2-r_1)/(r_3-r_2)}. \tag{3.27}$$

Taking $r_1 = p$, $r_2 = q_1$, $r_3 = q_2$ in (3.37), we obtain

$$E\|X\|^{q_1} \leqslant (E\|X\|^p)^{(q_2-q_1)/(q_2-p)}(E\|X\|^{q_2})^{(q_1-p)/(q_2-p)}, \tag{3.28}$$

which implies, for $0 < q_1 < q_2$,

$$\tilde{I} \geqslant a_1^{(q_2-p)/(q_2-q_1)} a_2^{(p-q_1)/(q_2-q_1)}. \tag{3.29}$$

Taking $r_1 = q_1$, $r_2 = q_2$, $r_3 = p$ in (3.7), we obtain

$$\tilde{I} \geqslant a_2^{(p-q_1)/(q_2-q_1)} a_1^{(p-q_2)/(q_2-q_1)} \quad \text{for } q_1 < q_2 < p. \tag{3.30}$$

Finally, putting $r_1 = q_1$, $r_2 = p$, $r_3 = q_2$ in (3.27), we have

$$\tilde{S} \leqslant a_1^{(q_2-p)/(q_2-q_1)} a_2^{(p-q_1)/(q_2-q_1)} \quad \text{for } q_1 < p < q_2. \tag{3.31}$$

The "optimal" rv X^* (for all inequalities (3.29)–(3.31) and $p \neq 0$) is given by

$$\Pr\{X^* = a_2^{1/(q_2-q_1)} a_1^{-1/(q_2-q_1)} \bar{e}\} = a_1^{q_2/(q_2-q_1)} a_2^{-q_1/(q_2-q_1)} := b,$$

$$\Pr\{X^* = 0\} = 1 - b.$$

Thus the equalities

$$(3.32)\qquad \tilde{I}=\begin{cases} a_1^{(q_2-p)/(q_2-q_1)}\,a_2^{-(q_1-p)/(q_2-q_1)} & \text{if } 0<p<q_1<q_2,\\ a_2^{(p-q_1)/(q_2-q_1)}\,a_1^{-(p-q_2)/(q_2-q_1)} & \text{if } 0<q_1<q_2<p,\end{cases}$$

$$(3.33)\qquad \tilde{S}=a_1^{(q_2-p)/(q_2-q_1)}\,a_2^{(p-q_1)/(q_2-q_1)}\quad \text{if } 0<q_1<p<q_2$$

and the inequality

$$(3.34)\qquad \tilde{I}\leqslant a_1^{q_2/(q_2-q_1)}\,a_2^{-q_1/(q_2-q_1)}\quad \text{if } p=0,\ q_1<q_2$$

are claimed.

Further, Ljapunov's inequality implies

$$(3.35)\qquad \tilde{I}\geqslant a_1^{p/q_1}\quad \text{for } q_1<p$$

and

$$(3.36)\qquad \tilde{S}\leqslant a_1^{p/q_1}\quad \text{for } p<q_1.$$

Thus, it is sufficient to determine a sequence of rv's $\{X_t\}_{t\geqslant t_0}$ such that

$$(3.37)\qquad E\|X_t\|^{q_1}=a_1,\quad E\|X_t\|^{q_2}=a_2,\quad \lim_{t\to\infty}E\|X_t\|^p=a_1^{p/q_1}.$$

Now we can use the claim in the proof of Theorem 2 with $f(t)=t^{q_2/q_1}$, $c_i=a_i (i=1,2)$ and define the sequence $\{X_t\}_{t\geqslant t_0}$ by

$$(3.38)\qquad \Pr\{X_t=(a_1-k(t))^{1/q_1}\bar{e}\}=\frac{t}{k(t)+t},$$

$$(3.39)\qquad \Pr\{X_t=(a_1+t)^{1/q_1}\bar{e}\}=\frac{k(t)}{k(t)+t},$$

where $k(t)$ satisfies (3.13)–(3.16). By (3.35)–(3.37), it follows that

$$(3.40)\qquad \tilde{I}=a_1^{p/q_1}\quad \text{if } 0<q_1<p<q_2,$$

$$(3.41)\qquad \tilde{S}=a_1^{p/q_1}\quad \text{if } 0<p<q_1<q_2.$$

If $p>q_2$, then the sequence $\{X_t\}$, $t\geqslant 0$, has the property $\lim_{t\to\infty}E\|X_t\|^p=\infty$. So,

$$(3.42)\qquad \tilde{S}=\infty\quad \text{if } 0<q_1<q_2<p.$$

Putting $r=r_2/r_1$, $\xi=\|X\|^{r_1}$ and $\eta=I\{X\neq\bar{0}\}$ in (3.6), we obtain

$$(3.43)\qquad E\|X\|^{r_1}\leqslant(E\|X\|^{r_2})^{r_1/r_2}(\Pr\{X\neq\bar{0}\})^{(r_2-r_1)/r_2}.$$

Taking $r_1=q_1$ and $r_2=q_2$, we get

$$(3.44)\qquad \tilde{I}\geqslant a_1^{q_2/(q_2-q_1)}\,a_2^{-q_1/(q_2-q_1)}\quad \text{for } 0=p<q_1<q_2.$$

By (3.34) and (3.43),

$$\tilde{I} = a_1^{q_2/(q_2-q_1)} a_2^{-q_1/(q_2-q_1)} \quad \text{for } 0 = p < q_1 < q_2. \tag{3.45}$$

Analogously, letting $r_1 = p$ and $r_2 = q_2$ in (3.43), we get

$$\tilde{S} \leqslant a_1^{(q_2-p)/q_2} a_2^{p/q_2} \quad \text{if } 0 = q_1 < q_2 < \mathrm{p}. \tag{3.46}$$

Also (3.3) with $r_1 = q_2$, $r_2 = p$ implies

$$\tilde{I} \geqslant a_2^{p/q_2} a_1^{(p-q_2)/q_2} \quad \text{if } 0 = q_1 < q_2 < p. \tag{3.47}$$

The "optimal" rv X^* for the last two cases is given by

$$\Pr\left\{X^* = \left(\frac{a_2}{a_1}\right)^{1/q_2}\right\} = a_1, \quad \Pr\{X^* = 0\} = 1-a.$$

Thus,

$$\tilde{S} = a_1^{(q_2-p)/q_2} a_2^{p/q_2}, \quad \tilde{I} = a_2^{p/q_2} a_1^{(p-q_2)/q_2} \quad \text{if } 0 = q_1 < q_2 < \mathrm{p}. \tag{3.48}$$

Further, the sequence $\{X_n\}_{n > 1/a_1}$ defined by

$$\Pr\{X_n = (na_2 - 1)^{1/q_2} \bar{e}\} = 1/n,$$

$$\Pr\{X_n = (na_2 - 1)^{-1/q_2} \bar{e}\} = a_1 - 1/n, \quad \Pr\{X_n = \bar{0}\} = 1 - a_1$$

possesses the properties:

$$\Pr\{X_n \neq \bar{0}\} = a_1, \quad E\|X_n\|^{q_2} = a_2,$$

$$\lim_{n\to\infty} E\|X_n\|^p = \begin{cases} \infty & \text{if } q_2 < p, \\ 0 & \text{if } p < q_2. \end{cases}$$

Hence,

$$\tilde{S} = \infty \quad \text{if } 0 = q_1 < q_2 < p, \tag{3.49}$$

$$\tilde{I} = 0 \quad \text{if } 0 = q_1 < p < q_2. \tag{3.50}$$

Finally, if $p = q_i$, then

$$\tilde{I} = \tilde{S} = a_i, \quad i = 1, 2. \tag{3.51}$$

Summarizing (3.32), (3.33), (3.40)–(3.42), (3.45), and (3.48)–(3.51) we obtain the desired explicit representations for $\tilde{I}$ and $\tilde{S}$.

4. General remarks. Combining Theorems A and 1 we obtain the following precise estimates of the extremal functionals $\mathscr{L}_H(P,Q)$ $(P,Q \in \mathscr{P}(U))$ and $\check{\mathscr{L}}_H(P,Q)$ (see Theorem A) in terms of the moments:

$$a = \int_U g(x)\, P(dx), \quad b = \int_U g(x)\, Q(dx). \tag{4.1}$$

THEOREM 4. *Let* $(U, \|\cdot\|)$ *be a separable normed space and* $H \in \mathscr{H}$ *(see (1.7)).*

(i) *If* $A(H,g)$ *and* $B(g)$ *hold, then*

$$\hat{\mathscr{L}}_H(P,Q) \geqslant H(\|g^{-1}(a) - g^{-1}(b)\|). \tag{4.2}$$

(ii) *If* $B(g)$ *and* $E(H,g)$ *hold, then*

$$\check{\mathscr{L}}_H(P,Q) \leqslant H(g^{-1}(a) + g^{-1}(b)). \tag{4.3}$$

Moreover, there exist P_i, $Q_i \in \mathscr{P}(U)$, $i = 1, 2$, *with*

$$a = \int_U g(x) P_i(dx), \quad b = \int_U g(x) Q_i(dx),$$

such that $\hat{\mathscr{L}}_H(P_1, Q_1) = H(\|g^{-1}(a) - g^{-1}(b)\|)$ *and* $\check{\mathscr{L}}_H(P_2, Q_2) = H(g^{-1}(a) + g^{-1}(b))$.

The following theorem is an extension for $p = q = 1$ of Theorem 1 to a non-normed space U such as the Skorokhod space $D[0,1]$, which is of special interest in probability and statistics (see [3]).

THEOREM 5. *Let* (U, d) *be a separable metric space,* $\mathscr{X} = \mathscr{X}(U)$ *the space of all* U*-valued rv's,* $u \in U$, $a \geqslant 0$, $b \geqslant 0$. *Assume that there exists a* $z \in U$ *such that* $d(z,u) \geqslant \max(a,b)$. *Then*

$$\min\{\boldsymbol{E} d(X,Y) : X, Y \in \mathscr{X}, \boldsymbol{E} d(X,u) = a, \boldsymbol{E} d(Y,u) = b\} = |a-b| \tag{4.4}$$

and

$$\max\{\boldsymbol{E} d(X,Y) : X, Y \in \mathscr{X}, \boldsymbol{E} d(X,u) = a, \boldsymbol{E} d(Y,u) = b\} = a+b. \tag{4.5}$$

Proof. Let $a \leqslant b$, $\gamma = d(z,u)$. By the triangle inequality the minimum in (4.4) is greater than $b-a$. On the other hand, if $\Pr(X = u, Y = u) = 1 - b/\gamma$, $\Pr(X = u, Y = z) = (b-a)/\gamma$, $\Pr(Z = z, Y = u) = 0$, $\Pr(X = z, Y = z) = a/\gamma$, then $\boldsymbol{E} d(X,u) = a$, $\boldsymbol{E} d(Y,u) = b$, $\boldsymbol{E} d(X,Y) = b-a$, which proves (4.4). Analogously, one shows (4.5).

We conclude by stating explicitly the following open problems.

(i) *Find the explicit expression of* $I(a, q_1, q_2; a_1, b_1, a_2, b_2)$ *and* $S(p, q_1, q_2; a_1, b_1, a_2, b_2)$ *for all* $p \geqslant 0$, $q_2 > 0$, $q_1 \geqslant 0$ (see (3.4), Corollary 2 and Theorem 3).

One could start with the following one-dimensional version of (i). Let $g_i: [0, \infty) \to \boldsymbol{R}$ $(i = 1, 2)$ and $h: \boldsymbol{R} \to \boldsymbol{R}$ be given continuous functions with h symmetric and strictly increasing on $[0, \infty)$. Let further X and Y be nonnegative random variables having given moments $a_i = \boldsymbol{E} g_i(x)$ and $b_i = \boldsymbol{E} g_i(Y)$, $i = 1, 2$. The problem is to evaluate.

$$I = \inf \boldsymbol{E} h(X-Y), \quad S = \sup \boldsymbol{E} h(X+Y). \tag{4.5}$$

If desired, one could think of $X = X(t)$ and $Y = Y(t)$ as functions on the unit interval (with Lebesgue measure), see [9], Chapter 3, and [17].

The five moments a_1, a_2, b_1, b_2 and $E\,h(X \pm Y)$ depend only on the joint distribution of the pair (X, Y), and the extremal values in (4.5) are realized by a probability measure supported by 6 points (see [17], Theorem 1, [9], Chapter 3, and [12]). Thus the problem can also be formulated as a nonlinear progamming problem to find

$$I = \inf \sum_{j=1}^{6} p_j h(u_j - v_j), \qquad S = \sup \sum_{j=1}^{6} p_j h(u_j - v_j),$$

subject to

$$p_j \geqslant 0,\ \sum_{j=1}^{6} p_j = 1,\ u_j \geqslant 0,\ v_j \geqslant 0, \qquad j = 1, \ldots, 6,$$

$$\sum_{j=1}^{6} p_j g_i(u_j) = a_j, \qquad \sum_{j=1}^{6} p_j g_i(v_j) = b_j, \qquad i = 1, 2.$$

Such a problem becomes simpler when the functions g_i and h on R_+ are convex (see, for example, [9], Chapter XIV).

(ii) *Find the explicit solutions of moment problems with one fixed pair of marginal moments for rv's with values in a separable metric space U.* (See Theorem 5).

Note that in the case where U is a normed space, the moment problem was easily reduced to the one-dimensional moment problem ($U = \boldsymbol{R}$). This is no longer possible for general (non-normed) spaces U rendering the moment problem (ii) quite different for that considered in Sections 2 and 3.

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REFERENCES

[1] N. I. Ahiezer and M. Krein, *Some questions in the theory of moments*, American Mathem. Society, Providence, 1962.

[2] S. K. Basu and G. Simons, *Moment spaces for IFR distributions, applications, and related material.* Contributions to Statistics: Essays in honour of Norman L. Johnson, North-Holland Publ. Co., 1983, p. 27–46.

[3] P. Billingsley, *Convergence and Probability Measures*, John Wiley and Sons, New York 1968.

[4] S. Cambanis and G. Simons, *Probability and expectation inequalities*, Z. Wahrsch. verw. Geb. 59 (1982), p. 1–25.

[5] S. Cambanis, G. Simons and W. Stout, *Inequalities for* $Ek(X, Y)$ *when the marginals are fixed*, ibidem 36 (1976), p. 285–294.

[6] W. K. K,. Haneveld, *Duality in Stochastic Linear and Dynamic Programming*, Centrum voor Wiskunde en Informatica, Amsterdam 1985.

[7] W. Hoeffding, *The extrema of the expected value of a function of independent random variables*, Ann. Math. Stat. 26 (1955), p. 268–275.

[8] W. Hoeffding and S. S. Shrikhande, *Bounds for the distribution function of a sum of independent, identically distributed random variables*, ibidem 27 (1956), p. 439–449.

[9] S. Karlin and W. J. Studden, *Tchebycheff Systems with Applications in Analysis and Statistics*, Interscience Publ., Wiley, New York 1966.

[10] H. G. Kellerer, *Duality theorems for marginal problems*, Z. Wahrsch. verw. Geb. 67 (1984), p. 399–432.

[11] J. M. B. Kemperman, *On a class of moment problems*, Proceedings Sixth Berkeley Symposium on Mathem. Stat. and Probab. 2 (1972), p. 101–128.

[12] – *On the role of duality in the theory of moments*. In: *Semi-Infinite Programming and Applications*, Lecture Notes in Economics and Mathem. Systems 215 (1983), p. 63–92.

[13] V. L. Levin and A. A. Miljutin, *The problem of mass transfer with a discontinuous cost function and a mass statement of the duality problem for convex extremal problems*, Russian Math. Surveys 34/3 (1978), p. 1–78.

[14] S. T. Rachev, *The Monge-Kantorovich mass transference problem and its stochastic applications*, Theory Probab. Appl. 29 (1984), p. 647–676.

[15] – *Extreme functionals in the space of probability measures*. In: *Stability Problems for Stochastic Models*, Lecture Notes in Mathem. 1155 (1985), p. 320–348.

[16] S. T. Rachev and R. M. Shortt, *Duality theorems for Kantorovich-Rubinstein and Wasserstein functionals* (submitted to J. Theor. Probab.).

[17] W. W. Rogosiński, *Moments of non-negative mass*, Proc. Royal Soc. London, Ser. A, 245 (1958), p. 1–27.

[18] L. Rüschendorf, *The Wasserstein distance and approximation theorems*, Z. Wahrsch, verw. Geb. 70 (1985), p. 117–129.

[19] J. A. Shohat and J. D. Tamarkin, *The Problems of Moments*, Amer. Math. Soc., Providence 1943.

[20] I. A. Sholpo, *ε-minimal metrics*, Theory Probab. Appl. 28 (1983), p. 854–855.

[21] V. M. Zolotarev, *Metric distances in spaces of random variables and their distributions*, Mat. USSR 30/3 (1976), p. 373–401.

[22] – *Probability metrics*, Theory Probab. App. 28 (1983), p. 278–302.

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