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VARIANCE COMPONENTS ADMISSIBLE ESTIMATION FROM SOME UNBALANCED DATA: FORMULAE FOR THE NESTED DESIGN*

BY

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Abstract. The paper gives unbiased and biased invariant quadratic estimators for variance components in unblanced nested classification random models. The estimators, as well as their quadratic risk functions, have simple closed forms that are easy to calculate, the estimators are admissible in their classes and the unbiased estimators reduce to best unbiased estimators in cases when all cells are filled and the variance of the error terms is set to zero. Numerous numerical risk comparisons of the given estimators with MINQE(U, I) as well as with the lower bounds of the mean squared error of unbiased estimators are also included.

1. Introduction. Let us consider the following model

$$X = A\beta + \varepsilon, \tag{1.1}$$

where A is an $(n \times q)$-matrix, β is a q-dimensional vector, i.e., $\beta \in \mathscr{R}^q$. The n-vector ε is a random vector with expectation zero and covariance $\sum_{i=1}^{p} \sigma_i V_i$, where the V_i are $n \times n$ n.n.d. matrices with V_p being the unit matrix and σ_i, $i = 1, \ldots, p$, are nonnegative numbers called *variance components*. We assume that the fourth moments of the vector X are as under normality. The vectors β and $\sigma = (\sigma_1, \ldots, \sigma_p)'$ are unknown. In the literature model (1.1) is called a *random model* when $A = \mathbf{1}_n$, $\mathbf{1}_n$ being an n-vector of ones. Otherwise, it is called a *mixed model.*

In this paper we are concerned with the problem of estimation of the vector σ of the variance components only. There is an extensive literature on this topic, and the reader may refer to a recent monograph of Rao and Kleffe (1987) for further details. Most research has been done on estimating σ by invariant quadratic estimators, i.e., by estimators of the form

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(1.2) $$(X' M L_1 M X, \ldots, X'ML_p MX)',$$

where $L_1, \ldots, L_p$ are symmetric $n \times n$ matrices, while M is the projection matrix on the subspace of $\mathscr{R}^n$ orthogonal to $\mathscr{R}(A)$, under the quadratic risk function. As usual $\mathscr{R}(\cdot)$ denotes here the range of a matrix argument.

There are several methods currently available (see for example Searle (1987)) to derive estimators for $\boldsymbol{\sigma}$. However, most of them ensure no optimal properties. A complete characterization of all admissible estimators, unbiased and biased, is known only in the case where the matrices $M_1 = MV_1 M, \ldots, M_p = MV_p M$ commute.

In this paper we give admissible unbiased and biased estimators for the vector $\boldsymbol{\sigma}$ of the variance components in model (1.1) in terms of the matrices $M_1, \ldots, M_p$ under the additional assumption that

(1.3) $$\mathscr{R}(M_1) \subsetneq \ldots \subsetneq \mathscr{R}(M_p).$$

As well known, this condition is fulfilled by all nested classification models. The calculation of the proposed estimators requires finding general inverses of a number of matrices. However, for the unbalanced $(p-1)$-way nested classification random model the proposed estimators can be expressed in simple closed forms by using techniques of the same nature as in the work of Swallow and Searle (1978).

The construction of the admissible estimators for the variance components presented in this paper is based on the well known step method (see Klonecki (1980), LaMotte (1980), and Klonecki and Zontek (1985)). The formulae for the model (1.1), subject to (1.3), are given in Section 3, and the explicit formulae for the unblanced nested classification random model in Section 4, where we also give the risk functions for some of the estimators. Finally, in the last section we present numerical results demonstrating the behaviour of the suggested estimators for a selected unblanced 2-way nested classification random model. There are admissible estimators that have more flat risk functions than other estimators and admissible estimators that become the best unbiased estimator (if such one exists) when the variance of the error terms is set to zero.

In a second paper of this series we shall present similar results for the unbalanced $(p-1)$-way crossed classification model.

2. Preliminaries. To characterize admissible invariant quadratic estimators under the quadratic loss one can apply the theory of linear estimation for linear models in finite-dimensional vector spaces.

In fact, introducing the notation $Y = MXX'M$, estimator (1.2) can be written as $L^{\#} Y$, where L is a linear operator mapping $\mathscr{R}^p$ into the space $\mathscr{S}_n$ of $n \times n$ symmetric matrices and defined by $\sum_{i=1}^{p} a_i L_i$ for every $\boldsymbol{a} = (a_1, \ldots, a_p)' \in \mathscr{R}^p$, while $L^{\#}$ is the adjoint operator to L under the usual inner products in $\mathscr{R}^p$

and $\mathscr{S}_n$ (to be denoted by $(\cdot,\cdot)$ and $[\cdot,\cdot]$, respectively). Let $\mathscr{L}$ stand for the set of all linear operators mapping $\mathscr{R}^p$ into $\mathscr{S}_n$.

The vector $\boldsymbol{\sigma}$ to be estimated can be also presented as $C^{\#}EY$ with C being a linear operator mapping $\mathscr{R}^p$ into $\mathscr{S}_n$ and associated with matrices $C_1, \ldots, C_p$ such that $[C_i, M_j] = \delta_{ij}$, $i, j = 1, \ldots, p$, where $M_i = MV_iM$. Assuming that such an operator exists is equivalent to assume that $\boldsymbol{\sigma}$ is estimable in the considered model.

The quadradic risk function $E(L^{\#}Y-\boldsymbol{\sigma}, L^{\#}Y-\boldsymbol{\sigma})$ becomes

$$\sum_{i=1}^{p}([L_i, W_{\boldsymbol{\sigma}} L_i] + [L_i - C_i, w_{\boldsymbol{\sigma}}(L_i - C_i)]), \tag{2.1}$$

where $W_{\boldsymbol{\sigma}} = 2M_{\boldsymbol{\sigma}} \otimes M_{\boldsymbol{\sigma}}$, $w_{\boldsymbol{\sigma}} = M_{\boldsymbol{\sigma}} \overline{\otimes} M_{\boldsymbol{\sigma}}$, while

$$M_{\boldsymbol{\sigma}} = EY = \sum_{i=1}^{p} \sigma_i M_i.$$

As usual, $(B_1 \otimes B_2)$ and $(B_1 \overline{\otimes} B_2)$ denote for B_1 and B_2 in $\mathscr{S}_n$ linear operators mapping $\mathscr{S}_n$ into itself and are for any matrix B_3 in $\mathscr{S}_n$ defined by $(B_1 \otimes B_2)B_3 = B_1B_3B_2$ and $(B_1 \overline{\otimes} B_2)B_3 = [B_2, B_3]B_1$, respectively.

Consider expression (2.1) as a functional $R(\cdot,\cdot)$ defined on the product $\mathscr{L} \times [\mathscr{T}]$, where $[\mathscr{T}]$ is the smallest closed convex cone containing $\mathscr{T} = \{(W_{\boldsymbol{\sigma}}, w_{\boldsymbol{\sigma}}): \boldsymbol{\sigma} \in \mathscr{R}^p_+\}$.

Estimator $L^{\#}Y$ is said to be *locally best at point* T in $[\mathscr{T}]$ among a subset $\mathscr{L}_0$ of $\mathscr{L}$ if $L \in \mathscr{L}_0$ and iff

$$R(L, T) = \min_{L^* \in \mathscr{L}_0} R(L^*, T).$$

One can show (LaMotte (1982)) that whatever be point T in $[\mathscr{T}]$, there is an estimator admissible within a given affine subset $\mathscr{L}_0$ of $\mathscr{L}$ in the class of all locally best estimators at T among $\mathscr{L}_0$. Since a necessary and sufficient condition for an estimator $L^{\#}Y$ to be locally best within $\mathscr{L}_0$ at every point T in $[\mathscr{T}]$ is available, LaMotte's result provides, at least theoretically, a straightforward multi-step method for constructing admissible estimators. The simplest situation occurs when admissibility of an estimator can be shown in a single step — this being equivalent in establishing that the estimator is unique locally best. It is the purpose of this paper to show that the step method can be successfully applied to construct a large class of admissible estimators, not being unique locally best, for the model defined by (1.1) and (1.3). More precisely, we shall show that one can construct for such models explicit formulae of admissible estimators, unbiased and biased, resulting from s steps, $2 \leqslant s \leqslant p$, with the i-th point being $T_{\boldsymbol{\sigma}^{(i)}}$, $\boldsymbol{\sigma}^{(i)} = (\sigma_{i1}, \ldots, \sigma_{iu_i}, 0, \ldots, 0)' \in \mathscr{R}^p_+$, with $1 \leqslant u_1 < u_2 < \ldots < u_s = p$ and $\sigma_{iu_i} = 1$. For convenience we present the coordinates of points $\boldsymbol{\sigma}^{(1)}, \ldots, \boldsymbol{\sigma}^{(s)}$ in the form of an $s \times p$ matrix (to be denoted henceforth by Σ) with the i-the row being $\boldsymbol{\sigma}^{(i)}$. Moreover, let

$$\bar{M}_i = \sum_{j=1}^{u_i} \sigma_{ij} M_i, \quad i = 1, \ldots, s. \tag{2.2}$$

In the case of unbiased estimation the corresponding equations, determining uniquely the admissible estimator associated with the matrix Σ, say $L_\Sigma^{\#} Y$, can be written as

$$E L_\Sigma^{\#} Y = \sigma, \tag{2.3}$$

$$W_{\sigma(i)} L_\Sigma(a) \in \mathscr{E} + \mathscr{N}(\Pi_{i-1}) \cap \mathscr{R}(\Pi_0) \quad \text{for all } a \in \mathscr{R}^p \quad (i = 1, \ldots, s), \tag{2.4}$$

where $\mathscr{E} = \operatorname{span}\{M_1, \ldots, M_p\}$, $\Pi_0 = M \otimes M$, $\Pi_i = M \otimes M - M_{u_i} M_{u_i}^+ \otimes M_{u_i} M_{u_i}^+$ for $i = 1, \ldots, s-1$, while $\mathscr{N}(\cdot)$ stands for null space.

For estimation without the condition of unbiasedness the s equations determining the admissible estimators $L_\Sigma^{\#} Y$ associated with matrix Σ are

$$\Pi_{i-1}(W_{\sigma(i)} + w_{\sigma(i)}) L_\Sigma = \Pi_{i-1} w_{\sigma(i)} C, \quad i = 1, \ldots, s. \tag{2.5}$$

The solutions L_Σ to equations (2.3) and (2.4) as well the solutions L_Σ to (2.5) for matrices Σ of the considered type will be presented in the next section. We close this section with some formulae that we will require in the sequel.

For the evaluation of the risk performance of the presented estimators we shall need the lower bounds of the quadratic risk functions for unbiased and biased estimators of the vector of the variance components. They are given for all $\sigma \in \mathscr{R}_+^p$, respectively, by

$$B_u(\sigma) = 2 \operatorname{tr}\{[W_\sigma^+(V_i), V_j]\} \tag{2.6}$$

and

$$B(\sigma) = \frac{2\sigma'\sigma}{2 + \operatorname{rank} M_\sigma}. \tag{2.7}$$

We shall also need the following formulae throughout the paper.

Let B_1, B_2 and B_3 be $n \times n$ nonnegative matrices and let $M = I_n - \frac{1}{n} 1_n 1_n'$.

If $a \in \mathscr{R}(B_1)$, then

$$(B_1 \pm aa')^+ = B_1^+ \mp \frac{1}{1 \pm a' B_1^+ a} B_1^+ aa' B_1^+, \tag{2.8}$$

provided the denominator is not zero.

If $1 \in \mathscr{R}(B_1)$, then

$$(MB_1M)^+ = B_1^+ - \frac{1}{1' B_1^+ 1} B_1^+ 11' B_1^+. \tag{2.9}$$

If $1 \in \mathscr{R}(B_1)$ and $\mathscr{R}(B_1) \subset \mathscr{R}(B_2)$, then

$$MB_1[MB_1(MB_2M)^+ B_1 M]^+ B_1 M = MB_1(B_1 B_2^+ B_1)^+ B_1 M. \tag{2.10}$$

Formulae (2.8) and (2.9), which are well known, entail (2.10).

If $\mathscr{R}(B_1) \subset \mathscr{R}(B_2)$ and if $\mathscr{R}(B_1) = \mathscr{R}(B_3)$, then

$$B_1[B_1(B_2+B_3)^+B_1]^+B_1 = B_1(B_1B_2^+B_1)^+B_1+B_3. \tag{2.11}$$

This formula may be verified by simple matrix algebra.

3. The main results. We now establish some results that can be used to obtain the linear operators L_Σ determined by (2.3), (2.4) and (2.5). To formulate them, we need to introduce the following definition and notation.

Let $\mathscr{S}_\Sigma = \bigcup \{L(a): a \in \mathscr{R}^p\} \cap \mathscr{R}(M_p \otimes M_p)$, where the sum extends over all solutions L of the relation (2.4) for the given matrix Σ.

For every $i = 1, \ldots, s$ let $W_i = \bar{M}_i \otimes \bar{M}_i$, $G_i = \bar{F}_i \otimes \bar{F}_i$, where $\bar{M}_i$ is defined by (2.2) while $\bar{F}_i = M_{i-1}(M_{i-1}\bar{M}_i^+M_{i-1})^+M_{i-1}$, $M_0 = 0$, and let $H_i = W_i^+ - W_i^+G_iW_i^+$.

LEMMA 3.1. *The set $\mathscr{S}_\Sigma$ is a p-dimensional subspace. The matrices*

$$A_i = \begin{cases} W_s^+G_s \;\; W_{s-1}^+ \;\; G_{s-1} \;\ldots\; W_2^+G_2H_1M_i, & i = 1, \ldots, u_1, \\ \qquad\quad W_s^+ \quad G_s \;\ldots\; W_3^+G_3H_2M_i, & i = u_1+1, \ldots, u_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \qquad\qquad\qquad W_s^+G_sH_{s-1}M_i, & i = u_{s-2}+1, \ldots, u_{s-1} \\ \qquad\qquad\qquad\qquad H_sM_i, & i = u_{s-1}+1, \ldots, u_s \end{cases} \tag{3.1}$$

form a basis for $\mathscr{S}_\Sigma$.

Proof. This lemma can be proved by induction. We do not give the details.

Now a somewhat surprising result shall be established. It states that without any additional assumptions imposed on the considered model the solutions to (2.4) span the same subspace for all $p \times p$ matrices Σ of the considered type.

LEMMA 3.2. *For every $p \times p$ matrix Σ formulae* (3.1) *present the same basis which is given by*

$$\begin{aligned} A_1 &= Q_2M_1^+Q_2, \\ A_2 &= Q_3[M_2^+ - M_2^+F_2M_2^+]Q_3, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ A_{p-1} &= Q_p[M_{p-1}^+ - M_{p-1}^+F_{p-1}M_{p-1}^+]Q_p, \\ A_p &= M_p - M_{p-1}M_{p-1}^+, \end{aligned} \tag{3.2}$$

where, for $2 \leqslant t \leqslant p$, $Q_t = M_p^+F_pM_{p-1}^+F_{p-1}\ldots F_tM_t^+$, while

$$F_t = M_{t-1}(M_{t-1}M_t^+M_{t-1})^+M_{t-1}.$$

Proof. The assertion follows by noting that (3.2) presents the basis (3.1) for $\Sigma = I_p$ and from formula (2.11) given in Section 2.

The following lemma states that the solutions of (2.4) span the same subspaces also for a class of $s \times p$ $(2 \leqslant s < p)$ matrices Σ. However, it goes only for some special models.

Rewrite expression (1.1) in the form

$$X = (U_1, U_2)\begin{pmatrix} A^*\beta+\varepsilon^* \\ \tau \end{pmatrix}, \tag{3.3}$$

where U_1, U_2 and A^* are known matrices, while ε^* and τ are uncorrelated random vectors with expectation 0 and covariance matrix equal to $\sum_{i=1}^{p^*} \sigma_i V_i^*$ and $\sum_{i=p^*+1}^{p} \sigma_i V_i^*$, say, respectively. Under this assumption

$$V_i = \begin{cases} U_1 V_i^* U_1' & \text{for } i = 1, \ldots, p^*, \\ U_2 V_i^* U_2' & \text{for } i = p^*+1, \ldots, p. \end{cases}$$

The projection matrix on the space orthogonal to $\mathscr{R}(A^*)$ will be denoted by M^*. Moreover, let Σ be any matrix of the considered type such that $u_{s^*} = p^*$ for some index $1 \leqslant s^* \leqslant p^*$, and let

$$\Sigma = \begin{pmatrix} \Sigma_{11} & 0 \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

be a partition of Σ such that Σ_{11} is an $s^* \times p^*$ matrix.

LEMMA 3.3. *Assume that U_1 is of full column rank. The subspace $\mathscr{S}_\Sigma$ spanned by the solutions to* (2.4) *does not depend on Σ_{11} if and only if there exists a best unbiased estimator for $\sigma^* = (\sigma_1, \ldots, \sigma_k)'$ in the submodel $X^* = A^*\beta+\varepsilon^*$.*

Proof. Suppose that $L^\# Y$ is an unbiased estimator of σ. The matrices $L_1, \ldots, L_p$ associated with L fulfill then the condition $[M_i, L_j] = \delta_{ij}$, $(i,j = 1, \ldots, p)$.

Since $A = U_1 A^*$ and since U_1 is of full column rank, it follows that $M^* = (MU_1)^+ MU_1$. Thus these equations can be rewritten as

$$[M^* V_1^* M^*, U_1' M L_j M U_1] = \delta_{ij}, \quad i,j = 1, \ldots, p^*.$$

This clearly shows that the linear operator L^*, associated with the matrices $U_1' M L_1 M U_1, \ldots, U_1' M L_{p^*} M U_1$, provides an unbiased estimator of σ^* within the submodel X^*. Moreover, since $\mathscr{R}(M_i) \subsetneq \mathscr{R}(M_j)$ for $i = 1, \ldots, p^*$ and $j = p^*+1, \ldots, p$, equations (2.4) with $i = 1, \ldots, s^*$ are equivalent to

$$\bar{M}_i^* U_1' M L_j M U_1 \bar{M}_i^* \in \mathscr{E}^* + \mathscr{N}(\Pi_{i-1}^*) \cap \mathscr{R}(M^* \otimes M^*),$$

for $i = 1, \ldots, s^*$ and $j = 1, \ldots, p^*$, where

$$\bar{M}_i^* = \sum_{j=1}^{p^*} \sigma_{ij} M^* V_j^* M^*, \qquad \Pi_i^* = (MU_1 \otimes MU_1)^+ \Pi_i,$$

while $\mathscr{E}^* = \operatorname{span}\{M^* V_1^* M^*, \ldots, M^* V_k^* M^*\}$. Now the assertion is evident.

LEMMA 3.4. *If L_Σ is the solution to* (2.5), *then*

$$\{L_\Sigma(a) : a \in \mathscr{R}^p\} \subset \mathscr{S}_\Sigma.$$

Proof. Since this result can be established analogously as Theorem 6.5 in Klonecki and Zontek (1985), we omit the proof.

We shall now present the basic results of the paper – explicit formulae for unbiased and biased admissible quadratic invariant estimators for the mixed model (1.1) fulfilling (1.3).

For any $n \times n$ symmetric matrices $A_1, \ldots, A_p$ define a p-vector Z by $Z = ([A_1, Y], \ldots, [A_p, Y])'$, and a $p \times p$ matrix K by $K = \{[A_i, M_j]\}'$. With this notation

$$EZ = K'\boldsymbol{\sigma} \quad \text{and} \quad \operatorname{cov} Z = 2\{\boldsymbol{\sigma}' V_{ij} \boldsymbol{\sigma}\},$$

where $V_{ij} = \{[A_i, M_u A_j M_v]\}$ for $i, j = 1, \ldots, p$.

THEOREM 3.1. *The admissible unbiased estimator associated with matrix $\boldsymbol{\Sigma}$ is given by*

$$(K')^{-1} Z, \tag{3.4}$$

where K and Z can be computed w.r.t. any basis of $\mathscr{S}_\Sigma$.

Proof. Since (3.4) is evidently an unbiased estimator, we need only to show that it fulfills relation (2.4). But this is also obvious, because it can be written in the form of

$$(K')^{-1} Z = \begin{pmatrix} [L_1, Y] \\ \ldots\ldots \\ [L_p, Y] \end{pmatrix},$$

where each L_i belongs to $\mathscr{S}_\Sigma$.

The next theorem gives a formula for the admissible biased estimator associated with a given matrix $\boldsymbol{\Sigma}$. It will be stated under the assumption that the vector Z and the matrix K are calculated w.r.t. the basis $A_1, \ldots, A_p$ given in Lemma 1.1. To present the formula we need to introduce additional notation.

For any matrix $\boldsymbol{\Sigma}$ of the above considered type let

$$\boldsymbol{\Sigma}^* = \begin{pmatrix} \sigma_{11} \ldots \sigma_{1u_1} & 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & 0 \\ 0 \ \ldots \ 0 & \sigma_{2u_1+1} \ldots \sigma_{2n_2} \quad 0 \ldots\ldots\ldots\ldots & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 0 & \sigma_{su_{s-1}+1} \ \ldots \ \sigma_{sn_s} \end{pmatrix}'.$$

THEOREM 3.2. *The admissible biased estimator associated with matrix Σ is given by*

$$[(2I+\Sigma K\Sigma^*)^{-1}\Sigma]'(\Sigma^*)'Z. \tag{3.5}$$

Proof. Suppose that L_Σ fulfills (2.5). By virtue of Lemma 2.2 the matrices $L_1, \ldots, L_p$ corresponding to L_Σ are linear combinations of matrices (3.1). This means that there exist uniquely determined numbers γ_{ij} that can be obtained from (2.5) such that

$$L_i = \sum_{j=1}^{p} \gamma_{ij} A_j.$$

In fact, substituting these expressions into (2.5) and noting that

$$(\bar{M}_1 \otimes \bar{M}_1) A_j = \begin{cases} M_j, & 1 \leqslant j \leqslant u_1, \\ \mathbf{0}, & u_1+1 \leqslant j \leqslant u_s. \end{cases}$$

$$\Pi_{t-1}(\bar{M}_t \otimes \bar{M}_t) A_j = \begin{cases} \mathbf{0}, & 1 \leqslant j \leqslant u_{t-1}, \\ \Pi_{t-1} M_j, & u_{t-1}+1 \leqslant j \leqslant u_t,\ 2 \leqslant t \leqslant s-1, \\ \mathbf{0}, & u_t+1 \leqslant j \leqslant u_s, \end{cases}$$

$$\Pi_{s-1}(\bar{M}_s \otimes \bar{M}_s) A_j = \begin{cases} \mathbf{0} & 1 \leqslant j \leqslant u_{s-1}, \\ \Pi_{s-1} M_j & u_{s-1}+1 \leqslant j \leqslant u_s, \end{cases}$$

we obtain the following linear equation in the coefficients γ_{ij}: $(2I+\Sigma^*\Sigma K)\{\gamma_{ij}\}' = \Sigma^*\Sigma$. Its solutions is $\{\gamma_{ij}\}' = (2I+\Sigma^*\Sigma K)^{-1}\Sigma^*\Sigma = \Sigma^*(2I+\Sigma K\Sigma^*)^{-1}\Sigma$. Hence $L_\Sigma^{\#} Y = \{\gamma_{ij}\} Z = [(2I+\Sigma K\Sigma^*)^{-1}\Sigma]'(\Sigma^*)'Z$ as asserted.

If Z and K are calculated w.r.t. an arbitrary basis $B_1, \ldots, B_p$ of $\mathscr{S}_\Sigma$ and if $\alpha = \{\alpha_{ij}\}$ is a $(p \times p)$-matrix of coefficients such that

$$A_i = \sum_{j=1}^{p} \alpha_{ij} B_j, \quad i = 1, \ldots, p,$$

where $A_1, \ldots, A_p$ is the basis of $\mathscr{S}_\Sigma$ given by (3.1), then the admissible biased estimator (3.5), associated with matrix Σ, takes the form

$$[(2I+\Sigma K\alpha'\Sigma^*)^{-1}\Sigma]'(\alpha'\Sigma^*)'Z. \tag{3.6}$$

Remark 3.1. Put model (1.1) in the form of (3.3). If vector Z is calculated w.r.t. the basis (3.1) associated with a matrix Σ of the considered type, then clearly $\Pr(Z_{p*+1} = 0, \ldots, Z_p = 0) = 1$ when $\Pr(\tau = 0) = 1$ and, moreover, under the assumptions of Lemma 3.2, the estimator (3.4) becomes the best unbiased estimator of $\boldsymbol{\sigma}^*$ in the relevant submodel $X^* = A^*\boldsymbol{\beta}+\boldsymbol{\varepsilon}^*$.

Remark 3.2. Admissible unbiased and biased estimators which are not unique locally best within their classes are called *limiting estimators*, for they are limits of unique locally best estimators. All the new admissible estimators presented in this paper are limiting estimators, but not every limiting estimator can be obtained in the described manner. It may be worth to recall (see Zontek (1988)) that under very general assumption the class of unique locally best estimators and their limits forms the minimal complete class.

Remark 3.3. For the unbalanced $(p-1)$-way nested classification random model with all subclasses filled the Henderson's estimators are not admissible among unbiased estimators. In fact, for such models they can be presented as $(K')^{-1}B^{\#}Y$, where B is associated with matrices $B_1 = M_1M_1^+$, $B_2 = M_2M_2^+ - M_1M_1^+, \ldots, B_p = M_p - M_{p-1}M_{p-1}^+$, while $K = \{[B_i, M_j]\}'$ and estimators of such a form can be neither unique locally best estimators nor limiting estimators.

4. Examples. We shall now apply the theory developed in Section 3 to obtain admissible estimators for the unbalanced $(p-1)$-way nested classification random model. The defining formula of this model is

$$x_{i_1i_2\ldots i_p} = \mu + \xi_{i_1} + \xi_{i_1i_2} + \ldots + \xi_{i_1i_2\ldots i_p}, \tag{4.1}$$

where $1 \leqslant i_1 \leqslant n_0$, while for any $2 \leqslant t \leqslant p$ and any $i_1i_2\ldots i_{t-1}$ the index i_t goes from 1 to $n_{i_1i_2\ldots i_{t-1}}$, while $n_{i_1i_2\ldots i_{p-1}} > 1$ for at least one $i_1i_2\ldots i_{p-1}$. As usual μ is an unknown parameter, while ξ_{i_1}, $\xi_{i_1i_2}, \ldots, \xi_{i_1i_2\ldots i_p}$ are mutually uncorrelated random variables with zero means and variances $\sigma_1, \sigma_2, \ldots, \sigma_p$, respectively. The problem consists in simultaneous estimation of the vector $\boldsymbol{\sigma}$ of variances $\sigma_1, \ldots, \sigma_p$.

To express the basis (3.2) corresponding to the $p \times p$ unit matrix in a simple form we need to introduce the following notation. Let $\mathscr{I}$ stand for a set defined by

$$\mathscr{I} = \{i_1i_2\ldots i_{p-1}, \ldots, i_1, 0\colon 1 \leqslant i_1 \leqslant n_0, \ldots, 1 \leqslant i_{p-1} \leqslant n_{i_1i_2\ldots i_{p-2}}\}.$$

Now, for $1 \leqslant t \leqslant p-1$ and for any $a = i_1i_2\ldots i_t \in \mathscr{I}$ and any $1 \leqslant i_{t+1} \leqslant n_a$ let $ai_{t+1} = i_1i_2\ldots i_ti_{t+1}$. Clearly, $ai_{t+1} \in \mathscr{I}$ for $t < p-1$. Next we define by a recursive formula a sequence of vectors denoted by $\boldsymbol{e}$ with labeling indices. Let $\boldsymbol{e}_{i_1i_2\ldots i_p} = \boldsymbol{1}_1$ for all possible indices $i_1i_2\ldots i_p$, and let

$$\boldsymbol{e}_a = \begin{pmatrix} \dfrac{1}{n_{a1}}\boldsymbol{e}_{a1} \\ \cdots \\ \dfrac{1}{n_{an_a}}\boldsymbol{e}_{an_a} \end{pmatrix}$$

for every $a \in \mathscr{I}$, where, for the convenience of notation, $n_{i_1i_2\ldots i_p} = 1$ for all indices $i_1i_2\ldots i_p$. Finally, let $E_a = \boldsymbol{e}_a\boldsymbol{e}_a'$ for the above defined vectors $\boldsymbol{e}_a$.

The basis (3.2) can then be presented in the following way:

$$A_{p+1-t} = \bigoplus_{i_1=1}^{n_0} \dots \bigoplus_{i_t=1}^{n_{i_1 i_2 \dots i_{t-1}}} \frac{1}{n^2_{i_1 i_2 \dots i_t}} E_{i_1 i_2 \dots i_t} - \bigoplus_{i_1=1}^{n_0} \dots$$

$$\dots \bigotimes_{i_{t-1}=1}^{n_{i_1 i_2 \dots i_{t-1}}} \frac{1}{n_{i_1 i_2 \dots i_{t-1}}} E_{i_1 i_2 \dots i_{t-1}}, \quad 1 \leqslant t \leqslant p.$$

First we shall give explicit formulae for the relevant vector Z and its expectation for the unbalanced 2-way nested classification random model. The expressions will be given in terms of the familar notation with n_0, n_{i_1} and $n_{i_1 i_2}$ replaced, respectively, by a, b_i, and n_{ij}. They are

$$(4.2) \qquad Z = \begin{bmatrix} \sum_{i=1}^{a} \left(\frac{1}{b^i} \sum_{j=1}^{b^i} \frac{1}{n_{ij}} \sum_{k=1}^{n^{ij}} x_{ijk} \right)^2 - a \left(\frac{1}{a} \sum_{i=1}^{a} \frac{1}{b^i} \sum_{j=1}^{b^i} \frac{1}{n_{ij}} \sum_{k=1}^{n^{ij}} x_{ijk} \right)^2 \\ \sum_{i=1}^{a} \sum_{j=1}^{b^i} \left(\frac{1}{n_{ij}} \sum_{k=1}^{n^{ij}} x_{ijk} \right)^2 - \sum_{i=1}^{a} b^i \left(\frac{1}{b^i} \sum_{j=1}^{b^i} \frac{1}{n_{ij}} \sum_{k=1}^{n^{ij}} x_{ijk} \right)^2 \\ \sum_{i=1}^{a} \sum_{j=1}^{b^i} \sum_{k=1}^{n^{ii}} x^2_{ijk} - \sum_{i=1}^{a} \sum_{j=1}^{b^i} n_{ij} \left(\frac{1}{n_{ij}} \sum_{k=1}^{n^{ij}} x_{ijk} \right)^2 \end{bmatrix}$$

and

$$(4.3) \qquad EZ = \begin{bmatrix} a-1 & \frac{1}{a}(a-1) \sum_{i=1}^{a} \frac{1}{b^i} & \frac{1}{a}(a-1) \sum_{i=1}^{a} \frac{1}{b_i^2} S_{i(-1)} \\ 0 & \sum_{i=1}^{a} b^i - a & \sum_{i=1}^{a} \left(1 - \frac{1}{b^i}\right) S_{i(-1)} \\ 0 & 0 & S^1 - \sum_{i=1}^{a} b^i \end{bmatrix} \boldsymbol{\sigma},$$

where $S_{it} = \sum_{j=1}^{b^i} n^t_{ij}$ and $S^t = \sum_{i=1}^{a} S_{it}$ $(t = \pm 1, \pm 2, 3)$.

Also the covariance of the Z can be written in a closed form expression

$$(4.4) \qquad \operatorname{cov} Z = 2 \begin{bmatrix} \boldsymbol{\sigma}' V_{11} \boldsymbol{\sigma} & \boldsymbol{\sigma}' V_{12} \boldsymbol{\sigma} & 0 \\ \boldsymbol{\sigma}' V_{12} \boldsymbol{\sigma} & \boldsymbol{\sigma}' V_{22} \boldsymbol{\sigma} & 0 \\ 0 & 0 & \boldsymbol{\sigma}' V_{33} \boldsymbol{\sigma} \end{bmatrix},$$

where

$$V_{11} = \begin{bmatrix} a-1 & \frac{a-1}{a}\sum_{i=1}^{a}\frac{1}{b_i} & \frac{a-1}{a}\sum_{i=1}^{a}\frac{1}{b_i^2}S_{i(-1)} \\ - & \frac{a-2}{a}\sum_{i=1}^{a}\frac{1}{b_i^2}+\frac{1}{a^2}\left(\sum_{i=1}^{a}\frac{1}{b_i}\right)^2 & \frac{a-2}{a}\sum_{i=1}^{a}\frac{1}{b_i^3}S_{i(-1)}+\frac{1}{a^2}\sum_{i=1}^{a}\frac{1}{b_i}\sum_{i=1}^{a}\frac{1}{b_i^2}S_{i(-1)} \\ - & - & \frac{a-2}{a}\sum_{i=1}^{a}\frac{1}{b_i^4}S_{i(-1)}^2+\frac{1}{a^2}\left(\sum_{i=1}^{a}\frac{1}{b_i^2}S_{i(-1)}\right)^2 \end{bmatrix},$$

$$V_{12} = \begin{bmatrix} 0 & 0 & 0 \\ - & 0 & 0 \\ - & - & \frac{a-1}{a}\left[\sum_{i=1}^{a}\frac{1}{b_i^2}S_{i(-2)}-\sum_{i=1}^{a}\frac{1}{b_i^3}S_{i(-1)}^2\right] \end{bmatrix},$$

$$V_{22} = \begin{bmatrix} 0 & 0 & 0 \\ - & \sum_{i=1}^{a} b_i-a & \sum_{i=1}^{a}\left(1-\frac{1}{b_i}\right)S_{i(-1)} \\ - & - & \sum_{i=1}^{a}\left(1-\frac{2}{b_i}\right)S_{i(-2)}+\sum_{i=1}^{a}\frac{1}{b_i^2}S_{i(-1)}^2 \end{bmatrix},$$

$$V_{33} = \begin{bmatrix} 0 & 0 & 0 \\ - & 0 & 0 \\ - & - & S_1-\sum_{i=1}^{a} b_i \end{bmatrix}.$$

If Z and K are defined as above, then $(K^{-1})'Z$ is the unbiased admissible estimator associated with every 3×3 matrix Σ by Theorem 3.1 and Lemma 3.2. Moreover, in view of Theorem 3.2,

$$[(2I+\Sigma K)^{-1}\Sigma]'Z, \tag{4.5}$$

where Σ may be any 3×3 matrix, furnishes then the biased admissible estimator associated with the matrix Σ.

When the considered model is partially unbalanced, i.e., when all b_i are equal to b, then $(K^{-1})'Z$ is also an unbiased estimator associated with every

2×3 matrix of the form

$$\Sigma = \begin{pmatrix} \sigma_{11} & 1 & 0 \\ \sigma_{21} & \sigma_{22} & 1 \end{pmatrix} \tag{4.6}$$

by Lemma 3.3.

The biased admissible estimator associated with this matrix has a form that differs from (4.5). It is given by $[(2I+\Sigma KK_1)^{-1}\Sigma]'K_1'Z$, where

$$K_1 = \begin{bmatrix} b/(\sigma_{11}b+1) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}',$$

while Z and K are given by (4.2) and (4.3), respectively. The matrix K_1 could be obtained from (3.6), but in this case it is easier to get it directly from (2.5).

Next we give the formula for vector Z and its expectation for the partially unbalanced 3-way nested classification random model with $n_0 = a$, $n_{i_1} = b$ and $n_{i_1 i_2} = c$ for all $1 \leqslant i_1 \leqslant a$ and $1 \leqslant i_2 \leqslant b$. The vector Z is given by

$$Z = \begin{bmatrix} \sum_{i=1}^{a}\left(\frac{1}{b}\sum_{j=1}^{b}\frac{1}{c}\sum_{k=1}^{c}\frac{1}{n_{ijk}}\sum_{l=1}^{n_{ijk}} x_{ijkl}\right)^2 - a\left(\frac{1}{a}\sum_{i=1}^{a}\frac{1}{b}\sum_{j=1}^{b}\frac{1}{c}\sum_{k=1}^{c}\frac{1}{n_{ijk}}\sum_{l=1}^{n_{ijk}} x_{ijkl}\right)^2 \\ \sum_{i=1}^{a}\sum_{j=1}^{b}\left(\frac{1}{c}\sum_{k=1}^{c}\frac{1}{n_{ijk}}\sum_{l=1}^{n_{ijk}} x_{ijkl}\right)^2 - \sum_{i=1}^{a} b\left(\frac{1}{b}\sum_{j=1}^{b}\frac{1}{c}\sum_{k=1}^{c}\frac{1}{n_{ijk}}\sum_{l=1}^{n_{ijk}} x_{ijkl}\right)^2 \\ \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\frac{1}{n_{ijk}}\sum_{l=1}^{n_{ijk}} x_{ijkl}\right)^2 - \sum_{i=1}^{a}\sum_{j=1}^{b} c\left(\frac{1}{c}\sum_{k=1}^{c}\frac{1}{n_{ijk}}\sum_{l=1}^{n_{ijk}} x_{ijkl}\right)^2 \\ \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\sum_{l=1}^{n_{ijk}} x_{ijkl}^2 - \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} n_{ijk}\left(\frac{1}{n_{ijk}}\sum_{l=1}^{n_{ijk}} x_{ijkl}\right)^2 \end{bmatrix}.$$

It is written in this way so as to imitate formula (4.2) By replacing in the above expression a, b, c and n_{ijk} by n_0, n_{i_1}, $n_{i_1 i_2}$, and $n_{i_1 i_2 i_3}$, respectively, and i, j and k, by i_1, i_2 and i_3, respectively, we obtain the corresponding formula for Z for the completely unbalanced 3-way nested classification random model.

The expectation of the vector Z is given by

$$\mathrm{E}Z = \begin{bmatrix} a-1 & \frac{a-1}{b} & \frac{a-1}{bc} & \frac{a-1}{ab^2c^2}S_{(-1)} \\ 0 & a(b-1) & \frac{a(b-1)}{c} & \frac{b-1}{bc^2}S_{(-1)} \\ 0 & 0 & ab(c-1) & \frac{c-1}{c}S_{(-1)} \\ 0 & 0 & 0 & S_1 - abc \end{bmatrix}' \boldsymbol{\sigma},$$

where now $S_t = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} n_{ijk}^t,\ t = \pm 1.$

As for the case of the 2-way nested classification random model the expressions for the admissible unbiased and biased estimators are obtained from Theorems 2.1 and 2.2, respectively.

Now we shall give for the partially unbalanced 2-way nested classification random model with all b_i equal to b the admissible unbiased estimator and the admissible biased estimator associated with matrix

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 1 \end{pmatrix}. \tag{4.7}$$

For this case formulae (3.1) give the following basis:

$$\begin{aligned} A_1 &= \bar{M}_2^+ \bar{F}_2 M_1^+ \bar{F}_2 \bar{M}_2^+ \\ A_2 &= \bar{M}_2^+ M_2 \bar{M}_2^+ - \bar{M}_2^+ \bar{F}_3 M_2^+ \bar{F}_3 \bar{M}_2^+, \\ A_3 &= (\bar{M}_2^+)^2 - \bar{M}_2^+ \bar{F}_3 (\bar{M}_2^+)^2 \bar{F}_3 \bar{M}_2^+. \end{aligned} \tag{4.8}$$

Formulae for these matrices in terms of a, b and n_{ij} can be easily derived from (2.8)–(2.10).

When σ_{21} and σ_{22} are set to zero, expressions (4.8) reduce to

$$A_1 = M_1^+, \quad A_2 = M_2 - M_1 M_1^+ M_2 M_1^+ M_1, \quad A_3 = M_3 - M_1 M_1^+.$$

The corresponding vector Z becomes

$$Z = \begin{bmatrix} \sum_{i=1}^{a}\left(x_{i..} - \frac{1}{a}\sum_{i=1}^{a} x_{i..}\right)^2 \\ \sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^2 (x_{ij.} - x_{..})^2 - \sum_{i=1}^{a}(x_{i..} - x_{...})^2 \sum_{j=1}^{b} n_{ij}^2 \\ \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}} (x_{ijk} - x_{i..})^2 \end{bmatrix} \tag{4.9}$$

and its expectation is

$$\mathrm{E}Z = \begin{bmatrix} a-1 & \frac{1}{a}(a-1)\sum_{i=1}^{a} S_{i1}^{-2} S_{i2} & \frac{1}{a}(a-1)\sum_{i=1}^{a} S_{i1}^{-1} \\ 0 & S_2 - \frac{2}{S_1}\left(S_3 - \sum_{i=1}^{a} S_{i1}^{-1} S_{i2}^2\right) - \sum_{i=1}^{a} S_{i1}^{-1} S_{i2}^2 & S_1 - \sum_{i=1}^{a} S_{i1}^{-1} S_{i2} \\ 0 & S_1 - \sum_{i=1}^{a} S_{i1}^{-1} S_{i2} & S_1 - a \end{bmatrix}' \sigma.$$

Having this formulae we can write down the expression for the admissible unbiased estimators as well as for the admissible biased estimator. To get the formula for the biased estimator observe that if $\sigma_{21} = \sigma_{22} = 0$, then

$$\Sigma^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus in view of (3.4) it has the form $[(2I+\Sigma K\Sigma^*)^{-1}\Sigma]'Z^*$, where $Z^* = \Sigma^* Z$, while Z is given by (4.9).

The expectation and the covariance of Z^* have simple forms in terms of the n's. These are

$$EZ^* = \begin{bmatrix} a-1 & \frac{1}{a}(a-1)\sum_{i=1}^{a} S_{i1}^{-2} S_{i2} & \frac{1}{a}(a-1)\sum_{i=1}^{a} S_{i1}^{-1} \\ 0 & S_1 - \sum_{i=1}^{a} S_{i1}^{-1} S_{i2} & S_1 - a \end{bmatrix}' \sigma$$

and

$$\operatorname{cov} Z^* = 2\begin{pmatrix} \sigma' V_{11}\sigma & 0 \\ 0 & \sigma' V_{22}\sigma \end{pmatrix},$$

where V_{11} is given by

$$\begin{bmatrix} a-1 & \frac{1}{a}(a-1)\sum_{i=1}^{a} S_{i1}^{-2} S_{i2} & \frac{1}{a}(a-1)\sum_{i=1}^{a} S_{i1}^{-1} \\ - & \frac{1}{a}(a-2)\sum_{i=1}^{a} S_{i1}^{-4} S_{i2}^2 + \frac{1}{a^2}(\sum_{i=1}^{a} S_{i1}^{-2} S_{i2})^2 & \frac{1}{a}(a-2)\sum_{i=1}^{a} S_{i1}^{-3} S_{i2} + \frac{1}{a^2}\sum_{i=1}^{a} S_{i1}^{-1}\sum_{i=1}^{a} S_{i1}^{-2} S_{i2} \\ - & - & \frac{1}{a}(a-2)\sum_{i=1}^{a} S_{i1}^{-2} + \frac{1}{a^2}(\sum_{i=1}^{a} S_{i1}^{-1})^2 \end{bmatrix},$$

while V_{22} by

$$V_{22} = \begin{bmatrix} 0 & 0 & 0 \\ - & S_2 - 2\sum_{i=1}^{a} S_{i1}^{-1} S_{i3} + \sum_{i=1}^{a} S_{i1}^{-2} S_{i2}^2 & S - \sum_{i=1}^{a} S_{i1}^{-1} S_{i2} \\ - & - & S_1 - a \end{bmatrix}.$$

5. Risk comparison. The expressions for the risk functions of the different estimators given in Section 3 are too complicated for a throughout analytical comparison and, therefore, an evaluation of their performance must be made on the basis of numerical comparison. We make a risk comparison only for an unbalanced 2-way nested classification model with all subclasses filled. To begin with we would like to make the following introductory comments.

If Σ is a 2×3 matrix such that $u_1 = 2$ or if it is a 3×3 matrix, then the risk of the corresponding unbiased estimators at points $\boldsymbol{\sigma} = (\sigma_1, 1-\sigma_1, 0)'$, $0 \leqslant \sigma_1 \leqslant 1$, depends only on the number of levels a and b, and at point $(1, 0, 0)'$ on a alone. Moreover, the variances of the unbiased estimators coincide with their relevant lower bounds at these points. When a and b are fixed and if the number of replications n_{ij} increases, i.e., if more observations are added, then the risks of all the considered estimators as well as the lower bounds (2.6) and (2.7) decrease at all points $\boldsymbol{\sigma}$ with $\sigma_3 > 0$, and, as numerous numerical results show, the largest decreases are at points nearest to $(0, 0, 1)'$. As one might also expect, the larger a and b, the smaller the dissimilarities between risks of unbiased and biased estimators. Our numerical results show that similarly as for the one-way classification model (see Swallow and Searle (1978)) the estimators benefit far more from adding observations in the form of more groups that from increasing group sizes.

For the numerical study we selected the following $N = \{n_{ij}\}$ pattern

$$N = \begin{bmatrix} 6 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

with 25 readings. The values of $\boldsymbol{\sigma}$ were selected similarly as in Zontek and Klonecki (1988), i.e., the risks have been calculated for $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, 1-\sigma_1-\sigma_2)'$ and for $\sigma_1, \sigma_2 = 0(.2)\ 1.0$, $0 \leqslant \sigma_1 + \sigma_2 \leqslant 1$. There is no loss of information in using this format, because the risk is a function of the variance components through being a function of the ratios σ_1/σ_3, σ_2/σ_3 and σ_3.

The following numerical results are provided. The entries in Table 1 present lower bounds of variances for unbiased estimators. Table 2 lists variances of the best unbiased estimator at point $(1/3, 1/3, 1)'$. Tables 3 gives risks of the unbiased estimator associated with matrix (4.7) with $\sigma_{21} = \sigma_{22} = 1/3$ and Table 4 the risks of the unbiased estimator associated with every 3×3 matrix as well as with every matrix of form (4.6). The latter has a slightly larger risk at the specified point $(1/3, 1/3, 1)'$ than the corresponding best unbiased estimator, but this disadvantages is outweighted by other desirable properties. In particular, when $\sigma_3 = 0$, its risk reaches the lower bound for unbiased estimators. It might be a reasonable choice when no *a priori* information about $\boldsymbol{\sigma}$ is available.

When biased estimators are allowed for considerations, a remarkable reduction in the mean square error, as presented by Table 5, takes place. Table 8 shows that the admissible biased estimator corresponding to the unit matrix is at every point of the parameter space better than any unbiased estimator. Its efficiencies w.r.t. the lower bounds given in Table 1 are, as it is shown in Table 9, never smaller than 13% and are above 50% over a large range of the values of the underlying variances that are most likely to be met in practice.

If the degree of unbalancedness is high, then there may exist unbiased estimators better than the admissible biased estimators corresponding to the unit matrix in some neighbourhood of $(0, 0, 1)'$. As one might expect this may occur as a result of a combination of three factors-number of total readings, number of levels and the degree of unbalancedness. For balanced nested classification random models (see Klonecki and Zontek (1981)) the admissible biased estimator corresponding to the unit matrix is better than the best unbiased estimator.

TABLE 1

Lower bound of mean squared error for unbiased estimation

$\sigma_1 \backslash \sigma_2$	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0.1894	0.2165	0.2526	0.3072	0.3831	0.4815
0.2	0.2105	0.2447	0.2936	0.3631	0.4548	
0.4	0.2951	0.3442	0.4082	0.4933		
0.6	0.4539	0.5180	0.5970			
0.8	0.6887	0.7659				
1.0	1.0000					

TABLE 2

Mean squared error of the unbiased estimator corresponding to $\Sigma = (1/3, 1/3, 1)$

$\sigma_1 \backslash \sigma_2$	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0.2137	0.2201	0.2556	0.3202	0.4139	0.5367
0.2	0.2193	0.2447	0.2992	0.3827	0.4954	
0.4	0.3011	0.3454	0.4188	0.5213		
0.6	0.4589	0.5222	0.6146			
0.8	0.6929	0.7751				
1.0	1.0029					

TABLE 3

Mean squared error of the unbiased estimator corresponding to $\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1/3 & 1 \end{pmatrix}$

$\sigma_1 \backslash \sigma_2$	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0.2142	0.2205	0.2559	0.3202	0.4135	0.5359
0.2	0.2200	0.2451	0.2991	0.3822	0.4942	
0.4	0.3015	0.3453	0.4180	0.5198		
0.6	0.4587	0.5211	0.6126			
0.8	0.6915	0.7727				
1.0	1.0000					

TABLE 4

Mean squared error of the unbiased estimator corresponding to $\Sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\sigma_1 \backslash \sigma_2$	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0.2397	0.2388	0.2626	0.3110	0.3839	0.4815
0.2	0.2392	0.2562	0.2978	0.3640	0.4548	
0.4	0.3150	0.3499	0.4093	0.4933		
0.6	0.4671	0.5198	0.5970			
0.8	0.6954	0.7659				
1.0	1.0000					

TABLE 5

Lower bound of mean squared error for biased estimation

$\sigma_1 \backslash \sigma_2$	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0.0769	0.0523	0.0400	0.0400	0.0523	0.2000
0.2	0.0523	0.0338	0.0277	0.0338	0.1360	
0.4	0.0400	0.0277	0.0277	0.1040		
0.6	0.0400	0.0338	0.1040			
0.8	0.0523	0.1360				
1.0	0.5000					

TABLE 6

Mean squared error of the biased estimator corresponding to $\Sigma = (1/3,\ 1/3,\ 1)$

$\sigma_1 \backslash \sigma_2$	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0.2509	0.1152	0.1182	0.2600	0.5406	0.9599
0.2	0.1155	0.0338	0.0909	0.2867	0.6213	
0.4	0.1182	0.0906	0.2017	0.4515		
0.6	0.2590	0.2853	0.4505			
0.8	0.5377	0.6182				
1.0	0.9546					

TABLE 7

Mean squared error of the biased estimator corresponding to $\Sigma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\sigma_1 \backslash \sigma_2$	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0.2165	0.2245	0.2858	0.4005	0.5686	0.7900
0.2	0.1240	0.1001	0.1295	0.2123	0.3484	
0.4	0.1352	0.0793	0.0768	0.1276		
0.6	0.2499	0.1621	0.1276			
0.8	0.4681	0.3484				
1.0	0.7900					

TABLE 8

Mean squared error of the biased estimator corresponding to $\Sigma = I_3$

$\sigma_1 \backslash \sigma_2$	0.0	0.2	0.4	0.6	0.8	1.0
0.0	0.1651	0.1445	0.1462	0.1701	0.2163	0.2847
0.2	0.1368	0.1248	0.1350	0.1675	0.2222	
0.4	0.1562	0.1527	0.1715	0.2125		
0.6	0.2231	0.2282	0.2556			
0.8	0.3377	0.3514				
1.0	0.5000					

TABLE 9

Efficiencies of the biased estimator corresponding to $\Sigma = I_3$ w.r.t. the lower bounds given in Table 1

$\sigma_1 \backslash \sigma_2$	0.0	0.2	0.4	0.6	0.8	1.0
0.0	12.82	33.25	42.13	44.61	43.53	40.87
0.2	35.00	49.00	54.02	53.87	51.14	
0.4	47.08	55.64	57.99	56.93		
0.6	50.84	55.95	57.20			
0.8	50.96	54.12				
1.0	50.00					

Numerical risk comparisons for N-patterns having greater degree of unbalancedness than the one here considered are available in Klonecki and Zontek (1988).

6. Conclusions. Since there is now available a large class of admissible estimators for nested random unbalanced models, the problem arises which of them to apply in practice. This problem requires further detailed, throughout investigation. The numerous numerical results obtained by us seem to suggest the following.

When unbiased estimation is preferred and when no *a priori* information about $\boldsymbol{\sigma}$ is available, the admissible estimator associated with unit matrix appears to be the best choice. It becomes, in the submodel obtained by setting in the original unbalanced model the variance of the error terms to zero, the limiting admissible estimator (see Zontek and Klonecki (1988)) which coincides with the best unbiased estimator when all cells are filled. For that reason it is very efficient for experiments (see Ahrens et al. (1980)) where the variance of the error terms is small in comparison with the other variances. This estimator is also distinguished for its flat risk function over the set of all possible parameter values. It is also a fortunate situation that of all the known admissible estimators, it and its covariance, are the easiest to compute. When there is available some *a priori* information about the estimated parameters, one can use either the MINQE(U, I) or the admissible estimator associated with matrix (4.7).

It is not as clear as above which one of the proposed admissible biased estimators to apply. When no *a priori* information is available, the estimator associated with the unit matrix appears again to be reasonable. Since it is better than all unbiased estimators under balanced nested classification random models, one might expect that when the unbalancedness of the accepted model is not to severe it is indeed the best choice. On the other hand, if we have some information about σ alternative possibilities provide the estimators associated with a 1×3 matrix or a 2×3 matrix of form (4.6), and in case all b_i are equal, also with a 2×3 matrix of form (4.7).

REFERENCES

[1] H. Ahrens, J. Kleffe and R. Tenzler, *Mean square error comparison for* MINQUE, ANOVA *and two alternative estimators under the unbalanced one-way random model*, Biom. J. 23 (1981), p. 323–342.

[2] R. H. Farrell, W. Klonecki and S. Zontek, *All admissible linear estimators of the vector of gamma scale parameters with application to random effects models*, Ann. Statist. (1989), p. 268–281.

[3] W. Klonecki, *Invariant quadratic unbiased estimation for variance components*, In: Preceedings, Six International Conference, Wisła, Poland, Springer (1980), p. 201–223.

[4] W. Klonecki and S. Zontek, *On admissible estimation of parametric functions for mixed linear models*, In: Proceedings of Berkeley Conference in Honor of Jerzy Neyman and Jack Kiefer, Vol. 2 (1985), p. 201–223.

[5] — *Variance components admissible estimation from some unbalanced data: Formulae for the nested design* (1), Preprint 441, Institute of Mathematics, Polish Academy of Sciences (1988).

[6] L. R. LaMotte, *Quadratic estimation of variance components*, Biometrics 29 (1973), p. 311–330.

[7] — *Some results on biased linear estimation applied to variance component estimation*, In: Proceedings, Six International Conference, Wisła, Poland, Springer (1980), p. 266–274.

[8] — *Admissibility in linear models*, Ann. Statist. 19 (1982), p. 245–256.

[9] C. R. Rao and J. Kleffe, *Estimation of Variance Components and Applications*, Elsevier Science Publisher (1988).

[10] S. R. Searle, *Linear Models for Unbalanced Data*, John Wiley & Sons (1987).

[11] W. H. Swallow and S. R. Searle, *Minimum variance quadratic unbiased estimation* (MIVQUE) *of variance components*, Technometrics 20 (1978), p. 265–272.

[12] S. Zontek, *Admissibility of limits of the unique locally best linear estimators with application to variance component models*, Prob. and Math. Statist. 9.2 (1988), p. 29–44.

[13] S. Zontek and W. Klonecki, *Limiting admissible estimators for variance components*, Preprint 423, Institute of Mathematics, Polish Academy of Sciences, (1988).

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