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ON COMPLETE CONVERGENCE FOR PARTIAL SUMS OF INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VARIABLES

BY

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Abstract. We give the strong law of large numbers of Hsu–Robbins type for a sequence of independent nonidentically distributed random variables.

1. Introduction. Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables and $S_n = \sum X_k$ for $k = 1, 2, \ldots, n$. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to satisfy the *law of large numbers of Hsu–Robbins type* with a sequence $\{b_n, n \geqslant 1\}$ of real numbers if for any given $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P[|S_n - b_n| \geqslant n\varepsilon] < \infty. \tag{1}$$

Necessary and sufficient (or only sufficient) conditions for (1) to hold were discussed in many papers (cf. [1]–[3], [6] and [7]). This paper contains extensions or generalizations of results given in [1]–[3], [5] and [11].

2. Preliminaries. A real-valued function $l(x)$, positive and measurable on $[A, \infty)$ for some $A > 0$, is said to be *slowly varying* if

$$\lim_{x\to\infty} \frac{l(x\lambda)}{l(x)} = 1 \quad \text{for each } \lambda > 0.$$

We need the following lemmas:

LEMMA 1 ([11]). *If $l(x) > 0$ is a slowly varying function as $x \to \infty$, then*

(i) $$\lim_{x\to\infty} \frac{l(x+u)}{l(x)} = 1 \quad \textit{for each } u > 0;$$

(ii) $$\lim_{k\to\infty} \sup_{2^k \leqslant x < 2^{k+1}} \frac{l(x)}{l(2^k)} = 1;$$

(iii) $$\lim_{x\to\infty} x^{\delta} l(x) = \infty, \quad \lim_{x\to\infty} x^{-\delta} l(x) = 0 \quad \textit{for each } \delta > 0;$$

(iv) $$c_1 2^{kr} l(\varepsilon 2^k) \leqslant \sum_{j=1}^{k} 2^{jr} l(\varepsilon 2^j) \leqslant c_2 2^{kr} l(\varepsilon 2^k)$$

for every positive r, ε, *positive integer* k *and some positive constants* c_1, c_2;

(v) $$c_3 2^{kr} l(\varepsilon 2^k) \leqslant \sum_{j=k}^{\infty} 2^{jr} l(\varepsilon 2^j) \leqslant c_4 2^{kr} l(\varepsilon 2^k)$$

for every $r < 0$, $\varepsilon > 0$ *and a positive integer* k *with some positive constants* c_3, c_4.

LEMMA 2 ([10]). *For every* $\varepsilon > 0$

(2) $$P[|X - \operatorname{med} X| \geqslant \varepsilon] \leqslant 2P[|X^s| \geqslant \varepsilon]$$

and

(3) $$P[\sup_{j \leqslant n} |X_j - \operatorname{med} X_j| \geqslant \varepsilon] \leqslant 2P[\sup_{j \leqslant n} |X_j^s| \geqslant \varepsilon].$$

LEMMA 3 ([4], [9]). *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent random variables with symmetric distributions. Then for every* $j = 1, 2, \ldots$ *and* $x > 0$

(4) $$P[|S_n| \geqslant 3^j x] \leqslant C_j \sum_{i=1}^{n} P[|X_i| \geqslant x] + D_j (P[|S_n| \geqslant x])^{2^j},$$

where C_j, D_j *are positive constants depending only on* j.

LEMMA 4 ([10]). *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent random variables with* $\mathrm{E}X_i = 0$, $i \geqslant 1$, *and let* $r \geqslant 1$. *Then*

(5) $$P[\sup_{k \leqslant n} |S_k| \geqslant c] \leqslant c^{-r} \mathrm{E}|S_n|^r \quad \text{for any given } c > 0.$$

LEMMA 5. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent random variables. Put* $X'_j = X_j I[|X_j| < n^{1/t}\delta]$, $\delta > 0$, $1 \leqslant j \leqslant n$, *where* $I[A]$ *stands for the indicator of the event* A, *and let* $S'_n = \sum X'_j$ *for* $j = 1, 2, \ldots, n$. *Suppose that for some* $0 < t < 2$

(6) $$|\mathrm{E}S'_n|/n^{1/t} \to 0, \qquad n \to \infty.$$

Then for any given $\varepsilon > 0$ *there exists a positive integer* n_0 *such that for* $n \geqslant n_0$

(7) $$P[|S_n| \geqslant 2n^{1/t}\varepsilon] \leqslant P[|S'_n - \mathrm{E}S'_n| \geqslant n^{1/t}\varepsilon] + \sum_{i=1}^{n} P[|X_i| \geqslant n^{1/t}\delta].$$

If there exists $\mathrm{E}X_i$, $i \geqslant 1$, then we often use (6) and (7) with X_j replaced by the centered random variables $X_j - \mathrm{E}X_j$, $j \geqslant 1$.

COROLLARY (cf. [2]). *Under the assumption of Lemma 5 for any given* $\varepsilon > 0$ *there exists a positive integer* n_0 *such that for* $n \geqslant n_0$

(8) $$P[|S_n| \geqslant 2n^{1/t}\varepsilon]$$
$$\leqslant \varepsilon^{-4} n^{-4/t} \Big(\sum_{j=1}^{n} \mathrm{E}|X'_j - \mathrm{E}X'_j|^4 + 2 \sum_{j=2}^{n} \sigma^2 X'_j \sum_{i=1}^{j-1} \sigma^2 X'_i \Big) + \sum_{i=1}^{n} P[|X_i| \geqslant n^{1/t}\delta].$$

3. Laws of large numbers of Hsu–Robbins type. The following theorem gives the complete convergence for the partial sums of independent random variables.

THEOREM 1. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent random variables and let* $l(x) > 0$ *be a slowly varying function as* $x \to \infty$. *If for any given* $\varepsilon > 0$, *some* $r \geqslant 1$, $0 < t < 2$ *and a nonnegative integer* j

$$\text{(i)} \qquad \sum_{n=1}^{\infty} n^{r-2} l(n) \sum_{i=1}^{n} P[|X_i| \geqslant n^{1/t}\varepsilon/(2\cdot 3^j)] < \infty,$$

$$\text{(ii)} \qquad \sum_{n=1}^{\infty} n^{r-2} l(n) (n^{-4/t} \sum_{i=1}^{n} \mathrm{E}X_i^4 I[|X_i| < n^{1/t}\varepsilon])^{2^j} < \infty,$$

$$\text{(iii)} \quad \sum_{n=1}^{\infty} n^{r-2} l(n) (n^{-4/t} \sum_{m=2}^{n} \mathrm{E}X_m^2 I[|X_m| < n^{1/t}\varepsilon] \sum_{i=1}^{m-1} \mathrm{E}X_i^2 I[|X_i| < n^{1/t}\varepsilon])^{2^j} < \infty,$$

$$\text{(iv)} \qquad n^{-2/t} \sum_{i=1}^{n} \mathrm{E}X_i^2 I[|X_i| < n^{1/t}\varepsilon] = O(1),$$

$$\text{(v)} \qquad n^{-1/t} \sum_{i=1}^{n} \mathrm{E}X_i I[|X_i| < n^{1/t}\varepsilon] = o(1),$$

then

$$(9) \qquad \sum_{n=1}^{\infty} n^{r-2} l(n) P[|\sum_{i=1}^{n} (X_i - b_i)| \geqslant n^{1/t}\varepsilon] < \infty$$

with $b_i = 0$ *for* $0 < t < 1$ *and* $b_i = \mathrm{E}X_i$ *for* $1 \leqslant t < 2$ *whenever there exist* $\mathrm{E}X_i$, $i \geqslant 1$, *and then* (i)–(v) *are taken with* X_i *replaced by* $X_i - \mathrm{E}X_i$.

Proof. Assume first that $\{X_n, n \geqslant 1\}$ is a sequence of symmetrically distributed random variables. Then inequalities (4), (8) and conditions (i)–(iii) give (9).

We shall assume that $\mathrm{E}X_k = 0$, $k \geqslant 1$, whenever mean values exist. To remove the symmetry assumption we argue as follows. Let $\{X_n^s, n \geqslant 1\}$ denote the sequence of symmetrized random variables, i.e. $X_k^s = X_k - X_k^*$, $k \geqslant 1$, where X_k and X_k^* are independent and have the same distribution function. Then, by (i)–(v), we see that

$$(10) \qquad \sum_{n=1}^{\infty} n^{r-2} l(n) \sum_{i=1}^{n} P[|X_i^s| \geqslant n^{1/t}\varepsilon/3^j]$$

$$\leqslant 2 \sum_{n=1}^{\infty} n^{r-2} l(n) \sum_{i=1}^{n} P[|X_i| \geqslant n^{1/t}\varepsilon/2\cdot 3^j] < \infty,$$

$$(11)\quad \sum_{n=1}^{\infty} n^{r-2} l(n)\big(n^{-4/t} \sum_{i=1}^{n} \mathrm{E}(X_i^s)^4 I[|X_i^s| < n^{1/t}\varepsilon/3^j]\big)^{2^j}$$

$$= \sum_{n=1}^{\infty} n^{r-2} l(n)\{n^{-4/t} \sum_{i=1}^{n} \mathrm{E}(X_i^s)^4 (I[|X_i^s| < n^{1/t}\varepsilon/3^j, |X_i^*| < n^{1/t}\varepsilon/3^j]$$

$$+ I[|X_i^s| < n^{1/t}\varepsilon/3^j, |X_i^*| > n^{1/t}\varepsilon/3^j])\}^{2^j}$$

$$\leqslant 2^{2^{j+1}-1} \sum_{n=1}^{\infty} n^{r-2} l(n)\big(n^{-4/t} \sum_{i=1}^{n} \mathrm{E}X_i^4 I[|X_i| < n^{1/t}\varepsilon]\big)^{2^j}$$

$$+ 2^{2^j-1}\varepsilon^{2^j} \sum_{n=1}^{\infty} n^{r-2} l(n)\big(\sum_{i=1}^{n} \mathrm{P}[|X_i| \geqslant n^{1/t}\varepsilon/3^j]\big)^{2^j} < \infty,$$

$$(12)\quad \sum_{n=1}^{\infty} n^{r-2} l(n)\big(n^{-4/t} \sum_{m=2}^{n} \mathrm{E}(X_m^s)^2 I[|X_m^s| < n^{1/t}\varepsilon/3^j]$$

$$\times \sum_{i=1}^{m-1} \mathrm{E}(X_i^s)^2 I[|X_i^s| < n^{1/t}\varepsilon/3^j]\big)^{2^j}$$

$$= \sum_{n=1}^{\infty} n^{r-2} l(n)\{n^{-4/t} \sum_{m=2}^{n} \mathrm{E}(X_m^s)^2 (I[|X_m^s| < n^{1/t}\varepsilon/3^j, |X_m^*| < n^{1/t}\varepsilon/3^j]$$

$$+ I[|X_m^s| < n^{1/t}\varepsilon/3^j, |X_m^*| > n^{1/t}\varepsilon/3^j]) \sum_{i=1}^{m-1} \mathrm{E}(X_i^s)^2 (I[|X_i^s| < n^{1/t}\varepsilon/3^j,$$

$$|X_i^*| < n^{1/t}\varepsilon/3^j] + I[|X_i^s| < n^{1/t}\varepsilon/3^j, |X_i^*| > n^{1/t}\varepsilon/3^j])\}^{2^j}$$

$$\leqslant C\{\sum_{n=1}^{\infty} n^{r-2} l(n)\big(n^{-4/t} \sum_{m=2}^{n} \mathrm{E}X_m^2 I[|X_m| < n^{1/t}\varepsilon] \sum_{i=1}^{m-1} \mathrm{E}X_i^2 I[|X_i| < n^{1/t}\varepsilon]\big)^{2^j}$$

$$+ \sum_{n=1}^{\infty} n^{r-2} l(n) \sum_{i=1}^{n} P[|X_i| \geqslant n^{1/t}\varepsilon/3^j]\} < \infty,$$

where C is a positive constant depending only on j. Therefore,

$$\sum_{n=1}^{\infty} n^{r-2} l(n) P[|S_n^s| \geqslant n^{1/t}\varepsilon] < \infty.$$

Hence, by the symmetrization inequality

$$P[|S_n/n^{1/t} - \operatorname{med}(S_n/n^{1/t})| \geqslant \varepsilon] \leqslant 2P[|S_n^s| \geqslant n^{1/t}\varepsilon],$$

we get

$$(13)\quad \sum_{n=1}^{\infty} n^{r-2} l(n) P[|S_n/n^{1/t} - \operatorname{med}(S_n/n^{1/t})| \geqslant \varepsilon] < \infty.$$

Thus

$$(14)\quad P[|S_n/n^{1/t} - \operatorname{med}(S_n/n^{1/t})| \geqslant \varepsilon] \to 0, \qquad n \to \infty.$$

Note that by (i)–(iii) and (v) we conclude that

$$\sum_{i=1}^{n} P[|X_i| \geqslant n^{1/t}\varepsilon] \to 0, \quad n \to \infty,$$

$$n^{-4/t} \sum_{i=1}^{n} EX_i^4 I[|X_i| < n^{1/t}\varepsilon] \to 0, \quad n \to \infty,$$

$$n^{-4/t} \sum_{m=2}^{n} EX_m^2 I[|X_m| < n^{1/t}\varepsilon] \sum_{i=1}^{m-1} EX_i^2 I[|X_i| < n^{1/t}] \to 0, \quad n \to \infty,$$

$$n^{-4/t} \Big(\sum_{i=1}^{n} EX_i I[|X_i| < n^{1/t}\varepsilon]\Big)^4 \to 0, \quad n \to \infty.$$

Therefore, by the Corollary we get

$$P[|S_n| \geqslant n^{1/t}\varepsilon] \to 0, \quad n \to \infty. \tag{15}$$

Hence, by (14) we have $\mathrm{med}(S_n/n^{1/t}) \to 0$, $n \to \infty$. Thus, by (13) we have proved that (9) holds.

Note that for the typical slowly varying function $l(x) = 1$, $l(x) = \log x$, one can get simpler formulas in Theorem 1.

Using Theorem 1 we can get the following assertions for a sequence $\{X_n, n \geqslant 1\}$ of independent identically distributed (i.i.d.) random variables.

COROLLARY 1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables and let $l(x) > 0$ be a slowly varying function as $x \to \infty$. Assume that for any given $\varepsilon > 0$ and some $r \geqslant 1$, $0 < t < 2$, and nonnegative integer j the following relations hold:*

$$\sum_{n=1}^{\infty} n^{r-1} l(n) P[|X_1| \geqslant n^{1/t}\varepsilon/(2\cdot 3^j)] < \infty, \tag{i}$$

$$\sum_{n=1}^{\infty} n^{r-2+2^j(1-4/t)} l(n) (EX_1^4 I[|X_1| < n^{1/t}\varepsilon])^{2^j} < \infty, \tag{ii}$$

$$\sum_{n=1}^{\infty} n^{r-2+2^j(2-4/t)} l(n) (EX_1^2 I[|X_1| < n^{1/t}\varepsilon])^{2^{j+1}} < \infty, \tag{iii}$$

$$n^{1-2/t} EX_1^2 I[|X_1| < n^{1/t}\varepsilon] = O(1), \tag{iv}$$

$$n^{1-1/t} E(X_1 - b) I[|X_1 - b| < n^{1/t}\varepsilon] = o(1), \tag{v}$$

where $b = 0$ when $0 < t < 1$, and $b = EX_1$ for $1 \leqslant t < 2$. Then

$$\sum_{n=1}^{\infty} n^{r-2} l(n) P[|S_n - nb| \geqslant n^{1/t}\varepsilon] < \infty.$$

COROLLARY 2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables and let $l(x) > 0$ be a slowly varying function as $x \to \infty$.*

If for $r > 1$ *and* $0 < t < 2$

$$E(|X_1|^{rt} l(|X_1|^t)) < \infty,$$

then

$$\sum_{n=1}^{\infty} n^{r-2} l(n) P[|S_n - nb| \geqslant n^{1/t}\varepsilon] < \infty, \tag{16}$$

where $b = 0$ *when* $0 < t < 1$ *and* $b = EX_1$ *for* $1 \leqslant t < 2$.

Moreover, if $l(x) > 0$ *is a monotone increasing, slowly varying function as* $x \to \infty$, *such that* $l(x) \to \infty$ *and*

$$E(|X_1|^t l(|X_1|^t)) < \infty, \quad 0 < t < 2,$$

then

$$\sum_{n=1}^{\infty} n^{-1} l(n) P[|S_n - nb| \geqslant n^{1/t}\varepsilon] < \infty, \tag{17}$$

where b *is as above.*

Proof of Corollary 2. To prove (16) and (17) it is enough to note that under the assumptions of Corollary 2 the conditions (i)–(v) of Corollary 1 hold.

Indeed, we see that using Lemma 1 we obtain

$$\sum_{n=1}^{\infty} n^{r-1} l(n) P[|X_1| \geqslant n^{1/t}\varepsilon/(2\cdot 3^j)]$$

$$\leqslant \sum_{k=0}^{\infty} 2^k (2^{k+1})^{r-1} l(2^{k+1}) P[|X_1| \geqslant 2^{k/t}\varepsilon/(2\cdot 3^j)]$$

$$\leqslant 2^{-1} \sum_{k=1}^{\infty} (2^k)^r l(2^k) P[|X_1| \geqslant 2^{(k-1)/t}\varepsilon/(2\cdot 3^j)]$$

$$= 2^{-1} \sum_{k=1}^{\infty} (2^k)^r l(2^k) \sum_{i=k}^{\infty} P[2^{(i-1)/t}\varepsilon/(2\cdot 3^j) \leqslant |X_1| < 2^{i/t}\varepsilon/(2\cdot 3^j)]$$

$$= 2^{-1} \sum_{i=1}^{\infty} P[2^{(i-1)/t}\varepsilon/(2\cdot 3^j) \leqslant |X_1| < 2^{i/t}\varepsilon/(2\cdot 3^j)] \sum_{k=1}^{i} (2^k)^r l(2^k)$$

$$\leqslant C \sum_{i=1}^{\infty} P[2^{(i-1)/t}\varepsilon/(2\cdot 3^j) \leqslant |X_1| < 2^{i/t}\varepsilon/(2\cdot 3^j)] (2^i)^r l(2^i)$$

$$\leqslant C' E|X_1|^{rt} l(|X_1|^t) < \infty,$$

which proves (i) for $r \geqslant 1$, $0 < t < 2$. Note that

$$\sum_{n=1}^{\infty} n^{r-2+2^j(1-4/t)} l(n) (EX_1^4 I[|X_1| < n^{1/t}\varepsilon])^{2^j}$$

$$\leqslant \sum_{n=1}^{\infty} n^{r-2+2^j(1-2/t)} l(n) (EX_1^2 I[|X_1| < n^{1/t}\varepsilon])^{2^j}.$$

Since $0 < t < 2$, we can take j large enough so that $r - 1 - 2^j(2/t - 1) < 0$.

In the case $rt \geqslant 2$ the assumption $\mathrm{E}|X_1|^{rt} l(|X_1|^t) < \infty$ implies $\mathrm{E}|X_1|^2 < \infty$ and we see that to prove (ii) it is enough to show that

$$\sum_{n=1}^{\infty} n^{r-2+2^j(1-2/t)} l(n) < \infty.$$

The series $\sum_{n=1}^{\infty} n^{r-2+2^j(1-2/t)} l(n)$ converges iff (cf. [8])

$$\sum_{k=1}^{\infty} (2^k)^{r-1+2^j(1-2/t)} l(2^k) < \infty,$$

and this inequality holds by using Lemma 1 for j such that $r-1-2^j(2/t-1) < 0$.

In the case $rt < 2$ we have

$$\mathrm{E}X_1^2 I[|X_1| < n^{1/t}\varepsilon] = \mathrm{E}|X_1|^{rt} |X_1|^{2-rt} I[|X_1| < n^{1/t}\varepsilon] \leqslant C n^{2/t-r} \mathrm{E}|X_1|^{rt}.$$

Hence

$$\sum_{n=1}^{\infty} n^{r-2+2^j(1-2/t)} l(n) (\mathrm{E}X_1^2 I[|X_1| < n^{1/t}\varepsilon])^{2^j}$$

$$\leqslant \sum_{n=1}^{\infty} n^{r-2+2^j(1-r)} l(n) (\mathrm{E}|X_1|^{rt})^{2^j} < \infty$$

(because $\sum (2^k)^{r-1+2^j(1-r)} l(2^k) < \infty$ for $k = 1, 2, \ldots$ and j such that $r-1+2^j(1-r) < 0$), which completes the proof of (ii) for $r \geqslant 1$, $0 < t < 2$.

The proof of (iii) is similar to that of (ii).

Now we prove that under the assumptions of Corollary 2 the condition (iv) holds.

If $rt \geqslant 2$, then $\mathrm{E}X_1^2 I[|X_1| < n^{1/t}\varepsilon] < \infty$, and

$$n^{1-2/t} \mathrm{E}X_1^2 I[|X_1| < n^{1/t}\varepsilon] = O(1).$$

If $rt < 2$, then

$$n^{1-2/t} \mathrm{E}X_1^2 I[|X_1| < n^{1/t}\varepsilon] = \varepsilon^{2-rt} n^{1-r} \mathrm{E}|X_1|^{rt} = O(1).$$

Thus we have proved that (iv) holds.

The condition (v) will be proved in two steps.

If $r = 1$, then for $0 < t < 1$ we have

$$n^{1-1/t} |\mathrm{E}X_1 I[|X_1| < n^{1/t}\varepsilon]| \leqslant n^{1-1/t} \big(l(\varepsilon^t n)\big)^{-1} (n^{1/t}\varepsilon)^{1-t} \mathrm{E}|X_1|^t l(|X_1|^t)$$

$$\leqslant C \mathrm{E}|X_1|^t l(|X_1|^t)/l(n) \to 0, \quad n \to \infty.$$

If $1 \leqslant t < 2$, then there exists $\mathrm{E}X_1$, and we can put $\mathrm{E}X_1 = 0$. Therefore

$$n^{1-1/t} |\mathrm{E}X_1 I[|X_1| < n^{1/t}\varepsilon]| = n^{1-1/t} |\mathrm{E}X_1 I[|X_1| \geqslant n^{1/t}\varepsilon]|$$

$$\leqslant n^{1-1/t} (n^{1/t}\varepsilon) \big(l(n\varepsilon^t)\big)^{-1} \mathrm{E}|X_1|^t l(|X_1|^t)$$

$$\leqslant C \big(l(n)\big)^{-1} \mathrm{E}|X_1|^t l(|X_1|) \to 0, \quad n \to \infty.$$

If $r > 1$, then for $0 < t < 1$ we have: in the case $rt < 1$

$$n^{1-1/t}|EX_1 I[|X_1| < n^{1/t}\varepsilon]| = n^{1-1/t}(n^{1/t}\varepsilon)^{1-rt}E|X_1|^{rt}$$
$$\leqslant \varepsilon^{1-rt}n^{1-r}E|X_1|^{rt} \to 0, \quad n \to \infty,$$

and in the case $rt \geqslant 1$

$$n^{1-1/t}|EX_1 I[|X_1| < n^{1/t}\varepsilon]| = n^{1-1/t}|EX_1 I[|X_1| \geqslant n^{1/t}\varepsilon]|$$
$$\leqslant n^{1-1/t}(n^{1/t}\varepsilon)^{1-rt}|EX_1|^{rt} = \varepsilon^{1-rt}n^{1-1/t}E|X_1| \to 0, \quad n \to \infty.$$

If $1 \leqslant t < 2$, then there exists EX and

$$n^{1-1/t}|EX_1 I[|X_1| < n^{1/t}\varepsilon]| = n^{1-1/t}|EX_1 I[|X_1| \geqslant n^{1/t}\varepsilon]|$$
$$\leqslant (\varepsilon)^{1-rt}n^{1-r}E|X_1|^{rt} \to 0, \quad n \to \infty.$$

Thus we have proved that (16) holds if $r > 1$ and (17) holds if $r = 1$. Note that by Corollary 2 we have the following direct consequences:

COROLLARY 3. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of i.i.d. random variables and let* $\alpha > 1/2$, $\varrho > 1/\alpha$. *If* $E|X_1|^\varrho < \infty$, *then for any given* $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{\alpha\varrho-2}P[|S_n - nb| \geqslant n^\alpha \varepsilon] < \infty, \tag{18}$$

where $b = 0$ *if* $\varrho < 1$ *and* $b = EX_1$ *if* $\varrho \geqslant 1$.

COROLLARY 4. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of i.i.d. random variables and let* $0 < t < 2$. *If* $E|X_1|^t \log^+|X_1| < \infty$, *then for any given* $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{-1}\log(n)P[|S_n - nb| \geqslant n^{1/t}\varepsilon] < \infty, \tag{19}$$

where $b = 0$ *if* $0 < t < 1$ *and* $b = EX_1$ *if* $1 \leqslant t < 2$.

Remark. Note that (18) is also true for $\varrho = 1/\alpha$, and (19) for $t \geqslant 2$. These facts can be proved in the same way as it has been done in [3].

Finally, we have the following result:

COROLLARY 5. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of i.i.d. random variables with* $EX_1 = b$. *If* $E|X_1|^r < \infty$, $r > 1$, *then for any given* $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{r-2}P[|S_n - nb| \geqslant n\varepsilon] < \infty.$$

By the condition (v) of Theorem 1 we have

$$\sum_{n=1}^{\infty} n^{r-2}l(n)P[\max_{m \leqslant n}|S_m| \geqslant n^{1/t}\varepsilon]$$
$$\leqslant \sum_{n=1}^{\infty} n^{r-2}l(n)\{P[\max_{m \leqslant n}|\sum_{i=1}^{m} X_i I[|X_i| < n^{1/t}\varepsilon]| \geqslant n^{1/t}\varepsilon] + \sum_{i=1}^{n} P[|X_i| \geqslant n^{1/t}\varepsilon]\}$$

$$\leqslant \sum_{n=1}^{\infty} n^{r-2} l(n) \{ P[\max_{m \leqslant n} | \sum_{i=1}^{m} (X_i I[|X_i| < n^{1/t}\varepsilon]$$

$$- EX_i I[|X_i| < n^{1/t}\varepsilon])| \geqslant n^{1/t}\varepsilon] + \sum_{i=1}^{n} P[|X_i| \geqslant n^{1/t}\varepsilon]\} \quad \text{for } n \geqslant n_0.$$

THEOREM 2. *Under the assumptions* (i)–(v) *of Theorem* 1 *for any given* $\varepsilon > 0$ *and some* $r \geqslant 1$ *and* $0 < t < 2$,

$$\sum_{n=1}^{\infty} n^{r-2} l(n) P[\max_{k \leqslant n} |S_k - b_k| \geqslant n^{1/t}\varepsilon] < \infty$$

with $b_k = 0$ *for* $0 < t < 1$ *and* $b_k = \sum EX_i$ *for* $i = 1, 2, \ldots, k$ *and* $1 \leqslant t < 2$ *whenever there exists* EX_i, $i \geqslant 1$, *and then* (i)–(v) *are taken with* X_i *replaced by* $X_i - EX_i$.

Now we see that one gets a stronger result than (9).

THEOREM 3. *Under the assumptions* (i)–(v) *of Theorem* 1 *for any given* $\varepsilon > 0$ *and some* $r \geqslant 1$ *and* $0 < t < 2$,

$$\sum_{n=1}^{\infty} n^{r-2} l(n) P[\sup_{k \geqslant n} k^{-1/t} | \sum_{i=1}^{k} (X_i - b_i)| \geqslant \varepsilon] < \infty \tag{20}$$

with $b_i = 0$ *for* $0 < t < 1$ *and* $b_i = EX_i$, $i \geqslant 1$, *for* $1 \leqslant t < 2$ *whenever there exists* EX_i, $i \geqslant 1$, *and then* (i)–(v) *are taken with* X_i *replaced by* $X_i - EX_i$.

Proof. Note that using Lemma 1 (iv) we have

$$\sum_{n=1}^{\infty} n^{r-2} l(n) P[\sup_{m \geqslant n} m^{-1/t} |S_m| \geqslant \varepsilon]$$

$$\leqslant \tfrac{1}{2} \sum_{k=1}^{\infty} (2^k)^{r-1} l(2^k) P[\sup_{n \geqslant 2^{k-1}} n^{-1/t} |S_n| \geqslant \varepsilon]$$

$$\leqslant \tfrac{1}{2} \sum_{k=1}^{\infty} (2^k)^{r-1} l(2^k) \sum_{m=k}^{\infty} P[\max_{2^{m-1} \leqslant n < 2^m} n^{-1/t} |S_n| \geqslant \varepsilon]$$

$$= \tfrac{1}{2} \sum_{m=1}^{\infty} P[\max_{2^{m-1} \leqslant n < 2^m} n^{-1/t} |S_n| \geqslant \varepsilon] \sum_{k=1}^{m} (2^k)^{r-1} l(2^k)$$

$$\leqslant C \sum_{m=1}^{\infty} (2^m)^{r-1} l(2^m) \{ P[\max_{2^{m-1} \leqslant n < 2^m} n^{-1/t} | \sum_{i=1}^{n} X_i I[|X_i| < 2^{m/t}\varepsilon]| \geqslant \varepsilon]$$

$$+ \sum_{i=1}^{2^m} P[|X_i| \geqslant 2^{m/t}\varepsilon/2]\}.$$

Note that, by (v),

$$P\Big[\max_{2^{m-1}\leqslant n<2^m} n^{-1/t}\Big|\sum_{i=1}^{n} X_i I[|X_i|<2^{m/t}\varepsilon]\Big|\geqslant\varepsilon\Big]$$

$$\leqslant P\Big[\max_{2^{m-1}\leqslant n<2^m} n^{-1/t}\Big|\sum_{i=1}^{n}(X_i I[|X_i|<2^{m/t}\varepsilon]-\mathrm{E}X_i I[|X_i|<2^{m/t}\varepsilon])\Big|\geqslant\varepsilon\Big]$$

$$+P\Big[\max_{2^{m-1}\leqslant n<2^m} n^{-1/t}\Big|\sum_{i=1}^{n}\mathrm{E}X_i I[|X_i|<2^{m/t}\varepsilon]\Big|\geqslant\varepsilon\Big]$$

$$\leqslant P\Big[\max_{2^{m-1}\leqslant n<2^m} n^{-1/t}\Big|\sum_{i=1}^{n}(X_i I[|X_i|<2^{m/t}\varepsilon]-\mathrm{E}X_i I[|X_i|<2^{m/t}\varepsilon])\Big|\geqslant\varepsilon\Big]$$

for $m\geqslant m_0$.

Therefore, using (5) and (6) we get for $m\geqslant m_0$

$$P\Big[\max_{2^{m-1}\leqslant n<2^m} n^{-1/t}\Big|\sum_{i=1}^{n} X_i I[|X_i|<2^{m/t}\varepsilon]\Big|\geqslant\varepsilon\Big]$$

$$\leqslant P\Big[\max_{2^{m-1}\leqslant n<2^m}\Big|\sum_{i=1}^{n} X_i I[|X_i|<2^{m/t}\varepsilon]\Big|\geqslant 2^{(m-1)/t}\varepsilon\Big]$$

$$\leqslant C\cdot(2^m)^{-4/t}\Big(\sum_{j=1}^{2^m}\mathrm{E}|X_j'-\mathrm{E}X_j'|^4+2\sum_{j=2}^{2^m}\sigma^2 X_j'\sum_{i=1}^{j-1}\sigma^2 X_i'\Big).$$

Now, by the considerations similar to those in the proof of Theorem 1 we can deduce (20).

COROLLARY 6. *Let $\{X_n, n\geqslant 1\}$ be a sequence of i.i.d. random variables and let $l(x)>0$ be a slowly varying function as $x\to\infty$. Suppose that the conditions (i)–(v) of Corollary 1 are satisfied with $r>1$. Then*

$$\sum_{n=1}^{\infty} n^{r-2} l(n) P[\sup_{k\geqslant n} k^{-1/t}|S_k-kb|\geqslant\varepsilon]<\infty.$$

Remark. If $\mathrm{E}|X_1|^{rt}<\infty$, $r>1$, $0<t<2$, $l(x)=1$, we obtain the classical result (cf. [3]).

Let us consider now the case $r=1$.

THEOREM 4. *Let $\{X_n, n\geqslant 1\}$ be a sequence of independent random variables and let $l(x)>0$ be a slowly varying function as $x\to\infty$ such that for any positive integer k*

$$\sum_{j=1}^{k} l(2^j)\leqslant ck\cdot l(2^k). \tag{21}$$

If for any given $\varepsilon>0$, $0<t<2$, and nonnegative integer j

$$\sum_{n=1}^{\infty} n^{-1} l(n)\log(n)\sum_{i=1}^{n} P[|X_i|\geqslant n^{1/t}\varepsilon/(2\cdot 3^j)]<\infty, \tag{i}$$

(ii) $$\sum_{n=1}^{\infty} n^{-1} l(n)\log(n)\Big(n^{-4/t} \sum_{i=1}^{n} \mathrm{E}X_i^4 I[|X_i| < n^{1/t}\varepsilon]\Big)^{2^j} < \infty,$$

(iii) $$\sum_{n=1}^{\infty} n^{-1} l(n)\log(n)\Big(n^{-4/t} \sum_{m=2}^{n} \mathrm{E}X_m^2 I[|X_m| < n^{1/t}\varepsilon] \times \sum_{i=1}^{m-1} \mathrm{E}X_i^2 I[|X_i| < n^{1/t}\varepsilon]\Big)^{2^j} < \infty,$$

(iv) $$n^{-2/t} \sum_{i=1}^{n} \mathrm{E}X_i^2 I[|X_i| < n^{1/t}\varepsilon] = O(1),$$

(v) $$n^{-1/t} \sum_{i=1}^{n} \mathrm{E}X_i I[|X_i| < n^{1/t}\varepsilon] = o(1),$$

then

(22) $$\sum_{n=1}^{\infty} n^{-1} l(n) P\Big[\sup_{k \geqslant n} k^{-1/t}\Big|\sum_{i=1}^{k} (X_i - b_i)\Big| \geqslant \varepsilon\Big] < \infty$$

with $b_i = 0$ *for* $0 < t < 1$ *and* $b_i = \mathrm{E}X_i$ *for* $1 \leqslant t < 2$ *whenever there exists* $\mathrm{E}X_i$, *and then* (i)–(v) *are taken with* X_i *replaced by* $X_i - \mathrm{E}X_i$.

Proof. Note that

$$\sum_{n=1}^{\infty} n^{-1} l(n) P[\sup_{m \geqslant n} m^{-1/t}|S_m| \geqslant \varepsilon]$$

$$\leqslant 2^{-1} \sum_{k=1}^{\infty} l(2^k) P[\sup_{n \geqslant 2^{k-1}} n^{-1/t}|S_n| \geqslant \varepsilon]$$

$$\leqslant 2^{-1} \sum_{k=1}^{\infty} l(2^k) \sum_{m=k}^{\infty} P[\max_{2^{m-1} \leqslant n < 2^m} n^{-1/t}|S_n| \geqslant \varepsilon]$$

$$\leqslant 2^{-1} \sum_{m=1}^{\infty} P[\max_{2^{m-1} \leqslant n < 2^m} n^{-1/t}|S_n| \geqslant \varepsilon] \sum_{k=1}^{m} l(2^k)$$

$$\leqslant C \sum_{m=1}^{\infty} ml(2^m)\Big\{P\Big[\max_{2^{m-1} \leqslant n < 2^m} n^{-1/t}\Big|\sum_{i=1}^{n} X_i I[|X_i| < 2^{m/t}\varepsilon]\Big| \geqslant \varepsilon\Big] + \sum_{i=1}^{2^m} P[|X_i| \geqslant 2^{m/t}\varepsilon/2]\Big\}$$

$$= Ct(\log 2)^{-1} \sum_{m=1}^{\infty} l(2^m)\log 2^{m/t}\Big\{P\Big[\max_{2^{m-1} \leqslant n < 2^m} n^{-1/t}\Big|\sum_{i=1}^{n} X_i I[|X_i| < 2^{m/t}\varepsilon]\Big| \geqslant \varepsilon\Big] + \sum_{i=1}^{2^m} P[|X_i| \geqslant 2^{(m+1)/t}\varepsilon/2]\Big\}.$$

Now, by the considerations similar to those in the proof of Theorems 1 and 3 we get (22).

COROLLARY 7. *Let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables and let $l(x) > 0$ be a slowly varying function satisfying* (21). *Assume that for any given $\varepsilon > 0$, $0 < t < 2$, and a nonnegative integer j the following conditions hold*:

(i) $$\sum_{n=1}^{\infty} l(n) P[|X_1| \geqslant n^{1/t}\varepsilon/(2\cdot 3^j)] < \infty,$$

(ii) $$\sum_{n=1}^{\infty} n^{2^j(1-4/t)-1} l(n)\log(n)(\mathrm{E}X_1^4 I[|X_1| < n^{1/t}\varepsilon])^{2^j} < \infty,$$

(iii) $$\sum_{n=1}^{\infty} n^{2^j(2-4/t)-1} l(n)\log(n)(\mathrm{E}X_1^2 I[|X_1| < n^{1/t}\varepsilon])^{2^{j+1}} < \infty,$$

(iv) $$n^{1-2/t}\mathrm{E}X_1^2 I[|X_1| < n^{1/t}\varepsilon] = O(1),$$

(v) $$n^{1-1/t}\mathrm{E}(X_1-b) I[|X_1-b| < n^{1/t}\varepsilon] = o(1),$$

where $b = 0$, when $0 < t < 1$ and $b = \mathrm{E}X_1$ for $1 \leqslant t < 2$. Then

$$\sum_{n=1}^{\infty} n^{-1} l(n) P[\sup_{k \geqslant n} k^{-1/t}|S_k - kb| \geqslant \varepsilon] < \infty.$$

Hence we can deduce the following result (cf. [3]):

COROLLARY 8. *Let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables. If $\mathrm{E}(|X_1|^t \log^+ |X_1|) < \infty$ for $0 < t < 2$, then for any given $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} n^{-1} P[\sup_{k \geqslant n} k^{-1/t}|S_k - kb| \geqslant \varepsilon] < \infty,$$

where b was defined earlier.

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