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SMALL BALL PROBLEMS FOR NON-CENTERED GAUSSIAN MEASURES

BY

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Abstract. Let X be a centered Gaussian random variable with values in a Hilbert space H. If $a \in H$, then we determine the asymptotic behaviour of $P\{\|X-a\| < \varepsilon\}$ as $\varepsilon \to 0$. This extends former results of G. N. Sytaya and V. M. Zolotarev in the centered case, i.e., for $a = 0$. More general, we describe the behaviour of $P\{\|X-f(t)a\| < R(t)\}$ as $t \to \infty$ for some $\boldsymbol{R}^+$-valued functions f and R. Basic tools are the Laplace transform and a modified saddle point method.

1. Introduction. Let X be a centered Gaussian random variable with values in a separable real Hilbert space H. Then one may ask for the probability that X attains small values, i.e., one asks for the behaviour of $P\{\|X\| < \varepsilon\}$ as $\varepsilon \to 0$. This so-called "Small Ball Problem" has been investigated by different methods. For example, in [6] the random variable X was approximated by X_n with range in an n-dimensional subspace, where n was chosen in a delicate way and depended on $\varepsilon > 0$. A completely different approach was used in [12] and [14]. Here the inversion formula for the Laplace transform of $\|X\|^2$ has been applied, together with some modified saddle point method. Recently it turned out that this technique leads also to solutions in related questions. For example, in [8] and [9] the behaviour of $P\{\|X-ta\| < R(t)\}$ as $t \to \infty$ ($a \in H$, R is some function either decreasing or increasing not too fast) could be determined by similar ideas. Results of this type have been used in many different problems.

The aim of this paper is to prove a general theorem which includes the results of [12] and [8] as special cases. More precisely, we determine the behaviour of

$$P\{\|X-f(t)^{1/2}a\|^2 < R(t)\} \quad \text{as } t \to \infty \tag{1.1}$$

for $a \in H$ and some functions f and R. If $f(t) = t^2$ we are in the situation of [8] and for $f \equiv 0$ and $R(t) = t^{-2}$ we rediscover the "Small Ball Problem" mentioned above. But even in these cases our main result (Theorem 3.1) gives some new insight. Namely, in both cases the asymptotic behaviour was formerly described by exactly one function $\gamma = \gamma(t)$ defined by the random variable X (and by a and by $R(t)$, respectively) in a rather difficult way. Here we show that one may also use different functions provided they are not too far from the original γ. So changing γ a little bit, sometimes it simplifies the concrete calculations considerably.

On the other hand, (1.1) contains at least one case of interest not treated before in this way. Namely, if $f \equiv 1$ and $R(t) = t^{-2}$, then (1.1) describes the behaviour of

$$P\{\|X-a\| < \varepsilon\} \quad \text{as } \varepsilon \to 0, \tag{1.2}$$

where a is an arbitrary element of H. Observe that the behaviour of (1.2) is known when a is in the reproducing kernel Hilbert space (RKHS) of X. Indeed, a result of Borell (cf. [3]) asserts that

$$\lim_{\varepsilon \to 0} \frac{P\{\|X-a\| < \varepsilon\}}{P\{\|X\| < \varepsilon\}} = \exp(-\|a\|_X^2/2),$$

where $\|a\|_X$ means the norm of a in the RKHS of X. Hence in this case (1.2) reduces to the investigation of balls centered at zero. So our results give new information about the behaviour of Gaussian random variables near points not belonging to the RKHS.

The organization of the paper is as follows: Section 2 includes all technical lemmas needed for the proof of Theorem 3.1 in Section 3. Also some improvements of the main result in [8] are given in Section 3. Section 4 contains applications of Theorem 3.1 to problem (1.2), and Section 5 is devoted to some concrete examples.

Let us fix the notation: If f, g are functions on (t_0, ∞), we write as usual $f \sim g$ provided that

$$\lim_{t \to \infty} f(t)/g(t) = 1$$

and $f \asymp g$ means that

$$0 < \liminf_{t \to \infty} f(t)/g(t) \leqslant \limsup_{t \to \infty} f(t)/g(t) < \infty.$$

Finally, we recall some well-known facts about Hilbert space valued Gaussian random variables. A random variable X with values in H is *Gaussian centered* if $\langle X, y\rangle$, the inner product in H, is a real centered Gaussian random variable for each $y \in H$. And there are $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant 0$, $\sum_{j=1}^{\infty} \lambda_j < \infty$, and an orthonormal basis $\{e_j; j \geqslant 1\} \subseteq H$ such that

$$X \stackrel{d}{=} \sum_{j=1}^{\infty} \lambda_j^{1/2} \xi_j e_j,$$

where $\{\xi_j; j \geqslant 1\}$ is a sequence of independent standard Gaussian random variables. Furthermore, for $a = \sum_{j=1}^{\infty} \alpha_j e_j$ we have $a \in \mathscr{H}_X$ (RKHS of X) iff

$$\sum_{j=1}^{\infty} \frac{\alpha_j^2}{\lambda_j} < \infty \quad \text{and} \quad \|a\|_X = \left(\sum_{j=1}^{\infty} \frac{\alpha_j^2}{\lambda_j} \right)^{1/2}.$$

2. Basic estimates. If $(\lambda_j)_{j=1}^{\infty}$ and $(\alpha_j)_{j=1}^{\infty}$ are sequences of real numbers with

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant 0, \quad \sum_{j=1}^{\infty} \lambda_j < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \alpha_j^2 < \infty,$$

then the functions

$$\psi(z) := \sum_{j=1}^{\infty} \frac{\alpha_j^2 z}{1 + 2\lambda_j z} \quad \text{and} \quad \chi(z) := \frac{1}{2} \sum_{j=1}^{\infty} \log(1 + 2\lambda_j z) \tag{2.1}$$

are well-defined and analytic on $\boldsymbol{C}^+ := \{z \in \boldsymbol{C};\ \mathrm{Re}(z) > 0\}$ and we easily get

$$\psi'(z) = \sum_{j=1}^{\infty} \frac{\alpha_j^2}{(1 + 2\lambda_j z)^2}, \quad \psi''(z) = -\sum_{j=1}^{\infty} \frac{4\alpha_j^2 \lambda_j}{(1 + 2\lambda_j z)^3}, \tag{2.2}$$

$$\chi'(z) = \sum_{j=1}^{\infty} \frac{\lambda_j}{1 + 2\lambda_j z}, \quad \chi''(z) = -\sum_{j=1}^{\infty} \frac{2\lambda_j^2}{(1 + 2\lambda_j z)^2}. \tag{2.3}$$

Throughout this paper f and R are fixed $\boldsymbol{R}^+$-valued functions defined on (t_0, ∞) for some $t_0 > 0$. Now we introduce the complex-valued function $B = B(t, \gamma, \sigma)$ which will play an important role later on:

$$\begin{aligned} B(t, \gamma, \sigma) := {} & iR(t)\sigma - f(t)\{\psi(\gamma + i\sigma) - \psi(\gamma)\} \\ & - \{\chi(\gamma + i\sigma) - \chi(\gamma)\} - \log(1 + i\sigma/\gamma), \end{aligned} \tag{2.4}$$

where $t > t_0$, $\gamma > 0$ and $\sigma \in \boldsymbol{R}$. Furthermore, let $\beta = \beta(t, \gamma)$ and $\eta = \eta(t, \gamma)$ be defined by

$$\beta(t, \gamma) := -f(t)\gamma^2 \psi''(\gamma) - \gamma^2 \chi''(\gamma) = f(t) \sum_{j=1}^{\infty} \frac{4\alpha_j^2 \lambda_j \gamma^2}{(1 + 2\lambda_j \gamma)^3} + \sum_{j=1}^{\infty} \frac{2\lambda_j^2 \gamma^2}{(1 + 2\lambda_j \gamma)^2} \tag{2.5}$$

and

$$\eta(t, \gamma) = \frac{R(t) - \chi'(\gamma) - f(t)\psi'(\gamma)}{\sqrt{-f(t)\psi''(\gamma) - \chi''(\gamma)}} = \frac{\gamma\big(R(t) - \chi'(\gamma) - f(t)\psi'(\gamma)\big)}{\sqrt{\beta(t, \gamma)}}. \tag{2.6}$$

LEMMA 2.1. *If $\gamma \geqslant \gamma_0$ for some $\gamma_0 > 0$ and $t > t_0$, then*

$$\Big| \int_{|\sigma| \geqslant \gamma/\sqrt{2}} e^{B(t,\gamma,\sigma)}\, d\sigma \Big| \leqslant c\gamma e^{-\beta/8}$$

for some $c > 0$ depending only on γ_0 and on λ_1.

Proof. Note that

$$\operatorname{Re} B(t, \gamma, \sigma) = -f(t) \sum_{j=1}^{\infty} \frac{2\alpha_j^2 \lambda_j \sigma^2}{(1+2\lambda_j \gamma)^3 + 4\lambda_j^2 \sigma^2 (1+2\lambda_j \gamma)} - \frac{1}{4} \sum_{j=1}^{\infty} \log\left(1 + \frac{4\lambda_j^2 \sigma^2}{(1+2\lambda_j \gamma)^2}\right) - \frac{1}{2} \log\left(1 + \frac{\sigma^2}{\gamma^2}\right). \tag{2.7}$$

For $|\sigma| \geqslant \gamma/\sqrt{2}$ it follows that

$$\frac{2\alpha_j^2 \lambda_j \sigma^2}{(1+2\lambda_j \gamma)^3 + 4\lambda_j^2 \sigma^2 (1+2\lambda_j \gamma)} \geqslant \frac{\alpha_j^2 \lambda_j \gamma^2}{(1+2\lambda_j \gamma)^3 + 2\lambda_j^2 \gamma^2 (1+2\lambda_j \gamma)} \geqslant \frac{1}{2} \frac{\alpha_j^2 \lambda_j \gamma^2}{(1+2\lambda_j \gamma)^3};$$

hence the first term of (2.7) can be estimated by

$$\frac{f(t)\gamma^2 \psi''(\gamma)}{8}.$$

Furthermore, using $\log(1+x) \geqslant x/2$ for $0 < x < 1$ and $|\sigma| \geqslant \gamma/\sqrt{2}$ we obtain

$$\log\left(1 + \frac{4\lambda_j^2 \sigma^2}{(1+2\lambda_j \gamma)^2}\right) \geqslant \log\left(1 + \frac{2\lambda_j^2 \gamma^2}{(1+2\lambda_j \gamma)^2}\right) \geqslant \frac{\lambda_j^2 \gamma^2}{(1+2\lambda_j \gamma)^2}.$$

Thus

$$-\frac{1}{4} \sum_{j=1}^{\infty} \log\left(1 + \frac{4\lambda_j^2 \sigma^2}{(1+2\lambda_j \gamma)^2}\right) \leqslant -\frac{1}{4} \sum_{j=2}^{\infty} \frac{\lambda_j^2 \gamma^2}{(1+2\lambda_j \gamma)^2} - \frac{1}{4} \log\left(1 + \frac{4\lambda_1^2 \sigma^2}{(1+2\lambda_1 \gamma)^2}\right)$$
$$\leqslant \frac{1}{8}\gamma^2 \chi''(\gamma) + \frac{1}{4} \frac{\lambda_1^2 \gamma^2}{(1+2\lambda_1 \gamma)^2} - \frac{1}{4} \log\left(1 + \frac{4\lambda_1^2 \sigma^2}{(1+2\lambda_1 \gamma)^2}\right).$$

Summing up, it follows that

$$\int_{|\sigma| \geqslant \gamma/\sqrt{2}} \exp\left(\operatorname{Re} B(t, \gamma, \sigma)\right) d\sigma$$
$$\leqslant e^{-\beta/8} e^{1/16} \int_{|\sigma| \geqslant \gamma/\sqrt{2}} \left(1 + \frac{4\lambda_1^2 \sigma^2}{(1+2\lambda_1 \gamma)^2}\right)^{1/4} \left(1 + \frac{\sigma^2}{\gamma^2}\right)^{-1/2} d\sigma \leqslant c\gamma e^{-\beta/8},$$

as claimed above. ■

LEMMA 2.2. *For $t > t_0$, $\gamma > 0$ and $\sigma \in \boldsymbol{R}$ the following estimates hold:*

$$-\tfrac{1}{2}(\sigma/\gamma)^2 (\beta + 1) \leqslant \operatorname{Re} B(t, \gamma, \sigma) \leqslant -\tfrac{1}{2}(\sigma/\gamma)^2 \left(1 - (\sigma/\gamma)^2\right) \beta. \tag{2.8}$$

Especially, if $|\sigma| \leqslant \gamma/\sqrt{2}$, then

$$\operatorname{Re} B(t, \gamma, \sigma) \leqslant -\frac{\beta}{4}\left(\frac{\sigma}{\gamma}\right)^2. \tag{2.9}$$

Proof. Using the inequality $\log(1+x) \leqslant x$, $x > 0$, we obtain

$$\operatorname{Re} B(t, \gamma, \sigma) \geqslant -\tfrac{1}{2}(\sigma/\gamma)^2 [-f(t)\gamma^2 \psi''(\gamma) - \gamma^2 \chi''(\gamma) + 1] = -\tfrac{1}{2}(\sigma/\gamma)^2 (\beta+1).$$

On the other hand, by the relation $x - x^2/2 \leqslant \log(x+1)$ we have

$$-\frac{1}{4}\sum_{j=1}^{\infty} \log\left(1 + \frac{4\lambda_j^2\sigma^2}{(1+2\lambda_j\gamma)^2}\right) \leqslant -\frac{1}{4}\sum_{j=1}^{\infty} \frac{4\lambda_j^2\sigma^2}{(1+2\lambda_j\gamma)^2}\left(1 - \frac{2\lambda_j^2\sigma^2}{(1+2\lambda_j\gamma)^2}\right) \leqslant -\frac{1}{2}\left(\frac{\sigma}{\gamma}\right)^2 \left(1 - \left(\frac{\sigma}{\gamma}\right)^2\right)(-\gamma^2\chi''(\gamma)).$$

Moreover,

$$\begin{aligned} -\left(\frac{\sigma}{\gamma}\right)^2 \sum_{j=1}^{\infty} & \frac{2\alpha_j^2\lambda_j\gamma^2}{(1+2\lambda_j\gamma)^3 + 4\lambda_j^2\sigma^2(1+2\lambda_j\gamma)} \\ &= -\frac{1}{2}\left(\frac{\sigma}{\gamma}\right)^2 \Bigg\{-\gamma^2\psi''(\gamma) + \sum_{j=1}^{\infty} 4\alpha_j^2\lambda_j\gamma^2 \\ &\qquad \times \left[\frac{1}{(1+2\lambda_j\gamma)^3 + 4\lambda_j^2\sigma^2(1+2\lambda_j\gamma)} - \frac{1}{(1+2\lambda_j\gamma)^3}\right]\Bigg\} \\ &= -\frac{1}{2}\left(\frac{\sigma}{\gamma}\right)^2 \left\{-\gamma^2\psi''(\gamma) - \left(\frac{\sigma}{\gamma}\right)^2 \sum_{j=1}^{\infty} \frac{4\alpha_j^2\lambda_j\gamma^2}{(1+2\lambda_j\gamma)^3} \frac{4\lambda_j^2\gamma^2}{(1+2\lambda_j\gamma)^2 + 4\lambda_j^2\sigma^2}\right\} \\ &\leqslant -\frac{1}{2}\left(\frac{\sigma}{\gamma}\right)^2 \left(1 - \left(\frac{\sigma}{\gamma}\right)^2\right)(-\gamma^2\psi''(\gamma)), \end{aligned}$$

and hence

$$\operatorname{Re} B(t, \gamma, \sigma) \leqslant -\tfrac{1}{2}(\sigma/\gamma)^2 (1 - (\sigma/\gamma)^2)\beta,$$

as asserted. ∎

LEMMA 2.3. *Let* $\eta = \eta(t, \gamma)$ *be as in* (2.6). *Then*

$$\left|\operatorname{Im} B(t, \gamma, \sigma) - \frac{\sigma\sqrt{\beta}}{\gamma}\eta(t, \gamma)\right| \leqslant \left|\frac{\sigma}{\gamma}\right| + \frac{1}{2}\left|\frac{\sigma}{\gamma}\right|^3 (1+\beta) \tag{2.10}$$

for all $t > t_0$, $\gamma > 0$ *and* $\sigma \in \mathbf{R}$.

Proof. Note that

$$\operatorname{Im} B(t, \gamma, \sigma) = R(t)\sigma - \arctan\left(\frac{\sigma}{\gamma}\right) - \frac{1}{2}\sum_{j=1}^{\infty} \arctan\left(\frac{2\lambda_j\sigma}{1+2\lambda_j\gamma}\right) - f(t)\sigma \sum_{j=1}^{\infty} \frac{\alpha_j^2}{(1+2\lambda_j\gamma)^2 + 4\lambda_j^2\sigma^2} = \frac{\sigma\sqrt{\beta}}{\gamma}\eta(t, \gamma) + g(t, \gamma, \sigma),$$

where

$$g(t,\gamma,\sigma) = -\frac{\sigma}{\gamma}+\left[\frac{\sigma}{\gamma}-\arctan\left(\frac{\sigma}{\gamma}\right)\right]+\frac{1}{2}\sum_{j=1}^{\infty}\left[\frac{2\lambda_j\sigma}{1+2\lambda_j\gamma}-\arctan\left(\frac{2\lambda_j\sigma}{1+2\lambda_j\gamma}\right)\right]$$
$$+f(t)\left\{\sum_{j=1}^{\infty}\alpha_j^2\sigma\left[\frac{1}{(1+2\lambda_j\gamma)^2}-\frac{1}{(1+2\lambda_j\gamma)^2+4\lambda_j^2\sigma^2}\right]\right\}.$$

Since $|x-\arctan x| \leqslant |x|^3/3$ and $1/x^2-1/(x^2+y^2) \leqslant y^2/x^4$, we have the estimate

$$|g(t,\gamma,\sigma)| \leqslant \left|\frac{\sigma}{\gamma}\right|+\frac{1}{3}\left|\frac{\sigma}{\gamma}\right|^3+\frac{1}{6}\left|\frac{\sigma}{\gamma}\right|^3\sum_{j=1}^{\infty}\left(\frac{2\lambda_j\gamma}{1+2\lambda_j\gamma}\right)^3+\left|\frac{\sigma}{\gamma}\right|^3 f(t)\sum_{j=1}^{\infty}\frac{4\alpha_j^2\lambda_j^2\gamma^3}{(1+2\lambda_j\gamma)^4}$$
$$\leqslant \left|\frac{\sigma}{\gamma}\right|+\frac{1}{2}\left|\frac{\sigma}{\gamma}\right|^3\{1-\gamma^2\chi''(\gamma)-f(t)\gamma^2\psi''(\gamma)\} = \left|\frac{\sigma}{\gamma}\right|+\frac{1}{2}\left|\frac{\sigma}{\gamma}\right|^3(1+\beta),$$

and this completes the proof. ∎

Next we choose $\gamma=\gamma(t)$ dependent on $t>0$. If γ is clearly understood, we shall write for simplicity $B(t,\sigma)$, $\beta(t)$ and $\eta(t)$ instead of $B(t,\gamma(t),\sigma)$, $\beta(t,\gamma(t))$ and $\eta(t,\gamma(t))$, respectively. Let us suppose now that $\gamma(t)$ has the following properties:

(2.11) $$\liminf_{t\to\infty}\gamma(t)>0 \quad\text{and}\quad \lim_{t\to\infty}\beta(t)=\lim_{t\to\infty}\beta(t,\gamma(t))=\infty.$$

LEMMA 2.4. *If* $\gamma(t)$ *satisfies* (2.11), *then*

$$\lim_{t\to\infty}\left\{\frac{\sqrt{\beta(t)}}{\gamma(t)}\int_{-\infty}^{\infty}e^{B(t,\sigma)}\,d\sigma-\sqrt{2\pi}\exp\left(-\eta(t)^2/2\right)\right\}=0.$$

Proof. First note that

(2.12) $$(\sqrt{\beta}/\gamma)\int_{|\sigma|\geqslant\gamma/\sqrt{2}}e^{B(t,\sigma)}\,d\sigma\to 0 \quad\text{as } t\to\infty$$

by (2.11) and Lemma 2.1. Next we choose a function $\delta=\delta(t)$ such that

$$\lim_{t\to\infty}\delta(t)=\infty \quad\text{and}\quad \lim_{t\to\infty}\delta(t)^3/\sqrt{\beta(t)}=0.$$

Observe that this is possible in view of (2.11). Applying (2.9) we get

$$\left|(\sqrt{\beta}/\gamma)\int_{\gamma/\sqrt{2}\geqslant|\sigma|\geqslant\delta\gamma\beta^{-1/2}}e^{B(t,\sigma)}\,d\sigma\right|\leqslant\int_{|u|\geqslant\delta}\exp(-u^2/4)\,du;$$

thus by (2.12) and by the choice of δ we obtain

$$\lim_{t\to\infty}(\sqrt{\beta}/\gamma)\int_{|\sigma|\geqslant\delta\gamma\beta^{-1/2}}e^{B(t,\sigma)}\,d\sigma=0.$$

On the other hand, the equalities

$$\sqrt{2\pi}\exp\left(-\eta(t)^2/2\right)=\int_{-\infty}^{\infty}\exp\left(iu\eta(t)-u^2/2\right)du$$

and

$$\lim_{t\to\infty} \int_{|u|\geqslant \delta(t)} \exp\big(iu\eta(t)-u^2/2\big)\,du = 0$$

imply that it suffices to prove

$$\lim_{t\to\infty} \int_{|u|\leqslant\delta} \left[\exp\left(B\left(\gamma, \frac{u\gamma}{\sqrt{\beta}}\right)\right)-\exp\big(iu\eta(t)-u^2/2\big)\right]du = 0. \tag{2.13}$$

From (2.10) we derive

$$\left|\operatorname{Im} B\left(t, \frac{u\gamma}{\sqrt{\beta}}\right)-u\eta(t)\right| \leqslant \frac{|u|}{\sqrt{\beta}}+\frac{1}{2}\frac{|u|^3(1+\beta)}{\beta^{3/2}} \leqslant c\frac{\delta^3}{\sqrt{\beta}} \tag{2.14}$$

for t large enough and $|u|\leqslant\delta$. Moreover, from (2.8) we also have

$$-\frac{\delta^2}{2\beta} \leqslant \operatorname{Re} B\left(t, \frac{u\gamma}{\sqrt{\beta}}\right)+\frac{u^2}{2} \leqslant \frac{\delta^4}{2\beta} \tag{2.15}$$

for $|u|\leqslant\delta$. Thus

$$\begin{aligned}
&\left|\exp\left(\operatorname{Re} B\left(t, \frac{u\gamma}{\sqrt{\beta}}\right)+i\operatorname{Im} B\left(t, \frac{u\gamma}{\sqrt{\beta}}\right)\right)-\exp\big(iu\eta(t)-u^2/2\big)\right| \\
&\leqslant \exp(-u^2/2)\left|\exp\left(i\operatorname{Im} B\left(t, \frac{u\gamma}{\sqrt{\beta}}\right)\right)-\exp\big(iu\eta(t)\big)\right| \\
&\quad +\left|\exp(-u^2/2)-\exp\left(\operatorname{Re} B\left(t, \frac{u\gamma}{\sqrt{\beta}}\right)\right)\right| \\
&\leqslant \exp(-u^2/2)\left\{\left|\operatorname{Im} B\left(t, \frac{u\gamma}{\sqrt{\beta}}\right)-u\eta(t)\right|+\left|1-\exp\left(\operatorname{Re} B\left(t, \frac{u\gamma}{\sqrt{\beta}}\right)+u^2/2\right)\right|\right\}
\end{aligned}$$

combined with (2.14) and (2.15) proves (2.13) since $\delta^3/\beta^{1/2}\to 0$. ■

3. Main result. Let X be a centered Gaussian random variable with values in a separable Hilbert space H. As mentioned above

$$X \overset{\mathrm{d}}{=} \sum_{j=1}^{\infty} \lambda_j^{1/2}\xi_j e_j$$

with $\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant 0$, $\sum_{j=1}^{\infty}\lambda_j<\infty$, and $(e_j)_{j=1}^{\infty}$ is an orthonormal basis in H. If $a=\sum_{j=1}^{\infty}\alpha_j e_j$, then we have

$$P\{\|X-f(t)^{1/2}a\|^2<R(t)\} = P\Big\{\sum_{j=1}^{\infty}|\lambda_j^{1/2}\xi_j-f(t)^{1/2}\alpha_j|^2<R(t)\Big\}. \tag{3.1}$$

Recall that $\xi_1, \xi_2, \ldots$ are independent $\mathcal{N}(0, 1)$-distributed. Our objective is to determine the behaviour of (3.1) as $t \to \infty$. To do so let us introduce the following definition:

A function $\gamma = \gamma(t)$ is *admissible* (for X, a, f and R) provided that

$$\lim_{t\to\infty} \beta(t, \gamma(t)) = \lim_{t\to\infty} -\gamma^2 (f(t)\psi''(\gamma)+\chi''(\gamma)) = \infty \tag{3.2}$$

and

$$\lim_{t\to\infty} \eta(t, \gamma(t)) = \lim_{t\to\infty} \frac{R(t)-f(t)\psi'(\gamma)-\chi'(\gamma)}{\sqrt{-f(t)\psi''(\gamma)-\chi''(\gamma)}} = 0. \tag{3.3}$$

THEOREM 3.1. *Let $\gamma = \gamma(t)$ be admissible with $\liminf_{t\to\infty} \gamma(t) > 0$. Then*

$$P\{\|X-f(t)^{1/2}a\|^2 < R(t)\} \sim \frac{1}{\sqrt{2\pi}} \frac{\exp(\gamma R(t)-f(t)\psi(\gamma)-\chi(\gamma))}{\sqrt{\beta(t)}} \tag{3.4}$$

as $t \to \infty$. Conversely, if for some $\gamma = \gamma(t)$ with $\liminf_{t\to\infty} \gamma(t) > 0$ we have (3.2) *as well as* (3.4)*, then necessarily γ satisfies* (3.3)*, i.e., γ is admissible.*

Proof. Given $z \in C^+$, the Laplace transform of $\|X-f(t)^{1/2}a\|^2$ at z is equal to

$$\mathbb{E}[\exp(-z\|X-f(t)^{1/2}a\|^2)] = \exp(-f(t)\psi(z)-\chi(z)).$$

By the inversion formula (cf. [4], Chapter II.1, Theorem 1),

$$P\{\|X-f(t)^{1/2}a\|^2 < R(t)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\Phi(t,z)}\,dz,$$

where

$$\Phi(t, z) = R(t)z-\log(z)-f(t)\psi(z)-\chi(z).$$

Defining $A(t, \gamma)$ by

$$A(t, \gamma) := R(t)\gamma-\log\gamma-\chi(\gamma)-f(t)\psi(\gamma) \tag{3.5}$$

we obtain

$$\Phi(t, \gamma+i\sigma) = A(t, \gamma)+B(t, \gamma, \sigma)$$

with $B(t, \gamma, \sigma)$ defined as in (2.4). Consequently,

$$P\{\|X-f(t)^{1/2}a\|^2 < R(t)\} = e^{A(t,\gamma)}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{B(t,\gamma,\sigma)}\,d\sigma \tag{3.6}$$

and this holds for any $t > t_0$ and any $\gamma > 0$. Choosing $\gamma = \gamma(t)$ satisfying (3.2) and (3.3), from Lemma 2.4 and (3.3) we have

$$\lim_{t\to\infty} \frac{1}{2\pi}\frac{\sqrt{\beta}}{\gamma}\int_{-\infty}^{\infty} e^{B(t,\sigma)}\,d\sigma = \frac{1}{\sqrt{2\pi}},$$

i.e.,

$$P\{\|X-f(t)^{1/2}a\| < R(t)\} \sim \frac{1}{\sqrt{2\pi}}\frac{\gamma}{\sqrt{\beta}}e^{A(t,\gamma(t))}$$
$$= \frac{1}{\sqrt{2\pi}}\frac{\exp\big(R(t)\gamma-\chi(\gamma)-f(t)\psi(\gamma)\big)}{\sqrt{\beta}}$$

as $t \to \infty$ and this proves (3.4).

Next suppose that $\gamma = \gamma(t)$ satisfies (3.2) as well as (3.4). Then (3.4)–(3.6) yield

$$\lim_{t\to\infty}\frac{\sqrt{\beta}}{\gamma}\int_{-\infty}^{\infty} e^{B(t,\gamma(t),\sigma)}\,d\sigma = \sqrt{2\pi}$$

for this $\gamma = \gamma(t)$ and, consequently, by Lemma 2.4 we conclude that

$$\lim_{t\to\infty}\exp\big(-\eta(t,\gamma(t))^2/2\big) = 1,$$

which clearly implies (3.3) and completes the proof. ∎

Remark 3.1. Suppose we have

$$R(t) \leqslant f(t)\sum_{j=1}^{\infty}\alpha_j^2+\sum_{j=1}^{\infty}\lambda_j-\varepsilon \tag{3.7}$$

for some $\varepsilon > 0$ and for t large enough. Then we may define the function $\gamma = \gamma(t)$ as unique solution of the equation

$$R(t) = f(t)\psi'(\gamma)+\chi'(\gamma) \tag{3.8}$$

for large t. Of course, this γ satisfies (3.3) by definition and, moreover,

$$\liminf_{t\to\infty}\gamma(t) > 0.$$

Hence the following is true:

COROLLARY 3.2. *Suppose that* (3.7) *holds and define* γ *by* (3.8). *If for this* γ

$$\lim_{t\to\infty} -\gamma^2\big(f(t)\psi''(\gamma)+\chi''(\gamma)\big) = \infty, \tag{3.9}$$

then

$$P\{\|X-f(t)^{1/2}a\|^2 < R(t)\} \sim \frac{1}{\sqrt{2\pi}}\frac{\exp\big(\gamma R(t)-f(t)\psi(\gamma)-\chi(\gamma)\big)}{\sqrt{\beta(t)}}$$

as $t \to \infty$.

Remark 3.2. Let us mention two cases where (3.7) holds and γ defined by (3.8) satisfies condition (3.9) as well:

$$\lim_{t\to\infty} R(t) = 0 \quad \text{and} \quad \operatorname{card}\{j;\ \lambda_j > 0\} = \infty. \tag{3.10}$$

Indeed, in this case we have $\lim_{t\to\infty}\gamma(t)=\infty$ and $\lim_{u\to\infty}-u^2\chi''(u)=\infty$.

$$\text{(3.11)}\qquad \lim_{t\to\infty} f(t)=\infty \quad\text{and}\quad \limsup_{t\to\infty} R(t)/f(t) < \sum_{j=1}^{\infty}\alpha_j^2 = \|a\|^2.$$

Here we have $\liminf_{t\to\infty}\gamma(t)>0$ and $\lim_{t\to\infty}-\gamma(t)^2 f(t)\,\psi''\big(\gamma(t)\big)=\infty$.

COROLLARY 3.3. *Suppose that either* (3.10) *or* (3.11) *holds. Then*

$$P\{\|X-f(t)^{1/2}a\|^2<R(t)\}\sim\frac{1}{\sqrt{2\pi}}\frac{\exp\big(\gamma R(t)-f(t)\psi(\gamma)-\chi(\gamma)\big)}{\sqrt{\beta(t)}}$$

as $t\to\infty$, *where* γ *is defined by* (3.8).

We want to state now three special cases of Theorem 3.1 explicitly. Let us begin with the "Small Ball Problem" for non-centered balls.

COROLLARY 3.4. *If* $a\in H$ *and* $\operatorname{card}\{j;\,\lambda_j>0\}=\infty$, *then*

$$\text{(3.12)}\qquad P\{\|X-a\|<\varepsilon\}\sim\frac{1}{\sqrt{2\pi}}\frac{\exp\big(\gamma\varepsilon^2-\psi(\gamma)-\chi(\gamma)\big)}{\sqrt{-\gamma^2\psi''(\gamma)-\gamma^2\chi''(\gamma)}}$$

iff

$$\text{(3.13)}\qquad \lim_{\varepsilon\to 0}\frac{\varepsilon^2-\psi'(\gamma)-\chi'(\gamma)}{\sqrt{-\psi''(\gamma)-\chi''(\gamma)}}=0.$$

Especially, (3.12) *holds for* $\gamma=\gamma(\varepsilon)$ *defined by* $\varepsilon^2=\psi'(\gamma)+\chi'(\gamma)$.

Remark 3.3. For $a=0$, i.e., $\psi\equiv 0$, the second part of Corollary 3.4 was proved in [12].

Next we apply Theorem 3.1 to $f(t)=t^{-2}$ and $R(t)=R^2/t^2$ for some $R>0$. Observe that (3.10) applies in this case.

COROLLARY 3.5. *As* $t\to\infty$

$$P\{\|tX-a\|<R\}\sim\frac{1}{\sqrt{2\pi}}\frac{\exp\big(R^2\gamma/t^2-\psi(\gamma)/t^2-\chi(\gamma)\big)}{\sqrt{-\gamma^2\psi''(\gamma)/t^2-\gamma^2\chi''(\gamma)}},$$

where $R^2=\psi'(\gamma)+t^2\chi'(\gamma)$.

Finally, we want to improve Theorem 1 in [8]. There is an additional $(1/\gamma)$-term in the definition of γ and it turns out that it is in fact not necessary. Note that (3.11) applies here.

COROLLARY 3.6. *If* $\limsup_{t\to\infty}R(t)/t<\|a\|$, *then*

$$\text{(3.14)}\qquad P\{\|X-ta\|<R(t)\}\sim\frac{1}{\sqrt{2\pi}}\frac{\exp\big(R^2(t)\gamma-t^2\psi(\gamma)-\chi(\gamma)\big)}{\sqrt{-t^2\gamma^2\psi''(\gamma)-\gamma^2\chi''(\gamma)}}$$

iff

$$\lim_{t\to\infty} \frac{R^2(t)-t^2\psi'(\gamma)-\chi'(\gamma)}{\sqrt{-t^2\psi''(\gamma)-\chi''(\gamma)}} = 0.$$

Especially, (3.14) *holds for* $\gamma = \gamma(t)$ *defined by* $R^2(t) = t^2\psi'(\gamma)+\chi'(\gamma)$.

Now let us treat the case $R(t) = R$ for some $R > 0$. Since $t \to \infty$ and $\chi'(\gamma) \to 0$, one might expect that γ defined by $R^2 = t^2\psi'(\gamma)$ would work as well. Our next proposition shows that this is only so for λ_j's going to zero not too slowly or, equivalently, if χ' tends to zero fast enough.

PROPOSITION 3.7. *Let* $\gamma = \gamma(t)$ *be defined by*

$$R^2 = t^2\psi'(\gamma). \tag{3.15}$$

Then γ *is admissible for* (3.14) (*with* $R(t) \equiv R$) *iff*

$$\lim_{u\to\infty} \sqrt{-\frac{\psi'(u)}{\psi''(u)}}\,\chi'(u) = 0. \tag{3.16}$$

Proof. Observe that γ defined by (3.15) is admissible iff

$$\lim_{t\to\infty} \frac{\chi'(\gamma)}{\sqrt{-t^2\psi''(\gamma)-\chi''(\gamma)}} = 0. \tag{3.17}$$

Using

$$-\chi''(\gamma) = 2\sum_{j=1}^{\infty} \frac{\lambda_j^2}{(1+2\lambda_j\gamma)^2} \leqslant 2\left(\sum_{j=1}^{\infty} \frac{\lambda_j}{1+2\lambda_j\gamma}\right)^2 = 2\left(\chi'(\gamma)\right)^2, \tag{3.18}$$

we see that condition (3.17) holds iff

$$\lim_{t\to\infty} \frac{\chi'(\gamma)}{\sqrt{-t^2\psi''(\gamma)}} = 0.$$

But by (3.15) this is equivalent to

$$\lim_{u\to\infty} \sqrt{-\frac{\psi'(u)}{\psi''(u)}}\,\chi'(u) = 0,$$

as asserted. ∎

Remark 3.4. Since $-u\psi''(u) \leqslant 2\psi'(u)$ we always have $-\psi'(u)/\psi''(u) \geqslant u/2$. Hence, in order that (3.16) holds we have to have

$$\lim_{u\to\infty} \sqrt{u}\,\chi'(u) = 0. \tag{3.19}$$

It is not difficult to see that $\sum_{j=1}^{\infty} \lambda_j^{1/2} < \infty$ implies (3.19). Conversely, if (3.19) is valid, then $\lim_{j\to\infty} j^2\lambda_j = 0$. Recall that we assumed that the sequence of the λ_j's is decreasing. Thus, if the λ_j's go to zero slower than j^{-2}, then (3.16) never holds and γ defined by (3.15) is never admissible (for any $a \in H$).

4. Applications to the "Small Ball Problem". Recall that for $a \in H$ we have

$$P\{\|X-a\| < \varepsilon\} \sim \frac{1}{\sqrt{2\pi}} \frac{\exp(\varepsilon^2 \gamma - \psi(\gamma) - \chi(\gamma))}{\sqrt{-\gamma^2 \psi''(\gamma) - \gamma^2 \chi''(\gamma)}} \tag{4.1}$$

with $\gamma = \gamma(\varepsilon)$ defined by

$$\varepsilon^2 = \psi'(\gamma) + \chi'(\gamma) = \sum_{j=1}^{\infty} \left[\frac{\alpha_j^2}{(1+2\lambda_j \gamma)^2} + \frac{\lambda_j}{1+2\lambda_j \gamma}\right]. \tag{4.2}$$

Even if the behaviour of the right-hand side in (4.1) can be calculated explicitly, this does not say very much about the dependence on ε because γ is defined by (4.2) in a very implicit way. Thus in general it is rather difficult to describe the behaviour of (4.1) as a function of ε even in the easiest examples. We shall do so in Section 5 only for one very special case. Much easier is to determine the behaviour of the logarithm of (4.1). Here the following is valid:

PROPOSITION 4.1. *We have*

$$\log P\{\|X-a\| < \varepsilon\} \sim -\int_{\varepsilon^2}^{K} \varrho^{-1}(u)\, du, \tag{4.3}$$

where

$$\varrho(u) := \psi'(u) + \chi'(u) = \sum_{j=1}^{\infty} \left[\frac{\alpha_j^2}{(1+2\lambda_j u)^2} + \frac{\lambda_j}{1+2\lambda_j u}\right] \tag{4.4}$$

and $K > 0$ is some positive constant.

Proof. First note that

$$-\tfrac{1}{2}\log(-\gamma^2 \psi''(\gamma) - \gamma^2 \chi''(\gamma)) = o([\gamma\chi'(\gamma) - \chi(\gamma)] + [\gamma\psi'(\gamma) - \psi(\gamma)]).$$

This follows from

$$\gamma\psi'''(\gamma) \leqslant -3\psi''(\gamma) \quad \text{and} \quad \gamma\chi'''(\gamma) \leqslant -2\chi''(\gamma)$$

by using de L'Hôspital's Rule. Hence the left-hand side of (4.3) behaves like

$$[\gamma\chi'(\gamma) - \chi(\gamma)] + [\gamma\psi'(\gamma) - \psi(\gamma)] = \int_0^{\gamma} u\varrho'(u)\, du = -\int_{\varepsilon^2}^{\varrho(0)} \varrho^{-1}(u)\, du.$$

This proves (4.3) as asserted. ∎

COROLLARY 4.2. *Assume* $\lim_{u\to\infty} u^q \varrho(u) = c$ *for some* $q \in (0, 1)$ *and some* $c > 0$. *Then*

$$\log P\{\|X-a\| < \varepsilon\} \sim -\frac{q}{1-q} c^{1/q} \varepsilon^{(2q-2)/q}. \tag{4.5}$$

Proof. Since $\varrho^{-1}(v) \sim c^{1/q} v^{-1/q}$ as $v \to 0$, (4.5) follows from (4.3) by applying de L'Hôspital's Rule. ∎

Our next objective is to ask for admissible γ's which are defined in a less complicated way as in (4.2). There are two natural choices for γ, namely, to define γ by

$$\varepsilon^2 = \chi'(\gamma) \tag{4.6}$$

or by

$$\varepsilon^2 = \psi'(\gamma). \tag{4.7}$$

THEOREM 4.3. (a) *If γ is defined by* (4.6), *then it is admissible iff*

$$\lim_{u\to\infty} \frac{\psi'(u)^2}{\chi''(u)} = 0. \tag{4.8}$$

(b) *Let γ be defined by* (4.7). *Then it is admissible iff*

$$\lim_{u\to\infty} \frac{\chi'(u)^2}{\psi''(u)} = 0. \tag{4.9}$$

Proof. (a) Observe that (3.13) holds in this case iff

$$\lim_{u\to\infty} \frac{\psi'(u)}{\sqrt{-\psi''(u)-\chi''(u)}} = 0. \tag{4.10}$$

Thus, (4.8) clearly implies (4.10). Now assume that (4.10) holds and (4.8) does not. Then we find $u_n \to \infty$ with

$$\frac{\psi'(u_n)^2}{-\chi''(u_n)} \geqslant c > 0 \tag{4.11}$$

for some $c > 0$. Since $-u_n^2 \chi''(u_n) \to \infty$ as $n \to \infty$, in view of (4.11) we get $u_n \psi'(u_n) \to \infty$ as well. Now (4.10) and (4.11) allow us to conclude that

$$\lim_{n\to\infty} -\frac{\psi''(u_n)}{\psi'(u_n)^2} = \infty. \tag{4.12}$$

But $-\psi''(u_n) \leqslant 2\psi'(u_n)/u_n$, so by (4.12) we have

$$\lim_{n\to\infty} \frac{1}{u_n \psi'(u_n)} = \infty,$$

which contradicts $u_n \psi'(u_n) \to \infty$. This completes the proof of part (a).

Part (b) follows easily from (3.18). ∎

THEOREM 4.4. (a) *Suppose* (4.8) *holds. Then*

$$P\{\|X-a\| < \varepsilon\} \sim \frac{1}{\sqrt{2\pi}} \frac{\exp(\gamma\chi'(\gamma)-\chi(\gamma)-\psi(\gamma))}{\sqrt{-\gamma^2\chi''(\gamma)}}, \quad \textit{where } \varepsilon^2 = \chi'(\gamma).$$

(b) *If* (4.9) *holds, then*

$$P\{\|X-a\| < \varepsilon\} \sim \frac{1}{\sqrt{2\pi}} \frac{\exp(\gamma\psi'(\gamma)-\psi(\gamma)-\chi(\gamma))}{\sqrt{-\gamma^2\psi''(\gamma)}}, \quad \textit{where } \varepsilon^2 = \psi'(\gamma).$$

Proof. (a) In view of Theorem 3.1 and Theorem 4.3 it remains to prove

$$\lim_{u\to\infty}\frac{\psi''(u)}{\chi''(u)}=0 \tag{4.13}$$

provided that (4.8) holds. Using $-\psi''(u)\leqslant 2\psi'(u)/u$ it follows that

$$\frac{\psi'(u)^2}{-\chi''(u)}\geqslant\frac{1}{4}\left(\frac{\psi''(u)}{\chi''(u)}\right)^2 u^2(-\chi''(u)).$$

But $u^2(-\chi''(u))\to\infty$, and hence (4.8) implies (4.13).

(b) By (3.18) we have

$$\lim_{u\to\infty}\frac{\chi''(u)}{\psi''(u)}=0$$

provided that (4.9) holds. This completes the proof. ∎

Remark 4.1. The e-term in the case (a) equals

$$-\frac{1}{2}\sum_{j=1}^{\infty}\log(1+2\lambda_j\gamma)+\gamma\sum_{j=1}^{\infty}\frac{\lambda_j-\alpha_j^2}{1+2\lambda_j\gamma}$$

and in the case (b) it may be written as follows:

$$-\frac{1}{2}\sum_{j=1}^{\infty}\log(1+2\lambda_j\gamma)-\sum_{j=1}^{\infty}\frac{\alpha_j^2\lambda_j\gamma^2}{(1+2\lambda_j\gamma)^2}.$$

COROLLARY 4.5. *Suppose that* (4.8) *holds. Then*

$$\frac{P\{\|X-a\|<\varepsilon\}}{P\{\|X\|<\varepsilon\}}\sim e^{-\psi(\gamma)},\qquad \textit{where } \varepsilon^2=\chi'(\gamma).$$

Remark 4.2. This easily generalizes to $a=\sum_{j=1}^{\infty}\alpha_j e_j$ and $b=\sum_{j=1}^{\infty}\beta_j e_j$ as follows:

$$\frac{P\{\|X-a\|<\varepsilon\}}{P\{\|X-b\|<\varepsilon\}}\sim\exp\left(\gamma\sum_{j=1}^{\infty}\frac{\beta_j^2-\alpha_j^2}{1+2\lambda_j\gamma}\right)$$

provided that the ψ's defined by a as well as by b satisfy (4.8).

As a special case of Corollary 4.5 we obtain the following result of Borell (cf. [3]):

COROLLARY 4.6. *If* $a\in\mathscr{H}_X$, *i.e.*, $\sum_{j=1}^{\infty}\alpha_j^2/\lambda_j<\infty$, *then*

$$\lim_{\varepsilon\to 0}\frac{P\{\|X-a\|<\varepsilon\}}{P\{\|X\|<\varepsilon\}}=\exp(-\|a\|_X^2/2).$$

Proof. Observe that $u\psi'(u)\to 0$ as $u\to\infty$ in this case. On the other hand, $-u^2\chi''(u)\to\infty$, thus (4.8) holds. Furthermore,

$$\lim_{\gamma\to\infty} \psi(\gamma) = \lim_{\gamma\to\infty} \sum_{j=1}^{\infty} \frac{\alpha_j^2 \gamma}{1+2\lambda_j \gamma} = \frac{1}{2} \|a\|_X^2,$$

which proves the corollary by Theorem 4.4 (a). ∎

To discuss the preceding results let us introduce the following three disjoint subsets of H. If $a \in H$ and ψ_a is defined by

$$\psi_a(z) := \sum_{j=1}^{\infty} \frac{\alpha_j^2 z}{1+2\lambda_j z}, \qquad a = \sum_{j=1}^{\infty} \alpha_j e_j,$$

then

$$\mathscr{A}_\chi := \left\{ a \in H;\ \lim_{u\to\infty} \frac{\psi_a'(u)^2}{\chi''(u)} = 0 \right\},$$

$$\mathscr{A}_\psi := \left\{ a \in H;\ \lim_{u\to\infty} \frac{\chi'(u)^2}{\psi_a''(u)} = 0 \right\} \quad \text{and} \quad \mathscr{A}_\varrho := H \backslash (\mathscr{A}_\chi \cup \mathscr{A}_\psi).$$

Recall that $a \in \mathscr{A}_\chi$ iff γ defined by (4.6) is admissible, $a \in \mathscr{A}_\psi$ iff γ defined by (4.7) is admissible, and, finally, $a \in \mathscr{A}_\varrho$ iff the γ's defined by (4.6) and by (4.7) are both not admissible. The set $\mathscr{A}_\chi$ contains those $a \in H$ for which the α_j's tend to zero fast. If $b = \sum_{j=1}^{\infty} \beta_j e_j$ and $|\beta_j| \leqslant |\alpha_j|$, $j = 1, 2, \ldots$, then $a \in \mathscr{A}_\chi$ implies $b \in \mathscr{A}_\chi$ as well. We have $\mathscr{H}_X \subseteq \mathscr{A}_\chi$, yet in general $\mathscr{H}_X \neq \mathscr{A}_\chi$. The set $\mathscr{A}_\psi$ contains those $a \in H$ for which the α_j's go to zero slowly. If $|\beta_j| \geqslant |\alpha_j|$, $j = 1, 2, \ldots$, and $a \in \mathscr{A}_\psi$, then $\sum_{j=1}^{\infty} \beta_j e_j$ belongs to $\mathscr{A}_\psi$ as well. Observe that $\mathscr{A}_\psi$ may be empty. Indeed, since $-u\psi''(u) \to 0$ as $u \to \infty$, we see that $\sqrt{u}\,\chi'(u) \to 0$ is necessary for $\mathscr{A}_\psi \neq \emptyset$. Thus, for λ_j's with $\liminf_{j\to\infty} j^2 \lambda_j > 0$ it follows that $\mathscr{A}_\psi = \emptyset$.

5. Examples. We investigate now the case

$$\alpha_j = j^{-\alpha/2} \quad \text{and} \quad \lambda_j = j^{-\beta}$$

with $\alpha, \beta > 1$. If ϱ is defined by (4.4), then

$$\varrho(u) = \sum_{j=1}^{\infty} \left[\frac{j^{2\beta-\alpha}}{(j^\beta + 2u)^2} + \frac{1}{j^\beta + 2u} \right]$$

in this case and

$$\varrho(u) \sim \begin{cases} \left(\frac{1}{4}\sum_{j=1}^{\infty} j^{2\beta-\alpha}\right) u^{-2} + K_\beta u^{1/\beta-1}, & \alpha > 2\beta+1, \\ (1/4\beta) u^{-2} \log(u) + K_\beta u^{1/\beta-1}, & \alpha = 2\beta+1, \\ c_{\alpha\beta} u^{1/\beta-\alpha/\beta} + K_\beta u^{1/\beta-1}, & \alpha < 2\beta+1, \end{cases} \tag{5.1}$$

where

$$K_\beta = 2^{1/\beta-1} \frac{\pi/\beta}{\sin(\pi/\beta)}, \tag{5.2}$$

$$c_{\alpha\beta} = 2^{1/\beta-\alpha/\beta} \beta^{-1} \Gamma((2\beta-\alpha+1)/\beta) \Gamma((\alpha-1)/\beta) \tag{5.3}$$

(cf. [8]). Observe that

$$c_{\alpha\beta} = \begin{cases} 2^{1/\beta-\alpha/\beta}\dfrac{\pi(\beta+1-\alpha)/\beta^2}{\sin(\pi(\beta+1-\alpha)/\beta)}, & 1<\alpha<\beta+1, \\ (2\beta)^{-1}, & \alpha=\beta+1, \\ 2^{1/\beta-\alpha/\beta}\dfrac{\pi/(\alpha-1-\beta)}{\sin(\pi(\alpha-\beta-1)/\beta)}, & \beta+1<\alpha<2\beta+1. \end{cases}$$

Especially, we have

$$c_{\alpha\beta} = (\beta+1-\alpha)K_{\alpha\beta}/\beta, \quad 1<\alpha<\beta+1,$$

where $K_{\alpha\beta}$ is defined by (5.5) below and, moreover,

$$c_{\beta\beta} = K_{\beta\beta}/\beta = K_\beta/\beta.$$

From (5.1) it follows that

$$\varrho(u) \sim \begin{cases} K_\beta u^{1/\beta-1}, & \alpha>\beta, \\ (1+1/\beta)K_\beta u^{1/\beta-1}, & \alpha=\beta, \\ c_{\alpha\beta}u^{1/\beta-\alpha/\beta}, & \alpha<\beta, \end{cases}$$

as $u\to\infty$. Then (4.5) implies the following:

PROPOSITION 5.1. *Let* $\lambda_j = j^{-\beta}$ *and* $\alpha_j = j^{-\alpha/2}$. *Then*

$$(5.4)\quad \log P\{\|X-a\|<\varepsilon\} \sim \begin{cases} -\dfrac{\beta-1}{2}\left(\dfrac{\pi}{\beta\sin(\pi/\beta)}\right)^{\beta/(\beta-1)}\varepsilon^{-2/(\beta-1)}, & \alpha>\beta, \\ -\dfrac{\beta-1}{2}\left(\dfrac{\pi(\beta+1)}{\beta^2\sin(\pi/\beta)}\right)^{\beta/(\beta-1)}\varepsilon^{-2/(\beta-1)}, & \alpha=\beta, \\ -\dfrac{\alpha-1}{2}\left(\dfrac{\pi}{\beta^2\sin(\pi(\beta+1-\alpha)/\beta)}\right)^{\beta/(\alpha-1)} & \\ \qquad\times\left((\beta+1-\alpha)\varepsilon^{-2}\right)^{(\beta+1-\alpha)/(\alpha-1)}, & \alpha<\beta, \end{cases}$$

as $\varepsilon\to 0$.

COROLLARY 5.2. *Let* α_j *and* λ_j *be as above. Then*

$$\frac{\log P\{\|X-a\|<\varepsilon\}}{\log P\{\|X\|<\varepsilon\}} \sim \begin{cases} 1, & \alpha>\beta, \\ ((\beta+1)/\beta)^{\beta/(\beta-1)}, & \alpha=\beta, \\ c\varepsilon^{2\beta(\alpha-\beta)/(\beta-1)(\alpha-1)}, & \alpha<\beta, \end{cases}$$

where $c>0$ *is the quotient of the coefficients in* (5.4) *for* $\alpha>\beta$ *and* $\alpha<\beta$, *respectively.*

Our next aim is to characterize those $\alpha > 1$ for which $a = \sum_{j=1}^{\infty} j^{-\alpha/2} e_j$ belongs to $\mathscr{A}_\chi$, $\mathscr{A}_\varrho$ or $\mathscr{A}_\psi$, respectively. To do so, observe that

$$-\chi''(u) \asymp u^{1/\beta - 2}$$

and

$$-\psi''(u) \asymp \begin{cases} u^{-3}, & \alpha > 2\beta+1, \\ u^{-3}\log(u), & \alpha = 2\beta+1, \\ u^{1/\beta - \alpha/\beta - 1}, & \alpha < 2\beta+1. \end{cases}$$

Combining this with the asymptotic of ψ' and χ' we obtain

$$\frac{\psi'(u)^2}{-\chi''(u)} \asymp \begin{cases} u^{-2-1/\beta}, & \alpha > 2\beta+1, \\ u^{-2-1/\beta}\log(u), & \alpha = 2\beta+1, \\ u^{1/\beta + \alpha/\beta - 1}, & \alpha < 2\beta+1, \end{cases}$$

and

$$\frac{\chi'(u)^2}{-\psi''(u)} \asymp \begin{cases} u^{2/\beta+1}, & \alpha > 2\beta+1, \\ u^{2/\beta+1}\left(\log(u)\right)^{-1}, & \alpha = 2\beta+1, \\ u^{1/\beta+\alpha/\beta-1}, & \alpha < 2\beta+1. \end{cases}$$

This implies

PROPOSITION 5.3. *If* $a = \sum_{j=1}^{\infty} j^{-\alpha/2} e_j$ *and* $\lambda_j = j^{-\beta}$, *then*
(i) $a \in \mathscr{H}_X$ *iff* $\alpha > \beta+1$,
(ii) $a \in \mathscr{A}_\chi$ *iff* $\alpha > \beta+1/2$,
(iii) $a \in \mathscr{A}_\varrho$ *iff* $\beta - 1 \leqslant \alpha \leqslant \beta + 1/2$,
(iv) $a \in \mathscr{A}_\psi$ *iff* $\alpha < \beta - 1$.

Let us discuss these results:

(i) If $a \in \mathscr{A}_\varrho$, i.e., $\beta - 1 \leqslant \alpha \leqslant \beta + 1/2$, then neither $\varepsilon^2 = \chi'(\gamma)$ nor $\varepsilon^2 = \psi'(\gamma)$ can be used to describe the behaviour of $P\{\|X-a\| < \varepsilon\}$. But observe that

$$\lim_{u\to\infty} \frac{\psi''(u)}{\chi''(u)} = 0 \text{ for } \alpha > \beta \quad \text{and} \quad \lim_{u\to\infty} \frac{\chi''(u)}{\psi''(u)} = 0 \text{ for } \alpha < \beta.$$

Hence, for example, if $\beta < \alpha < \beta + 1/2$, then

$$P\{\|X-a\| < \varepsilon\} \sim \frac{1}{\sqrt{2\pi}} \frac{\exp\left(\varepsilon^2\gamma - \psi(\gamma) - \chi(\gamma)\right)}{\sqrt{-\gamma^2\chi''(\gamma)}}$$

with $\varepsilon^2 = \psi'(\gamma) + \chi'(\gamma)$.

(ii) As already mentioned, $\mathscr{A}_\psi$ may be empty. Here this happens for $1 < \beta \leqslant 2$ in view of $\alpha > 1$.

(iii) To our opinion, the most interesting case is $a \in \mathscr{A}_\chi \backslash \mathscr{H}_X$, i.e.,

$\beta+\frac{1}{2}<\alpha\leqslant\beta+1$. Then γ may be defined by $\varepsilon^2=\chi'(\gamma)$ and

$$P\{\|X-a\|<\varepsilon\}\sim e^{-\psi(\gamma)}P\{\|X\|<\varepsilon\}.$$

We want to investigate this case now more thoroughly. To do so let us introduce the following notation:

$$K_{\alpha\beta}:=2^{1/\beta-\alpha/\beta}\frac{\pi/\beta}{\sin(\pi(\beta+1-\alpha)/\beta)},\qquad \alpha\neq\beta+1,$$

$$K_\beta:=K_{\beta\beta}=2^{1/\beta-1}\frac{\pi/\beta}{\sin(\pi/\beta)}\qquad \text{(cf. (5.2))}, \tag{5.5}$$

$$C_s:=\begin{cases}\sum_{j=1}^{\infty}\{1/j^s-(1-s)^{-1}[1/(j+1)^{s-1}-1/j^{s-1}]\}, & 0<s<1,\\ \sum_{j=1}^{\infty}\{1/j-\log(1+1/j)\}, & s=1.\end{cases}$$

Observe that $C_1:=C$ is Euler's constant and C_s, $0<s<1$, has been investigated by de la Vallée Poussin (cf. [13], p. 39).

THEOREM 5.4. *Suppose that* $\beta+\frac{1}{2}<\alpha\leqslant\beta+1$. *Then*

$$\frac{P\{\|X-a\|<\varepsilon\}}{P\{\|X\|<\varepsilon\}}\sim\begin{cases}\exp\{-K_{\alpha\beta}K_\beta^{(\beta-\alpha+1)/(\beta-1)}\varepsilon^{-2(\beta-\alpha+1)/(\beta-1)}\\ \qquad+(2\beta-2\alpha+2)^{-1}-C_{\alpha-\beta}/2\}, & \alpha<\beta+1,\\ K_\beta^{-1/(2\beta-2)}2^{-1/2\beta}e^{-C/2}\varepsilon^{1/(\beta-1)}, & \alpha=\beta+1.\end{cases}$$

Proof. We have to investigate the behaviour of $e^{-\psi(\gamma)}$ with $\varepsilon^2=\chi'(\gamma)$. Because of

$$\psi(\gamma)=\sum_{j=1}^{\infty}\frac{j^{-\alpha}}{1+2j^{-\beta}\gamma}=\int_1^\infty\frac{x^{-\alpha}\gamma}{1+2x^{-\beta}\gamma}dx-\sum_{j=1}^{\infty}\int_0^1 f_\gamma'(x+j)(1-x)\,dx$$

with

$$f_\gamma(x)=\frac{x^{-\alpha}\gamma}{1+2x^{-\beta}\gamma}$$

and by

$$|f_\gamma'(x+j)|\leqslant\frac{\alpha+\beta}{2}j^{\beta-\alpha-1},\qquad \alpha>\beta,$$

Lebesgue's D.C.T. applies and leads to

$$\lim_{\gamma\to\infty}\left(\psi(\gamma)-\int_1^\infty\frac{x^{-\alpha}\gamma}{1+2x^{-\beta}\gamma}dx\right)=-\frac{\beta-\alpha}{2}\sum_{j=1}^{\infty}\int_0^1(x+j)^{\beta-\alpha-1}(1-x)\,dx=C_{\alpha-\beta}/2.$$

Moreover, if $\alpha=\beta+1$, then

$$\int_1^\infty \frac{x^{-\alpha}\gamma}{1+2x^{-\beta}\gamma}dx = \frac{1}{2\beta}\log(1+2\gamma) = \frac{1}{2\beta}\log(2)+\frac{1}{2\beta}\log(\gamma)+o(1).$$

For $\alpha < \beta+1$ we have

$$\lim_{\gamma\to\infty}\int_0^1 \frac{x^{-\alpha}\gamma}{1+2x^{-\beta}\gamma}dx = (2\beta-2\alpha+2)^{-1}$$

because of

$$\frac{x^{-\alpha}\gamma}{1+2x^{-\beta}\gamma} \leqslant \frac{1}{2}x^{\beta-\alpha}, \qquad \beta-\alpha > -1.$$

Thus, in this case

$$\psi(\gamma) = \int_0^\infty \frac{x^{-\alpha}\gamma}{1+2x^{-\beta}\gamma}dx - \frac{1}{2}(\beta-\alpha+1)^{-1}+\frac{1}{2}C_{\alpha-\beta}+o(1)$$

and

$$\int_0^\infty \frac{x^{-\alpha}\gamma}{1+2x^{-\beta}\gamma}dx = K_{\alpha\beta}\gamma^{1-\alpha/\beta+1/\beta}.$$

Combining these results we obtain

$$\text{(5.6)}\quad -\psi(\gamma) = \begin{cases} -(1/2\beta)\log(2)-(1/2\beta)\log(\gamma)-C/2+o(1), & \alpha=\beta+1,\\ -K_{\alpha\beta}\gamma^{1-\alpha/\beta+1/\beta}+(2\beta-2\alpha+2)^{-1}-C_{\alpha-\beta}/2+o(1), & \alpha<\beta+1.\end{cases}$$

Our next aim is to replace γ by ε in (5.6). Using Euler–Maclaurin's summation formula, by similar arguments as above we get

$$\text{(5.7)}\qquad \gamma\chi'(\gamma) = \sum_{j=1}^\infty \frac{j^{-\beta}\gamma}{1+2j^{-\beta}\gamma} = K_\beta\gamma^{1/\beta}-\frac{1}{4}+o(1).$$

Using (5.7) we derive

$$\text{(5.8)}\qquad \gamma^{-1}K_\beta^{\beta/(\beta-1)}\varepsilon^{-2\beta/(\beta-1)} = \left(1-\frac{\gamma^{-1/\beta}}{4K_\beta}+o(\gamma^{-1/\beta})\right)^{-\beta/(\beta-1)}$$

$$= 1+\frac{\beta}{\beta-1}\frac{\gamma^{-1/\beta}}{4K_\beta}+o(\gamma^{-1/\beta}).$$

Let $\delta>0$ be defined by

$$\delta := 1-\alpha/\beta+1/\beta.$$

Then (5.8) lets us conclude that

$$\begin{aligned}\gamma^{\delta}-(K_{\beta}^{\beta/(\beta-1)}\varepsilon^{-2\beta/(\beta-1)})^{\delta} &= \gamma^{\delta}\big(1-(\gamma^{-1}K_{\beta}^{\beta/(\beta-1)}\varepsilon^{-2\beta/(\beta-1)})^{\delta}\big)\\ &= \delta\gamma^{\delta}(1-\gamma^{-1}K_{\beta}^{\beta/(\beta-1)}\varepsilon^{-2\beta/(\beta-1)})+o(\gamma^{\delta-1/\beta})\\ &= -\delta\frac{\beta}{\beta-1}\frac{\gamma^{\delta-1/\beta}}{4K_{\beta}}+o(\gamma^{\delta-1/\beta}).\end{aligned}$$

Now, $\delta-1/\beta = 1-\alpha/\beta < 0$, i.e., we may replace $\gamma^{1-\alpha/\beta+1/\beta}$ by

$$(K_{\beta}^{\beta/(\beta-1)}\varepsilon^{-2\beta/(\beta-1)})^{1-\alpha/\beta+1/\beta}$$

in (5.6), and this completes the proof. ∎

Remark 5.1. We have

$$K_{\beta}^{-1/2(\beta-1)}2^{-1/2\beta} = \left(\frac{\sin(\pi/\beta)}{\pi/\beta}\right)^{1/2(\beta-1)},$$

i.e., for $\alpha = \beta+1$ we obtain

$$\frac{P\{\|X-a\| < \varepsilon\}}{P\{\|X\| < \varepsilon\}} \sim \left(\frac{\sin(\pi/\beta)}{\pi/\beta}\right)^{1/2(\beta-1)} e^{-C/2}\varepsilon^{1/(\beta-1)}.$$

For example, if $\beta = 2$, then

$$\begin{aligned}P\left\{\sum_{j=1}^{\infty}\left|\frac{\xi_j}{j}-\frac{1}{j^{3/2}}\right|^2 < \varepsilon^2\right\} &\sim (2/\pi)^{1/2}e^{-C/2}\varepsilon P\left\{\sum_{j=1}^{\infty}\frac{|\xi_j|^2}{j^2} < \varepsilon^2\right\}\\ &\sim \frac{4e^{-C/2}}{\pi}\varepsilon\exp(-\pi^2/8\varepsilon^2)\end{aligned}$$

(cf. [1]). Thus, if

$$h(t) = \sqrt{2}\sum_{j=1}^{\infty}\frac{\sin(j\pi t)}{j^{3/2}\pi}$$

and $\{B(t);\, 0 \leqslant t \leqslant 1\}$ is a Brownian bridge, then

$$P\Big\{\int_0^1 |B(t)-h(t)|^2\,dt < \varepsilon^2\Big\} \sim 4e^{-C/2}\varepsilon\exp(-1/8\varepsilon^2) \quad \text{as } \varepsilon\to 0.$$

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