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CONDITIONAL VARIANCE FOR STABLE RANDOM VECTORS*

BY

STAMATIS CAMBANIS (CHAPEL HILL, NORTH CAROLINA)
AND
STERGIOS FOTOPOULOS (PULLMAN, WASHINGTON)

Abstract. For a symmetric α-stable random vector $(X_1, \ldots, X_n, X_{n+1})$ with $1 < \alpha < 2$ and spectral measure Γ, we find a necessary and sufficient condition in terms of Γ for the conditional variance $\operatorname{Var}(X_{n+1} \mid X_1, \ldots, X_n)$ to be finite. We express the conditional variance in terms of Γ, and we develop an additivity property when $X_1, \ldots, X_n$ are independent. These results are then applied to stable processes: scale mixtures of Gaussian processes, harmonizable and moving averages.

1. INTRODUCTION

For jointly normally distributed random variables, conditional expectations are always linear and conditional variances are always constants (i.e., degenerate random variables). Here, we consider random variables $X_1, \ldots, X_n, X_{n+1}$ that are jointly symmetric α-stable with $\alpha < 2$ ($\alpha = 2$ corresponds to the normality). In this case, conditional expectations do not always exist (when $0 < \alpha \leqslant 1$), and when they do, they are not always linear. Existence (when $0 < \alpha \leqslant 1$) and linearity (when $0 < \alpha \leqslant 2$) of conditional expectations have been considered for the bivariate case $n = 1$ in [4], [5], [9] and for the multivariate case $n \geqslant 2$ in [2], [7]. Here, we consider conditional variances, which are not always finite and when they are, they are not generally constant (i.e., degenerate); we focus on the case where $1 < \alpha < 2$.

We give a necessary and sufficient condition for $\mathrm{E}(X_{n+1}^2 \mid X_1, \ldots, X_n)$ to be finite, and we express the conditional second moment in terms of the joint spectral measure (Theorem 1). The bivariate case $n = 1$ was considered in [3], [9], [10]. We also relate the finiteness of $\mathrm{E}(X_{n+1}^2 \mid X_1, \ldots, X_n)$ with that of

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$E(X_{n+1}^2 \mid X_k)$, $k = 1, \ldots, n$ (the Corollary). When the random variables $X_1, \ldots, X_n$ are independent, we show that $\mathrm{Var}(X_{n+1} \mid X_1, \ldots, X_n)$ is a sum of n terms, each depending on X_k and proportional but not equal to $\mathrm{Var}(X_{n+1} \mid X_k)$, $k = 1, \ldots, n$.

Several examples of stable processes are considered in Section 3. For scale mixtures of Gaussian processes, conditional variances are shown to be always finite and they are expressed in terms of a fixed functional form (Theorem 3). Harmonizable stationary processes are also shown to always have finite conditional variances (Theorem 4); this extends a more special result in [6] which was established for all $0 < \alpha < 2$. Finally, for moving averages, necessary and sufficient, and, simpler, sufficient conditions are given for finiteness of conditional variances, and they are shown to be satisfied by the two-sided Ornstein–Uhlenbeck process (Theorem 5).

All proofs are collected in Section 4.

Throughout the paper, points in $\boldsymbol{R}^n$ are denoted by $x^{(n)} = (x_1, \ldots, x_n)$, the usual inner product by $\langle x^{(n)}, y^{(n)} \rangle = \sum_{k=1}^n x_k y_k$, and the Euclidean norm by $\|x^{(n)}\|^2 = \langle x^{(n)}, x^{(n)} \rangle$. We will also denote points in $\boldsymbol{R}^{n+1}$ by $x^{(n+1)} = (x^{(n)}, x_{n+1})$, where $x^{(n)} \in \boldsymbol{R}^n$. If $X^{(n+1)}$ is a symmetric α-stable (SαS) random vector with $0 < \alpha \leqslant 2$, then its characteristic function is of the form

$$\begin{aligned} (1.1) \qquad \phi_{X^{(n+1)}}(t^{(n+1)}) &= \mathrm{E}\exp(i\langle t^{(n+1)}, X^{(n+1)} \rangle) \\ &= \exp\Big\{-\int_{S_{n+1}} |\langle t^{(n+1)}, y^{(n+1)} \rangle|^{\alpha} \Gamma(dy^{(n+1)})\Big\}, \end{aligned}$$

where Γ is a finite Borel measure on the unit sphere $S_{n+1} = \{s^{(n+1)} \in \boldsymbol{R}^{n+1}: \|s^{(n+1)}\| = 1\}$ of $\boldsymbol{R}^{(n+1)}$, called the *spectral measure* of $X^{(n+1)}$.

We also use the bracket power notation $x^{\langle p \rangle} = |x|^p \operatorname{sign}(x)$ for $x \in \boldsymbol{R}^1$; and the symbol c for a generic finite positive constant whose value may change from expression to expression, while c_k denotes a specific constant.

2. RESULTS

When X_{n+1} is independent of $X^{(n)}$, or when its conditional distribution given $X^{(n)}$ is stable with index α, then $\mathrm{E}(|X_{n+1}|^p \mid X^{(n)}) < \infty$ a.s. only for $0 < p < \alpha$, so $\mathrm{Var}(X_{n+1} \mid X^{(n)}) = \infty$. Here are some specific examples.

INDEPENDENCE. When X_{n+1} is independent of $X^{(n)}$, then

$$\mathrm{E}(|X_{n+1}|^p \mid X^{(n)}) = \mathrm{E}(|X_{n+1}|^p) = \begin{cases} < \infty, & 0 < p < \alpha, \\ \infty, & \alpha \leqslant p, \end{cases} \quad \text{a.s.}$$

AR(m) PROCESS. $X_n - \alpha_1 X_{n-1} - \ldots - \alpha_m X_{n-m} = Z_n$, where $\{Z_n;\ n \in \boldsymbol{Z}\}$ is an independent SαS sequence of random variables. Then, for $n \geqslant m$, $X^{(n+1)}$ is an SαS random vector and

$$\mathrm{E}(|X_{n+1}|^p \mid X^{(n)} = x^{(n)}) = \mathrm{E}(|\alpha_1 x_n + \ldots + \alpha_n x_{n+1-m} + Z_{n+1}|^p)$$
$$= \begin{cases} < \infty, & 0 < p < \alpha, \\ \infty, & \alpha \leqslant p, \end{cases} \quad \text{a.s.}$$

ORNSTEIN–UHLENBECK PROCESS. $X(t) = \int_{-\infty}^{t} e^{-\lambda(t-u)} dZ(u)$, $t \in \boldsymbol{R}$, where Z has independent SαS stationary increments. For any fixed $t_1 < \ldots < t_n < t_{n+1}$, we can write

$$X(t_{n+1}) = \int_{-\infty}^{t_n} \exp\{-\lambda(t_{n+1}-u)\} dZ(u) + \int_{t_n}^{t_{n+1}} \exp\{-\lambda(t_{n+1}-u)\} dZ(u)$$
$$= \exp\{-\lambda(t_{n+1}-t_n)\} X(t_n) + \exp\{-\lambda t_{n+1}\} \int_{t_n}^{t_{n+1}} e^{\lambda u} dZ(u),$$

so that the (nonstationary) sequence $Y_n = e^{\lambda t_n} X(t_n)$ satisfies $Y_{n+1} = Y_n + Z_{n+1}$, where $Z_{n+1} = \int_{t_n}^{t_{n+1}} e^{-\lambda u} dZ(u)$ are independent SαS. It follows that $\mathrm{E}(|Y_{n+1}|^p \mid Y^{(n)}) < \infty$ a.s. for $0 < p < \alpha$ and $= +\infty$ a.s. for $\alpha \leqslant p$, and thus

$$\mathrm{E}(|X_{(t_{n+1})}|^p \mid X(t_1), \ldots, X(t_n)) = \begin{cases} < \infty, & 0 < p < \alpha, \\ \infty, & \alpha \leqslant p, \end{cases} \quad \text{a.s.}$$

ONE-SIDED LINEAR PROCESS. A slightly more general example with the same kind of behaviour is provided by $X_n = \sum_{i=0}^{\infty} b_i Z_{n-i}$, where $\{b_0 = 1, b_n;\ n \in \boldsymbol{N}\}$ is a sequence of constants with $\sum_{j=0}^{\infty} |b_j|^\alpha < \infty$ and $\{Z_n;\ n \in \boldsymbol{Z}\}$ is a sequence of independent and identically distributed SαS random variables. It may easily be seen that

$$\mathrm{E}(|X_{n+1}|^p \mid X^{(n)}) = \mathrm{E}\{\mathrm{E}(|Z_{n+1} + \sum_{j=1}^{\infty} b_j Z_{n+1-j}|^p \mid Z_j,\ j \leqslant n) \mid X^{(n)}\}$$
$$= \begin{cases} < \infty, & 0 < p < \alpha, \\ +\infty, & \alpha \leqslant p, \end{cases} \quad \text{a.s.}$$

On the other hand, when X_{n+1} is dependent on $X^{(n)}$, then $\mathrm{E}(|X_{n+1}|^p \mid X^{(n)}) < \infty$ a.s. for all p. So the question arises under what type of weaker condition of dependence of X_{n+1} on $X^{(n)}$ is it possible to have $\mathrm{Var}(X_{n+1} \mid X^{(n)}) < \infty$ a.s. This question is answered in Theorem 1.

We will assume that $X_1, \ldots, X_n, X_{n+1}$ are linearly independent. As in Lemma 2 (due to Samorodnitsky and Taqqu [9]), this implies that

$$\phi_{X^{(n+1)}}(t^{(n+1)}) \leqslant \exp\{-c \|t^{(n+1)}\|^\alpha\}$$

for all $t^{(n+1)} \in \boldsymbol{R}^{n+1}$ and some positive, finite constant c, and thus $\phi(t^{(n+1)}) \in L^1(\boldsymbol{R}^{n+1})$ and $X^{(n+1)}$ has a continuous probability density function $f_{X^{(n+1)}}(x^{(n+1)})$. The regular conditional second moment of X_{n+1} given

$X^{(n)} = x^{(n)}$ is then given for all $x^{(n)} \in \boldsymbol{R}^n$ by

$$\mathrm{E}(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) = \frac{1}{f_{X^{(n)}}(x^{(n)})} \int_{-\infty}^{\infty} x_{n+1}^2 f_{X^{(n+1)}}(x^{(n)}, x_{n+1})\, dx_{n+1},$$

where $f_{X^{(n)}}(x^{(n)})$ is the density of $X^{(n)}$. Throughout, whenever we write $\mathrm{E}(X_{n+1}^2 \mid X^{(n)} = x^{(n)})$, we mean this regular version. We first give a necessary and sufficient condition for the finiteness of the second conditional moment, and we express it explicitly in terms of Γ.

THEOREM 1. *Let $X_1, \ldots, X_{n+1}$ be linearly independent and jointly $S\alpha S$ random variables with $1 < \alpha < 2$. Then $\mathrm{E}(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) < \infty$ for all $x^{(n)} \in \boldsymbol{R}^n$ if and only if*

$$\int_{\boldsymbol{R}^n} \phi_{X^{(n)}}(t^{(n)}) \int_{S_{n+1}} \frac{y_{n+1}^2 \Gamma(dy^{(n+1)})}{|\langle t^{(n)}, y^{(n)} \rangle|^{2-\alpha}} dt^{(n)} < \infty. \tag{2.1}$$

Also, for all $x^{(n)} \in \boldsymbol{R}^n$,

$$\mathrm{E}(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) = \frac{\alpha^2}{(2\pi)^n f_{X^{(n)}}(x^{(n)})} \int_{\boldsymbol{R}^n} dt^{(n)} \exp(-i\langle t^{(n)}, x^{(n)} \rangle)\, \phi_{X^{(n)}}(t^{(n)}) \tag{2.2}$$

$$\times \left\{ \left(1 - \frac{1}{\alpha}\right) \int_{S_{n+1}} \frac{y_{n+1}^2 \Gamma(dy^{(n+1)})}{|\langle t^{(n)}, y^{(n)} \rangle|^{2-\alpha}} - \Big(\int_{S_{n+1}} \langle t^{(n)}, y^{(n)} \rangle^{\langle \alpha - 1 \rangle} y_{n+1} \Gamma(dy^{(n+1)}) \Big)^2 \right\}.$$

When $n = 1$, $\phi_{X_1}(x_1) = \exp(-\sigma_1^\alpha |t_1|^\alpha)$ and the necessary and sufficient condition (2.1) becomes equivalent to $\int_{S_2} |y_1|^{\alpha-2} \Gamma(dy_1, dy_2) < \infty$, as in [10].

When $X_1, \ldots, X_{n+1}$ have a spherically symmetric distribution:

$$\phi_{X^{(n+1)}}(t^{(n+1)}) = \exp(-\sigma^2 \|t^{(n+1)}\|^\alpha)$$

and Γ is surface measure on S_{n+1}, then $\mathrm{E}(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) < \infty$ for all $x^{(n)} \in \boldsymbol{R}^n$ if and only if

$$\int_{S_{n+1}} \frac{y_{n+1}^2}{\|y^{(n)}\|^{2-\alpha}} \Gamma(dy^{(n+1)}) < \infty, \tag{2.3}$$

as follows from Lemma 3. Since Γ is surface measure and $2 - \alpha < 1$, condition (2.3) is satisfied. Thus, *if in addition to the assumptions of Theorem 1, the r.v.'s $X_1, \ldots, X_{n+1}$ have a spherically symmetric distribution, then $\mathrm{E}(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) < \infty$ for all $x^{(n)} \in \boldsymbol{R}^n$.*

It follows from Lemmas 2 and 3 that *condition* (2.3) *is sufficient for condition* (2.1).

A stronger sufficient condition than (2.3) is obtained from $\|y^{(n)}\| \geqslant |y_k|$, $k = 1, \ldots, n$:

$$\int_{S_{n+1}} \frac{y_{n+1}^2}{|y_k|^{2-\alpha}} \Gamma(dy^{(n+1)}) < \infty \quad \text{for some } k = 1, \ldots, n. \tag{2.4}$$

This is slightly stronger than the necessary and sufficient condition for $\mathrm{E}(X_{n+1}^2 \mid X_k = x_k) < \infty$ for all $x_k = \boldsymbol{R}^1$, which is expressed below in terms of the spectral measure Γ of the full vector $(X_1, \ldots, X_{n+1})$ for comparison purposes.

COROLLARY. *Let $X_1, \ldots, X_n, X_{n+1}$ be linearly independent and jointly $S\alpha S$ with $1 < \alpha < 2$, and let $k = 1, \ldots, n$.*

(a) $\mathrm{E}(X_{n+1}^2 \mid X_k = x_k) < \infty$ *for all* $x_k \in \boldsymbol{R}^1$ *if and only if*

(2.5)
$$\int_{S_{n+1} \cap \{y_k \neq 0\}} \frac{y_{n+1}^2}{|y_k|^{2-\alpha}} \Gamma(dy^{(n+1)}) < \infty \quad \text{and} \quad \int_{S_{n+1} \cap \{y_k = 0\}} |y_{n+1}|^\alpha \Gamma(dy^{(n+1)}) = 0.$$

(b) *If, in addition, $n > 1$ and $X_1, \ldots, X_n$ are independent, then* $\mathrm{E}(X_{n+1}^2 \mid X_k = x_k) < \infty$ *for all $x_k \in \boldsymbol{R}^1$ if and only if X_{n+1} is independent of X_j, $j \neq k$, $j = 1, \ldots, n$, and*

$$\int_{S_{n+1} \cap \{y_k^2 + y_{n+1}^2 = 1, y_k \neq 0\}} \frac{y_{n+1}^2}{|y_k|^{2-\alpha}} \Gamma(dy^{(n+1)}) < \infty. \tag{2.6}$$

(c) *If in addition $n > 1$ and $X_1, \ldots, X_n$ are independent but $X_1, \ldots, X_n$, X_{n+1} are not independent, then at most one of*

$$\{\mathrm{E}(X_{n+1}^2 \mid X_k = x_k) < \infty \text{ for all } x_k \in \boldsymbol{R}^1\}, \quad k = 1, \ldots, n, \tag{2.7}$$

may be true, while if X_{n+1} depends on more than one of $X_1, \ldots, X_n$, then none of (2.7) *is true.*

It is clear from this Corollary that $\mathrm{E}(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) < \infty$ for all $x^{(n)} \in \boldsymbol{R}^n$ does not necessarily imply that $\mathrm{E}(X_{n+1}^2 \mid X_k = x_k) < \infty$ for all $x_k \in \boldsymbol{R}^1$ and some $k = 1, \ldots, n$. (Of course, in the converse direction, $\mathrm{E}(X_{n+1}^2 \mid X_k = x_k) < \infty$ for all $x_k \in \boldsymbol{R}^1$ and some $k = 1, \ldots, n$ always implies that $\mathrm{E}(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) < \infty$ for all $x^{(n)} \in \boldsymbol{R}^n$, since $\mathrm{E}(X_{n+1}^2 \mid X_k = x_k) = [f_{X_k}(x_k)]^{-1} \int_{\boldsymbol{R}^{n-1}} \mathrm{E}(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) f_{X^{(n)}}(x^{(n)}) \prod_{j=1, j \neq k}^n dx_j$.) When $X_1, \ldots, X_n$ are spherically distributed, in view of the Corollary, we have $\mathrm{E}(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) < \infty$ for all $x^{(n)} \in \boldsymbol{R}^n$ and $\mathrm{E}(X_{n+1}^2 \mid X_k = x_k) < \infty$ for all $x_k \in \boldsymbol{R}^1$ and $k = 1, \ldots, n$.

Whenever the conditional second moment is finite, so is the conditional variance whose expression can be found from Theorem 1 via

$$\mathrm{Var}(X_{n+1} \mid X^{(n)}) = \mathrm{E}(X_{n+1}^2 \mid X^{(n)}) + \{\mathrm{E}(X_{n+1} \mid X^{(n)})\}^2,$$

and the expression of the regular conditional mean, which is given likewise for all $x^{(n)} \in \boldsymbol{R}^n$ by

$$\mathrm{E}(X_{n+1} \mid X^{(n)} = x^{(n)}) = \frac{\alpha i}{(2\pi)^n f_{X^{(n)}}(x^{(n)})} \int_{\boldsymbol{R}^n} dt^{(n)} \exp(-i\langle t^{(n)}, x^{(n)} \rangle) \phi_{X^{(n)}}(t^{(n)}) \times \Big\{ \int_{S_{n+1}} \langle t^{(n)}, y^{(n)} \rangle^{\langle \alpha - 1 \rangle} y_{n+1} \Gamma(dy^{(n+1)}) \Big\} \tag{2.8}$$

and is not necessarily linear [2]. In general, this is a complicated expression, and it would be interesting to know if it can ever lead to a (degenerate) constant value; this is never true when $n = 1$, as is clear from [10]. Now we show that when $X_1, \ldots, X_n$ are independent, then $\operatorname{Var}(X_{n+1} \mid X_1, \ldots, X_n)$ is a sum of n terms, each of which depends on X_k, $k = 1, \ldots, n$, and is proportional but not equal to $\operatorname{Var}(X_{n+1} \mid X_k)$.

THEOREM 2. *Let $X_1, \ldots, X_n, X_{n+1}$ be linearly independent, jointly SαS r.v.'s with $1 < \alpha < 2$, and let $X_1, \ldots, X_n$ be independent.*

(a) $\mathrm{E}(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) < \infty$ *for all* $x^{(n)} \in \boldsymbol{R}^n$ *if and only if*

$$\Gamma(S_{n+1} \cap \{y_{n+1} = \pm 1\}) = 0 \tag{2.9}$$

and

$$\int_{S_{n+1} \cap \{y_k^2 + y_{n+1}^2 = 1, y_k \neq 0\}} \frac{y_{n+1}^2}{|y_k|^{2-\alpha}} \Gamma(dy^{(n)}) < \infty, \qquad k = 1, \ldots, n. \tag{2.10}$$

(b) *If* $\mathrm{E}(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) < \infty$ *for all* $x^{(n)} \in \boldsymbol{R}^n$, *then*

$$\operatorname{Var}(X_{n+1} \mid X_1 = x_1, \ldots, X_n = x_n) = \sum_{k=1}^{n} D_{n+1|k}^2(\alpha)\, S_1^2(x_k/\sigma_k;\, \alpha) \tag{2.11}$$

for all $x^{(n)} \in \boldsymbol{R}^n$, *where the universal "standard deviation" function* $S_1(x;\alpha)$ *is up to factor* $\alpha(\alpha-1)$ *as in* [10]:

$$S_1^2(x;\alpha) = \alpha(\alpha-1) \frac{\int_0^\infty \cos(xt)\exp(-t^\alpha)\, t^{\alpha-2}\, dt}{\int_0^\infty \cos(xt)\exp(-t^\alpha)\, dt} \tag{2.12}$$

and the coefficients $D_{n+1|k}^2$ *depend on the joint distribution of* X_k *and* X_{n+1} *as follows*:

$$D_{n+1|k}^2(\alpha) = \frac{\alpha(\alpha-1)}{\sigma_k^{2(\alpha-1)}} \Big\{\sigma_k^\alpha \int_{S_{n+1} \cap \{y_k^2 + y_{n+1}^2 = 1, y_k \neq 0\}} |y_k|^{2-\alpha} y_{n+1}^2\, \Gamma(dy^{(n+1)}) - \Big(\int_{S_{n+1}} y_k^{\langle \alpha-1 \rangle} y_{n+1}\, \Gamma(dy^{(n+1)})\Big)^2\Big\} \tag{2.13}$$

and $\sigma_k^\alpha = \int_{S_{n+1}} |y_k|^\alpha \Gamma(dy^{(n+1)})$ *is the scale parameter of* X_k: $\phi_{X_k}(t) = \exp(-\sigma_k^\alpha |t|^\alpha)$.

Note that the dependence coefficients $D_{n+1|k}^2(\alpha)$ differ crucially from the dependence coefficients $C_{n+1|k}^2(\alpha)$ in the expression of $\operatorname{Var}(X_{n+1} \mid X_k) = C_{n+1|k}^2(\alpha)\, S_1(X_k/\sigma_k;\alpha)$ given in [10] only by the constraint $y_k \neq 0$ in the first integral within brackets. This is crucial because, according to the theorem, without the constraint $y_k \neq 0$ we obtain the coefficients $C_{n+1|k}^2(\alpha)$ whose value is infinite!

3. APPLICATIONS TO STOCHASTIC PROCESSES

3.1. Scale mixtures of Gaussian processes. These are defined by $X(t) = A^{1/2}G(t)$, $t \in T$, where A is totally right skewed $(\alpha/2)$-stable, $0 < \alpha < 2$, i.e., positive with $\mathrm{E}e^{-uA} = \exp(-u^{\alpha/2})$, $u \geqslant 0$, independent of the Gaussian process $G(t)$, $t \in T$, with mean 0 and covariance function $R(t, s)$. With $X^{(n+1)} = (X(t_1), \ldots, X(t_{n+1}))$, we have

$$
\begin{aligned}
(3.1)\qquad \phi_{X^{(n+1)}}(s^{(n+1)}) &= \mathrm{E}\exp\{i\langle s^{(n+1)}, X^{(n+1)}\rangle\} \\
&= \mathrm{E}\exp\{iA^{1/2}\langle s^{(n+1)}, G^{(n+1)}\rangle\} \\
&= \mathrm{E}\exp\left\{-\frac{1}{2}A\langle s^{(n+1)}, \Sigma_{n+1}s^{(n+1)}\rangle\right\} \\
&= \exp\left\{-\frac{1}{2^{\alpha/2}}\langle s^{(n+1)}, \Sigma_{n+1}s^{(n+1)}\rangle^{\alpha/2}\right\},
\end{aligned}
$$

where $\Sigma_{n+1} = \{R(t_k, t_j)\}_{k,j=1}^{n+1}$, so the finite dimensional distributions are SαS. We will assume, without loss of generality, that $X(t_1), \ldots, X(t_n)$ are linearly independent, i.e., the covariance matrix Σ_n is positive definite. It was shown in [2] that, for all $0 < \alpha < 2$, multiple regressions exist:

$$\mathrm{E}\{|X(t_{n+1})| \mid X(t_1), \ldots, X(t_n)\} < \infty \ \text{a.s.},$$

and are linear:

$$\mathrm{E}\{X(t_{n+1}) \mid X(t_1), \ldots, X(t_n)\} = a_1X(t_1) + \ldots + a_nX(t_n),$$

and the regression coefficients are those of the Gaussian process:

$$\mathrm{E}\{G(t_{n+1}) \mid G(t_1), \ldots, G(t_n)\} = a_1G(t_1) + \ldots + a_nG(t_n).$$

Here, we show that multiple conditional variances are also finite when $1 < \alpha < 2$ and we derive their expression. The case $n = 1$ was considered in [10].

THEOREM 3. *With the notation above, when* $1 < \alpha < 2$, *we have*

$$\mathrm{E}\{X^2(t_{n+1}) \mid X(t_1) = x_1, \ldots, X(t_n) = x_n\} < \infty \quad \textit{for all } x^{(n)} \in \mathbf{R}^n$$

and

$$
\begin{aligned}
(3.2)\qquad &\mathrm{Var}\{X(t_{n+1}) \mid X(t_1) = x_1, \ldots, X(t_n) = x_n\} \\
&= [R(t_{n+1}, t_{n+1}) - \Sigma_{n+1,n}\Sigma_n^{-1}\Sigma_{n+1,n}']\, S_n^2(\langle x^{(n)}, \Sigma_n^{-1}x^{(n)}\rangle^{1/2}; \alpha),
\end{aligned}
$$

where $\Sigma_{n+1,n} = (R(t_{n+1}, t_1), \ldots, R(t_{n+1}, t_n))$ *and, for* $n \geqslant 2$,

$$(3.3)\qquad S_n^2(x;\alpha) = \frac{\alpha}{2}\,\frac{\int_0^\infty r^{n+\alpha-3}\exp(-r^\alpha)\,dr\int_0^\pi(\sin\theta)^{n-2}\cos(\sqrt{2}\,xr\cos\theta)\,d\theta}{\int_0^\infty r^{n-1}\exp(-r^\alpha)\,dr\int_0^\pi(\sin\theta)^{n-2}\cos(\sqrt{2}\,xr\cos\theta)\,d\theta},$$

whereas $S_1^2(x;\alpha)$ *is as in* (2.12).

Notice that the conditional variance in (3.2) is proportional to the corresponding conditional variance of the Gaussian process G and depends on $x^{(n)}$ via the quadratic form $\langle x^{(n)}, \Sigma_n x^{(n)}\rangle$ and the fixed function $S_n^2(\cdot;\alpha)$.

3.2. Harmonizable stationary processes. These are represented by

$$(3.4)\qquad X(t) = \operatorname{Re}\int_{-\infty}^{\infty} e^{it\lambda}\,dZ(\lambda), \qquad -\infty < t < \infty,$$

where Z has complex, independent, rotationally invariant, α-stable increments and finite control measure m, and their finite dimensional characteristic functions are given by

$$(3.5)\qquad \mathrm{E}\exp\{\sum_{k=1}^{n+1} s_k X(t_k)\} = \exp\{-\int_{-\infty}^{\infty}|\sum_{k=1}^{n+1} s_k\exp(it_k\lambda)|^\alpha\}\,dm(\lambda)$$

$$= \exp\{-\int_{-\infty}^{\infty}|\langle s^{(n+1)}, \Sigma_{n+1}(\lambda)s^{(n+1)}\rangle|^{\alpha/2}\}\,dm(\lambda),$$

where $\Sigma_{n+1}(\lambda) = \{\cos[(t_k-t_j)\lambda]\}_{k,j=1}^{n+1}$. It was shown in [2] that for all $0<\alpha<2$ multiple regressions exist: $\mathrm{E}\{|X(t_{n+1})| \mid X(t_1), \ldots, X(t_n)\} < \infty$ a.s.; however, their (nonlinear) expression is not currently known when $n \geqslant 2$. Here we show that multiple conditional variances are also finite when $1<\alpha<2$. The case $n=1$ was established in [10], and the case $n=2$ with special times $t_2-t_1 = t_3-t_2$ and all $0<\alpha<2$ was established in [6].

THEOREM 4. *With* $X(t)$, *as in* (3.4)–(3.5), *and* $1<\alpha<2$, *we have*

$$\operatorname{Var}\{X(t_{n+1}) \mid X(t_1), \ldots, X(t_n)\} < \infty \text{ a.s.}$$

At present, the functional form of the conditional variance is not known when $n \geqslant 2$. The case $n=1$ was developed in [10].

3.3. Moving average processes. These are stationary SαS processes of the form

$$(3.6)\qquad X(t) = \int_{-\infty}^{\infty} f(t-u)\,dZ(u), \qquad -\infty < t < \infty,$$

where the process Z has stationary independent SαS increments and $f\in L^\alpha$, so that the finite dimensional characteristic functions are, with $X^{(n+1)}$

$= (X(t_1), \ldots, X(t_{n+1})),$

(3.7)

$$\phi_{X^{(n+1)}}(s^{(n+1)}) = \mathrm{E}\exp\{i\langle s^{(n+1)}, X^{(n+1)}\rangle\} = \exp\{-\int_{-\infty}^{\infty} |\sum_{k=1}^{n+1} f(t_k-u)|^{\alpha} du\}.$$

THEOREM 5. *For a moving average process as in* (3.6)–(3.7), *with* $1 < \alpha < 2$ *and* $\{X(t_k), k = 1, \ldots, n+1\}$ *linearly independent, we have*

$$\mathrm{E}\{X^2(t_{n+1}) \mid X(t_1) = x_1, \ldots, X(t_n) = x_n\} < \infty \quad \textit{for all } x^{(n)} \in \boldsymbol{R}^n$$

if and only if

$$\int_{\boldsymbol{R}^n} ds^{(n)} \exp\{-\int_{-\infty}^{\infty} |\sum_{k=1}^{n} s_k f(t_k-u)|^{\alpha} du\} \int_{-\infty}^{\infty} \frac{f^2(t_{n+1}-v)\,dv}{|\sum_{k=1}^{n} f(t_k-v)|^{\alpha}} < \infty \tag{3.8}$$

or if

$$\int_{-\infty}^{\infty} \frac{f^2(u)\,du}{\{\sum_{k=1}^{n} f^2(t_k-t_{n+1}+u)\}^{1-\alpha/2}} < \infty. \tag{3.9}$$

Specifically, for $t_1 < \ldots < t_n < t_{n+1}$ *and the stable two-sided Ornstein–Uhlenbeck process* $X(t) = \int_{-\infty}^{\infty} e^{-\lambda|t-u|} dZ(u)$, *we have*

$$\mathrm{Var}\{X(t_{n+1}) \mid X(t_1) = x_1, \ldots, X(t_n) = x_n\} < \infty \quad \textit{for all } x^{(n)} \in \boldsymbol{R}^n.$$

The functional form of the conditional variance, when it is finite, is not known even for $n = 2$.

4. PROOFS

Proof of Theorem 1. This follows the line of the proof in [10]. The regular conditional ch.f. of X_{n+1} given $X^{(n)} = x^{(n)}$ is expressed in terms of the joint ch.f. $\phi_{X^{(n+1)}}(t^{(n+1)})$ as follows:

$$\psi(t_{n+1}; x^{(n)}) \triangleq \mathrm{E}(\exp(it_{n+1}X_{n+1}) \mid X^{(n)} = x^{(n)})$$
$$= \frac{1}{(2\pi)^n f_{X^{(n)}}(x^{(n)})} \int_{\boldsymbol{R}^n} \exp\{-i\langle t^{(n)}, x^{(n)}\rangle\} \phi_{X^{(n+1)}}(t^{(n)}, t_{n+1})\,dt^{(n)}$$

for all $t_{n+1} \in \boldsymbol{R}^1$ and $x^{(n)} \in \boldsymbol{R}^n$ (Zabell [11]). It follows that

$$\mathrm{E}(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) < \infty \quad \text{for all } x^{(n)} \in \boldsymbol{R}^n$$

if and only if

$$\lim_{t_{n+1}\to 0} \frac{1}{t_{n+1}^2}\{2-\psi(t_{n+1}; x^{(n)})-\psi(-t_{n+1}; x^{(n)})\}$$

exists and is finite for all $x^{(n)} \in \boldsymbol{R}^n$.

We can write

$$I(t_{n+1};x^{(n)}) \triangleq \frac{1}{t_{n+1}^2}\{2-\psi(t_{n+1};x^{(n)})-\psi(-t_{n+1};x^{(n)})\}$$

$$= \frac{1}{(2\pi)^n f_{X^{(n)}}(x^{(n)})}\int_{R^n} \exp\{-i\langle t^{(n)},x^{(n)}\rangle\}$$

$$\times \frac{1}{t_{n+1}^2}\{2\phi_{X^{(n+1)}}(t^{(n)},0)-\phi_{X^{(n+1)}}(t^{(n)},t_{n+1})$$

$$-\phi_{X^{(n+1)}}(t^{(n)},-t_{n+1})\}\,dt^{(n)}$$

$$= \frac{1}{(2\pi)^n f_{X^{(n)}}(x^{(n)})}\int_{R^n} \exp\{-i\langle t^{(n)},x^{(n)}\rangle\}\,\phi_{X^{(n+1)}}(t^{(n)},0)\{J_1(t^{(n+1)})$$

$$+J_2(t^{(n+1)})\}\,dt^{(n)}$$

$$\triangleq I_1(t_{n+1};x^{(n)})+I_2(t_{n+1};x^{(n)}),$$

where

$$J_1(t^{(n+1)}) = \frac{1}{t_{n+1}^2}\{1-\Delta(t^{(n)},t_{n+1})-\exp\{-\Delta(t^{(n)},t_{n+1})\}$$

$$+1-\Delta(t^{(n)},-t_{n+1})-\exp\{-\Delta(t^{(n)},-t_{n+1})\}\},$$

$$J_2(t^{(n+1)}) = \frac{1}{t_{n+1}^2}\{\Delta(t^{(n)},t_{n+1})+\Delta(t^{(n)},-t_{n+1})\},$$

$$\Delta(t^{(n+1)}) = \int_{S_{n+1}} \{|\langle t^{(n+1)},y^{(n+1)}\rangle|^\alpha-|\langle t^{(n)},y^{(n)}\rangle|^\alpha\}\,\Gamma(dy^{(n+1)}).$$

We first show that the limit of $I_1(t_{n+1};x^{(n)})$ as $t_{n+1}\to 0$ always exists and is finite for all $x^{(n)}\in R^n$. By Taylor's expansion, we have

$$1-\Delta(t^{(n+1)})-\exp\{-\Delta(t^{(n+1)})\} = -\tfrac{1}{2}\Delta^2(t^{(n+1)})\exp\{-\theta(t^{(n+1)})\},$$

where $|\theta(t^{(n+1)})|\leqslant|\Delta(t^{(n+1)})|$. Also, by Lemma 1, we have for $|t_{n+1}|\leqslant 1$, say,

$$|\Delta(t^{(n+1)})| \leqslant \int_{S_{n+1}} \{|t_{n+1}y_{n+1}|^\alpha+\alpha|\langle t^{(n)},y^{(n)}\rangle|^{\alpha-1}|t_{n+1}y_{n+1}|\}\,\Gamma(dy^{(n+1)})$$

$$\leqslant c(1+\|t^{(n)}\|^{\alpha-1}),$$

and using Lemma 2 we obtain the following upper bound:

$$|\phi_{X^{(n+1)}}(t^{(n)},0)J_1(t^{(n+1)})| \leqslant \exp\{-c_5\|t^{(n)}\|^\alpha\}\frac{c^2}{4}(1+\|t^{(n)}\|^{\alpha-1})^2\exp\{c(1+\|t^{(n)}\|^{\alpha-1})\},$$

which is in $L^1(\boldsymbol{R}^n)$ as a function of $t^{(n)}$. The elementary argument on pp. 90–91 of [10] gives

$$\lim_{t_{n+1}\to 0} J_1(t^{(n+1)}) = \lim_{t_{n+1}\to 0} \frac{1}{t_{n+1}^2}\{\Delta^2(t^{(n)}, t_{n+1}) + \Delta^2(t^{(n)}, -t_{n+1})\}$$

$$= -\alpha^2\{\int_{S_{n+1}} \langle t^{(n)}, y^{(n)}\rangle^{\langle\alpha-1\rangle} y_{n+1}\Gamma(dy^{(n+1)})\}^2.$$

Thus, the dominated convergence theorem gives

$$\lim_{t_{n+1}\to 0} I_1(t_{n+1}; x^{(n)}) = \frac{-\alpha^2}{(2\pi)^n f_{X^{(n)}}(x^{(n)})} \int_{\boldsymbol{R}^n} \exp\{-i\langle t^{(n)}, x^{(n)}\rangle\} \phi_{X^{(n+1)}}(t^{(n)}, 0)$$

$$\times\{\int_{S_{n+1}} \langle t^{(n)}, y^{(n)}\rangle^{\langle\alpha-1\rangle} y_{n+1}\Gamma(dy^{(n+1)})\}^2 dt^{(n)}.$$

Since this limit exists and is finite for all $x^{(n)}\in\boldsymbol{R}^n$, it follows that $\mathrm{E}(X_{n+1}^2 \mid X^{(n)} = x^{(n)})$ exists and is finite for all $x^{(n)}\in\boldsymbol{R}^n$ if and only if the limit of $I_2(t_{n+1}; x^{(n)})$ as $t_{n+1}\to 0$ exists and is finite for all $x^{(n)}\in\boldsymbol{R}^n$.

Assuming first that $\lim_{t_{n+1}\to 0} I_2(t_{n+1}; x^{(n)})$ exists and is finite for all $x^{(n)}\in\boldsymbol{R}^n$, we obtain by Fatou's lemma (since, by Lemma 1, $|x+y|^\alpha+|x-y|^\alpha-2|x|^\alpha \geqslant 0$ for all $x, y\in\boldsymbol{R}^1$),

$$\begin{aligned}
\infty &> (2\pi)^n f_{X^{(n)}}(0) \lim_{t_{n+1}\to 0} I_2(t_{n+1}; 0)\\
&\geqslant \int_{\boldsymbol{R}^n} dt^{(n)} \phi_{X^{(n+1)}}(t^{(n)}, 0) \int_{S_{n+1}} \Gamma(dy^{(n+1)}) \lim_{t_{n+1}\to 0} \frac{1}{t_{n+1}^2}\{|\langle t^{(n)}, y^{(n)}\rangle + t_{n+1}y_{n+1}|^\alpha\\
&\quad + |\langle t^{(n)}, y^{(n)}\rangle - t_{n+1}y_{n+1}|^\alpha - 2|\langle t^{(n)}, y^{(n)}\rangle|^\alpha\}\\
&= \alpha(\alpha-1)\int_{\boldsymbol{R}^n} dt^{(n)} \phi_{X^{(n+1)}}(t^{(n)}, 0) \int_{S_{n+1}} \Gamma(dy^{(n+1)}) |\langle t^{(n)}, y^{(n)}\rangle|^{\alpha-2} y_{n+1}^2.
\end{aligned}$$

Thus, condition (2.1) follows.

Now assume that, conversely, condition (2.1) is satisfied. Using Lemma 1 (ii) with $r = 2$, we have

$$|J_2(t^{(n+1)})| \leqslant c \int_{S_{n+1}} |\langle t^{(n)}, y^{(n)}\rangle|^{\alpha-2} y_{n+1}^2 \Gamma(dy^{(n+1)}),$$

so the integrand of $I_2(t_{n+1}; x^{(n)})$ is upper-bounded in absolute value for all t_{n+1} by

$$c\phi_{X^{(n+1)}}(t^{(n)}, 0) \int_{S_{n+1}} |\langle t^{(n)}, y^{(n)}\rangle|^{\alpha-2} y_{n+1}^2 \Gamma(dy^{(n+1)}),$$

which is in $L^1(\boldsymbol{R}^n)$ as a function of $t^{(n)}$ in view of condition (2.1). Thus, the dominated convergence theorem gives

$$(2\pi)^n f_{X^{(n)}}(x^{(n)}) \lim_{t_{n+1}\to 0} I_2(t_{n+1}; x^{(n)})$$

$$= \int_{R^n} \exp\{-i\langle t^{(n)}, x^{(n)}\rangle\} \phi_{X^{(n+1)}}(t^{(n)}, 0)\{\lim_{t_{n+1}\to 0} J_2(t^{(n+1)})\} dt^{(n)}$$

$$= \alpha(\alpha-1) \int_{R^n} \exp\{-i\langle t^{(n)}, x^{(n)}\rangle\} \phi_{X^{(n+1)}}(t^{(n)}, 0)$$

$$\times\{\int_{S_{n+1}} |\langle t^{(n)}, y^{(n)}\rangle|^{\alpha-2} y_{n+1}^2 \Gamma(dy^{(n+1)})\} dt^{(n)},$$

i.e., this limit exists and is finite, leading to $E(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) < \infty$. In this case we have

$$E(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) = \lim_{t_{n+1}\to 0} I(t_{n+1}; x^{(n)}) = \lim_{t_{n+1}\to 0} \{I_1(t_{n+1}; x^{(n)}) + I_2(t_{n+1}; x^{(n)})\},$$

which gives the final expression (2.2).

Proof of the Corollary. (a) From Theorem 1 or [10] we know that

$$E(X_{n+1}^2 \mid X_k = x_k) < \infty \quad \text{for all } x_k \in R^1$$

if and only if

$$\int_{S_2} \frac{v^2}{|u|^{2-\alpha}} \Gamma_{k,n+1}(du, dv) < \infty, \tag{4.1}$$

where $\Gamma_{k,n+1}$ is the spectral measure of the joint distribution of X_k, X_{n+1} in

$$E\exp\{i(t_k X_k + t_{n+1} X_{n+1})\} = \exp\{-\int_{S_2} |t_k u + t_{n+1} v|^\alpha \Gamma_{k,n+1}(du, dv)\}.$$

The relationship between $\Gamma_{k,n+1}$ and Γ is

$$\Gamma_{k,n+1} = \hat{\Gamma} \circ h^{-1}, \quad \text{where } \hat{\Gamma}(dy^{(n+1)}) = (y_k^2 + y_{n+1}^2)^{\alpha/2} \Gamma(dy^{(n+1)})$$

and h maps $S_{n+1} \cap \{y_k^2 + y_{n+1}^2 > 0\}$ onto S_2 by $h(y^{(n+1)}) = (y_k, y_{n+1}) \times (y_k^2 + y_{n+1}^2)^{-1/2}$ $(= (u, v))$. Now condition (4.1) is equivalent to

$$0 = \Gamma_{k,n+1}(S_2 \cap \{u = 0\}) = \hat{\Gamma}(S_{n+1} \cap \{y_k = 0\}) = \int_{S_{n+1} \cap \{y_k = 0\}} |y_{n+1}|^\alpha \Gamma(dy^{(n+1)})$$

and

$$\infty > \int_{S_2 \cap \{u \neq 0\}} \frac{v^2}{|u|^{2-\alpha}} \Gamma_{k,n+1}(du, dv) = \int_{S_{n+1} \cap \{y_k \neq 0\}} \frac{y_{n+1}^2}{|y_k|^{2-\alpha}} \Gamma(dy^{(n+1)}).$$

(b) Note from Miller [7] that $X_1, \ldots, X_n$ are jointly independent if and only if they are pairwise independent, if and only if the joint spectral measure Γ of $X_1, \ldots, X_n, X_{n+1}$ is concentrated on the n circles $y_k^2 + y_{n+1}^2 = 1$,

$k = 1, \ldots, n$, of S_{n+1}. Then the second condition in (2.5) is written as

$$\sum_{\substack{j=1\\ j\neq k}}^{n} \int_{S_{n+1}\cap\{y_j^2+y_{n+1}^2=1,\, y_j\neq 0\}} |y_{n+1}|^\alpha \Gamma(dy^{(n+1)}) = 0,$$

which is equivalent to saying that on each punctured circle $y_j^2+y_{n+1}^2 = 1$, $y_j \neq 0$, we have $y_{n+1} = 0$ a.e. $[\Gamma]$, i.e., that X_j and X_{n+1} are independent. Also, the first condition in (2.5) is equivalent to (2.6).

Proof of Theorem 2. (a) Assume that $\mathrm{E}(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) < \infty$ for all $x^{(n)} \in \boldsymbol{R}^n$, so that condition (2.1) in Theorem 1 is satisfied. It follows that for almost every $t^{(n)} \in \boldsymbol{R}^n$

$$\int_{S_{n+1}} \frac{y_{n+1}^2 \Gamma(dy^{(n+1)})}{|\langle t^{(n)}, y^{(n)}\rangle|^{2-\alpha}} < \infty. \tag{4.2}$$

Thus,

$$\Gamma(S_{n+1}\cap\{y_1 = \ldots = y_n = 0\}) < \infty, \qquad \text{i.e.,} \qquad \Gamma(S_{n+1}\cap\{y_{n+1} = \pm 1\}) < \infty.$$

Then in view of the independence of $X_1, \ldots, X_n$, the integral in (4.2) over $S_{n+1}\cap\{y^{(n)} \neq 0\}$ may be partitioned into $\bigcup_{k=1}^{n} S_{n+1}\cap\{y_k^2+y_{n+1}^2 = 1,\, y_k \neq 0\}$, leading to

$$\sum_{k=1}^{n} \frac{1}{|t_k|^{2-\alpha}} \int_{S_{n+1}\cap\{y_k^2+y_{k+1}^2=1,\, y_k\neq 0\}} \frac{y_{n+1}^2}{|y_k|^{2-\alpha}} \Gamma(dy^{(n+1)}) < \infty,$$

and thus to the conclusion.

Conversely, if (2.9) and (2.10) hold, then the double integral in condition (2.1) of Theorem 1 equals

$$\sum_{k=1}^{n} \int_{\boldsymbol{R}^n} \phi_{X^{(n)}}(t^{(n)}) \frac{1}{|t_k|^{2-\alpha}} dt^{(n)} \int_{S_{n+1}\cap\{y_k^2+y_{n+1}^2=1,\, y_k\neq 0\}} \frac{y_{n+1}^2}{|y_k|^{2-\alpha}} \Gamma(dy^{(n+1)}).$$

(b) Using the independence of $X_1, \ldots, X_n$, we can simplify the expressions appearing in (2.2) of Theorem 1 as follows:

$$\int_{S_{n+1}} \frac{y_{n+1}^2 \Gamma(dy^{(n+1)})}{|\langle t^{(n)}, y^{(n)}\rangle|^{2-\alpha}} = \sum_{k=1}^{n} \frac{1}{|t_k|^{2-\alpha}} \int_{S_{n+1}\cap\{y_k^2+y_{n+1}^2=1,\, y_k\neq 0\}} \frac{y_{n+1}^2}{|y_k|^{2-\alpha}} \Gamma(dy^{(n+1)})$$

$$\triangleq \sum_{k=1}^{n} |t_k|^{\alpha-2} b_k,$$

$$\int_{S_{n+1}} \langle t^{(n)}, y^{(n)}\rangle^{\langle\alpha-1\rangle} y_{n+1} \Gamma(dy^{(n+1)}) = \sum_{k=1}^{n} t_k^{\langle\alpha-1\rangle} \int_{S_{n+1}} y_k^{\langle\alpha-1\rangle} y_{n+1} \Gamma(dy^{(n+1)})$$

$$\triangleq \sum_{k=1}^{n} t_k^{\langle\alpha-1\rangle} a_k.$$

Also, using the inversion relation:

$$\frac{1}{2\pi f_{X_k}(x_k)} \int_{-\infty}^{\infty} \exp(it_k x_k)\exp(-\sigma_k^\alpha |t_k|^\alpha)\, dt_k = 1$$

and $\phi_{X^{(n)}}(t^{(n)}) = \prod_{j=1}^n \exp(-\sigma_j^\alpha |t_j|^\alpha)$, we can write (2.2) of Theorem 1 in the form

$$\begin{aligned} E(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) = \alpha(\alpha-1) \sum_{k=1}^n \frac{b_k}{2\pi f_{X_k}(x_k)} & \\ \times \int_{-\infty}^{\infty} \exp(-it_k x_k)\exp(-\sigma_k^\alpha |t_k|^\alpha)\, |t_k|^{\alpha-2}\, dt_k & \\ -\alpha^2 \sum_{k=1}^n \frac{a_k^2}{2\pi f_{X_k}(x_k)} & \\ \times \int_{-\infty}^{\infty} \exp(-it_k x_k)\exp(-\sigma_k^\alpha |t_k|^\alpha)\, t_k^{2(\alpha-1)} dt_k & \\ -\alpha^2 \sum\sum_{k \neq j = 1}^n \frac{a_k a_j}{(2\pi)^2 f_{X_k}(x_k) f_{X_j}(x_j)} & \\ \times \int_{-\infty}^{\infty} \exp(-it_k x_k)\exp(-\sigma^k |t_k|^\alpha)\, t_k^{\langle\alpha-1\rangle}\, dt & \\ \times \int_{-\infty}^{\infty} \exp(-it_j x_j)\exp(-\sigma_j^\alpha |t_j|^\alpha)\, t_j^{\langle\alpha-1\rangle}\, dt_j. & \end{aligned}$$

In order to simplify further, note that from (2.12) we have

$$\frac{1}{2\pi f_{X_k}(x_k)} \int_{-\infty}^{\infty} \exp(-it_k x_k)\exp(-\sigma_k^\alpha |t_k|^\alpha)\, |t_k|^{\alpha-2}\, dt_k = \frac{1}{\alpha(\alpha-1)} S_1(x_k/\sigma_k;\, \alpha)\, \sigma_k^{2-\alpha}.$$

Using $(\exp(-\sigma^\alpha |t|^\alpha))' = -\exp(-\sigma^\alpha |t|^\alpha)\, \alpha\sigma^\alpha t^{\langle\alpha-1\rangle}$ and integrating by parts, we obtain

$$\frac{1}{2\pi f_{X_k}(x_k)} \int_{-\infty}^{\infty} \exp(-it_k x_k)\exp(-\sigma_k^\alpha |t_k|^\alpha)\, t_k^{\alpha-1}\, dt_k = -i\frac{x_k}{\alpha\sigma_k^\alpha}.$$

Also, integrating out $(\exp(-\sigma^\alpha |t|^\alpha))'' = \exp(-\sigma^\alpha |t|^\alpha)\, \alpha^2 \sigma^{2\alpha} t^{2(\alpha-1)} - \exp(-\sigma^\alpha |t|^\alpha) \times \alpha(\alpha-1)\sigma^\alpha |t|^{\alpha-2}$ and using integration by parts twice, we find

$$\begin{aligned} &\frac{\alpha^2}{2\pi f_{X_k}(x_k)} \int_{-\infty}^{\infty} \exp(-it_k x_k)\exp(-\sigma_k^\alpha |t_k|^\alpha)\, t_k^{2(\alpha-1)} dt_k \\ &= \frac{\alpha(\alpha-1)}{\sigma_k^\alpha} \frac{1}{2\pi f_{X_k}(x_k)} \int_{-\infty}^{\infty} \exp(-it_k x_k)\exp(-\sigma_k^\alpha |t_k|^\alpha)\, |t_k|^{\alpha-2}\, dt_k - x_k^2/\sigma_k^{2\alpha} \\ &= \sigma_k^{2-2\alpha} S_1(x_k/\sigma_k;\, \alpha) - x_k^2/\sigma_k^{2\alpha}. \end{aligned}$$

It follows that

$$(4.3)\quad \mathrm{E}(X_{n+1}^2 \mid X^{(n)} = x^{(n)}) = \sum_{k=1}^{n} b_k \sigma_k^{2-\alpha} S_1(x_k/\sigma_k;\, \alpha)$$

$$-\sum_{k=1}^{n} a_k^2 \left\{\sigma_k^{2-2\alpha} S_1\left(\frac{x_k}{\sigma_k};\, \alpha\right) - \frac{x_k^2}{\sigma_k^{2\alpha}}\right\} + \sum \sum_{k \neq j = 1}^{n} \frac{a_k x_k}{\sigma_k^\alpha} \frac{a_j x_j}{\sigma_j^\alpha}$$

$$= \sum_{k=1}^{n} (b_k \sigma_k^{2-\alpha} - a_k^2 \sigma_k^{2-2\alpha}) S_1\left(\frac{x_k}{\sigma_k};\, \alpha\right) + \sum \sum_{k,j=1}^{n} \frac{a_k x_k}{\sigma_k} \frac{a_j x_j}{\sigma_j}.$$

Likewise, the expression (2.8) for $\mathrm{E}(X_{n+1} \mid X^{(n)} = x^{(n)})$ can be simplified as follows:

$$(4.4)\quad \mathrm{E}(X_{n+1} \mid X^{(n)} = x^{(n)}) = i\alpha \sum_{k=1}^{n} \frac{a_k}{2\pi f_{X_k}(x_k)}$$

$$\times \int_{-\infty}^{\infty} \exp(-it_k x_k) \exp(-\sigma_k^\alpha |t_k|^\alpha)\, t_k^{\langle \alpha - 1 \rangle} dt_k$$

$$= \sum_{k=1}^{n} \frac{a_k x_k}{\sigma_k}.$$

Now (4.3) and (4.4) imply (2.11) and (2.13).

Proof of Theorem 3. From (3.1) we have

$$\int_{S_{n+1}} |\langle s^{(n+1)}, y^{(n+1)} \rangle|^\alpha \Gamma(dy^{(n+1)}) = 2^{-\alpha/2} \langle s^{(n+1)}, \Sigma_{n+1} s^{(n+1)} \rangle^{\alpha/2},$$

and differentiating twice with respect to s_{n+1}, we find:

$$\int_{S_{n+1}} \langle s^{(n+1)}, y^{(n+1)} \rangle^{\langle \alpha - 1 \rangle} y_{n+1} \Gamma(dy^{(n+1)})$$

$$= 2^{-\alpha/2} \langle s^{(n+1)}, \Sigma_{n+1} s^{(n+1)} \rangle^{\alpha/2 - 1} \langle s^{(n+1)}, \Sigma_{n+1,\cdot} \rangle,$$

where $\Sigma_{n+1,\cdot}$ is the $(n+1)$-st row of Σ_{n+1}, and

$$2^{\alpha/2} \int_{S_{n+1}} \frac{y_{n+1}^2 \Gamma(dy^{(n+1)})}{|\langle s^{(n+1)}, y^{(n+1)} \rangle|^{2-\alpha}}$$

$$= -(2-\alpha) \langle s^{(n+1)}, \Sigma_{n+1} s^{(n+1)} \rangle^{\alpha/2 - 2} \langle s^{(n+1)}, \Sigma_{n+1,\cdot} \rangle^2$$

$$+ \langle s^{(n+1)}, \Sigma_{n+1} s^{(n+1)} \rangle^{\alpha/2 - 1} R(t_{n+1}, t_{n+1})$$

$$\leqslant \frac{R(t_{n+1}, t_{n+1})}{\langle s^{(n+1)}, \Sigma_{n+1} s^{(n+1)} \rangle^{1 - \alpha/2}}.$$

Putting $s_{n+1} = 0$, we find that the left-hand side of (2.1) is upper-bounded by

$$2^{-\alpha/2} R(t_{n+1}, t_{n+1}) \int_{R^n} \exp\left\{-\frac{1}{2^{\alpha/2}} \langle s^{(n)}, \Sigma_n s^{(n)}\rangle^{\alpha/2}\right\} \langle s^{(n)}, \Sigma_n s^{(n)}\rangle^{\alpha/2-1} ds^{(n)},$$

which is shown to be finite as in (4.3) of [2]. This establishes (2.1).

To find the expression of the conditional second moment and variance in this case, we may use a direct argument as follows, instead of (2.2). We have

$$\begin{aligned} \mathrm{E}(X^2(t_{n+1}) \mid X^{(n)}) &= \mathrm{E}\{\mathrm{E}(AG^2(t_{n+1}) \mid A, G^{(n)}) \mid X^{(n)}\} \\ &= \mathrm{E}\{A\mathrm{E}(G^2(t_{n+1}) \mid G^{(n)}) \mid X^{(n)}\} \end{aligned}$$

and for the Gaussian process G:

$$\begin{aligned} \mathrm{E}(G^2(t_{n+1}) \mid G^{(n)}) &= \mathrm{Var}(G(t_{n+1}) \mid G^{(n)}) + [\mathrm{E}(G(t_{n+1}) \mid G^{(n)})]^2 \\ &= [R(t_{n+1}, t_{n+1}) - \Sigma_{n+1,n}\Sigma_n^{-1}\Sigma'_{n+1,n}] + \langle a^{(n)}, G^{(n)}\rangle^2. \end{aligned}$$

Thus,

$$\begin{aligned} &\mathrm{E}\{X^2(t_{n+1}) \mid X^{(n)}\} \\ &\qquad = [R(t_{n+1}, t_{n+1}) - \Sigma_{n+1,n}\Sigma_n^{-1}\Sigma'_{n+1,n}]\mathrm{E}(A \mid X^{(n)}) + \langle a^{(n)}, X^{(n)}\rangle^2 \end{aligned}$$

and

(4.5) $$\mathrm{Var}\{X(t_{n+1}) \mid X^{(n)}\} = [R(t_{n+1}, t_{n+1}) - \Sigma_{n+1,n}\Sigma_n^{-1}\Sigma'_{n+1,n}]\,\mathrm{E}(A \mid X^{(n)}).$$

To find $\mathrm{E}(A \mid X^{(n)})$ we now use a standard argument. For all $u \geqslant 0$ and $v^{(n)} \in R^n$ we have

$$\begin{aligned} \mathrm{E}\exp\{-uA + i\langle v^{(n)}, X^{(n)}\rangle\} &= \mathrm{E}\exp\{-uA + iA^{1/2}\langle v^{(n)}, G^{(n)}\rangle\} \\ &= \mathrm{E}\exp\{-uA - \tfrac{1}{2}A\langle v^{(n)}, \Sigma_n v^{(n)}\rangle\} = \exp\{-[u + \tfrac{1}{2}\langle v^{(n)}, \Sigma_n v^{(n)}\rangle]^{\alpha/2}\}. \end{aligned}$$

Differentiating with respect to u and putting $u = 0$ we obtain

$$\mathrm{E}\{A\exp\{i\langle v^{(n)}, X^{(n)}\rangle\}\} = \frac{\alpha}{2}[\tfrac{1}{2}\langle v^{(n)}, \Sigma_n v^{(n)}\rangle]^{\alpha/2-1}\exp\{-[\tfrac{1}{2}\langle v^{(n)}, \Sigma_n v^{(n)}\rangle]^{\alpha/2}\}.$$

Since the left-hand side can also be written as $\mathrm{E}\{\exp\{i\langle v^{(n)}, X^{(n)}\rangle\}\,\mathrm{E}(A \mid X^{(n)})\}$, it follows that

$$\begin{aligned} \mathrm{E}(A \mid X^{(n)} = x^{(n)}) = \frac{1}{(2\pi)^n f_{X^{(n)}}(x^{(n)})} &\int_{R^n} \exp\{-i\langle v^{(n)}, x^{(n)}\rangle\} \\ &\times \frac{\alpha}{2^{\alpha/2}}\langle v^{(n)}, \Sigma_n v^{(n)}\rangle^{\alpha/2-1}\exp\{-2^{-\alpha/2}\langle v^{(n)}, \Sigma_n v^{(n)}\rangle^{\alpha/2}\}\,dv^{(n)}. \end{aligned}$$

Likewise,

$$f_{X^{(n)}}(x^{(n)}) = \frac{1}{(2\pi)^n}\int_{R^n} \exp\{-i\langle v^{(n)}, x^{(n)}\rangle\}\exp\{-2^{-\alpha/2}\langle v^{(n)}, \Sigma_n v^{(n)}\rangle^{\alpha/2}\}\,dv^{(n)},$$

so, finally,

$$\mathrm{E}(A \mid X^{(n)} = x^{(n)})$$
$$= \frac{\alpha}{2^{\alpha/2}} \frac{\int_{\mathbf{R}^n} \exp\{-i\langle v^{(n)}, x^{(n)}\rangle - 2^{-\alpha/2}\langle v^{(n)}, \Sigma_n v^{(n)}\rangle^{\alpha/2}\}\langle v^{(n)}, \Sigma_n v^{(n)}\rangle^{\alpha/2-1}\, dv^{(n)}}{\int_{\mathbf{R}^n} \exp\{-i\langle v^{(n)}, x^{(n)}\rangle - 2^{-\alpha/2}\langle v^{(n)}, \Sigma_n v^{(n)}\rangle^{\alpha/2}\}\, dv^{(n)}}.$$

Now let $\frac{1}{2}\Sigma_n = B'B$ be a full rank factorization and put $u^{(n)} = Bv^{(n)}$ and $r = \|u^{(n)}\|$. The above expression is then written as follows:

$$\mathrm{E}(A \mid X^{(n)} = x^{(n)}) = \frac{\alpha}{2} \frac{\int_{\mathbf{R}^n} \exp\{-i\langle B^{-1}u^{(n)}, x^{(n)}\rangle - r^{\alpha}\}\, r^{\alpha-2}\, du^{(n)}}{\int_{\mathbf{R}^n} \exp\{-i\langle B^{-1}u^{(n)}, x^{(n)}\rangle - r^{\alpha}\}\, du^{(n)}}.$$

Changing variables from $u^{(n)} \in \mathbf{R}^n$ to $r \geqslant 0$ and $v^{(n)} \in S_n$, we have, with γ_n being surface measure on S_n,

(4.6) $$\mathrm{E}(A \mid X^{(n)} = x^{(n)})$$
$$= \frac{\alpha}{2} \frac{\int_0^\infty dr\, r^{n+\alpha-3} \exp(-r^{\alpha}) \int_{S_n} \gamma_n(dv^{(n)}) \exp\{-ir\langle B^{-1}v^{(n)}, x^{(n)}\rangle\}}{\int_0^\infty dr\, r^{n-1} \exp(-r^{\alpha}) \int_{S_n} \gamma_n(dv^{(n)}) \exp\{-ir\langle B^{-1}v^{(n)}, x^{(n)}\rangle\}}.$$

Finally, changing coordinates with $y^{(n)} = r(B^{-1})'x^{(n)}$, we have for $n = 1$

$$\int_{S_1} \exp\{-ivy\}\, \gamma_1(dv) = \cos(|y|),$$

leading to the expression of the conditional variance in [10] and in (3.2), (3.4), and for $n \geqslant 2$

(4.7) $$\int_{S_n} \exp\{-i\langle v^{(n)}, y^{(n)}\rangle\}\, \gamma_n(dv^{(n)}) = \int_0^\pi \exp\{-i\,\|y^{(n)}\|\cos\theta\}(\sin\theta)^{n-2}\, d\theta$$
$$= \int_0^\pi \cos[\|y^{(n)}\|\cos\theta](\sin\theta)^{n-2}\, d\theta,$$

where $\|y^{(n)}\|^2 = 2r^2\langle x^{(n)}, \Sigma_n^{-1}x^{(n)}\rangle$. The final expression (3.2)–(3.3) in Theorem 3 follows from (4.6) and (4.7).

Proof of Theorem 4. Following the same steps as in the proof of Theorem 3 leads to

$$\int_{S_{n+1}} \frac{|y_{n+1}^2|^{\alpha}\Gamma(dy^{(n+1)})}{|\langle s^{(n)}, y^{(n)}\rangle|^{2-\alpha}} \leqslant 2^{-\alpha/2} \int_{-\infty}^{\infty} \langle s^{(n)}, \Sigma_n(\lambda)s^{(n)}\rangle^{\alpha/2-1}\, dm(\lambda).$$

Then, condition (2.1) of Theorem 1 may be shown by applying the same arguments as in the proof of Theorem 3 in [2].

Proof of Theorem 5. By (3.7) we have

$$\int_{S_{n+1}} |\langle s^{(n+1)}, y^{(n+1)}\rangle|^{\alpha}\Gamma(dy^{(n+1)}) = \int_{-\infty}^{\infty} \Big|\sum_{k=1}^{n+1} s_k f(t_k - u)\Big|^{\alpha} du.$$

Differentiating twice with respect to s_{n+1} and plugging in (2.1), we obtain (3.8). Now the linear independence of $X(t_k)$, $k = 1, \ldots, n$, by Lemma 2 implies

$$\phi_{X^{(n)}}(s^{(n)}) \leqslant \exp\{-c\,\|s^{(n)}\|^{\alpha}\} \quad \text{for all } s^{(n)} \in \boldsymbol{R}^n.$$

Thus, a sufficient condition for (3.8) is

$$\int_{\boldsymbol{R}^n} ds^{(n)} \exp\{-c\,\|s^{(n)}\|^{\alpha}\} \int_{-\infty}^{\infty} \frac{f^2(t_{n+1}-v)\,dv}{\left|\sum_{k=1}^{n} s_k f(t_k-v)\right|^{\alpha}} < \infty,$$

and going to polar coordinates this reduces to (3.9).

When $f(t) = e^{-\lambda|t|}$, in order to check condition (3.9), it suffices to check the finiteness of the integrals over $(-\infty, t_{n+1}-t_n)$ and $(t_{n+1}-t_1, \infty)$, the remaining integrals being clearly finite. But

$$\int_{t_{n+1}-t_1}^{\infty} \frac{e^{-2\lambda u}\,du}{\left\{\sum_{k=1}^{n} \exp\{-2\lambda(u-t_{n+1}+t_k)\}\right\}^{1-\alpha/2}}$$
$$= \left\{\sum_{k=1}^{n} \exp\{2\lambda(t_{n+1}-t_k)\}\right\}^{\alpha/2-1} \int_{t_{n+1}-t_1}^{\infty} e^{-\alpha\lambda u}\,du < \infty,$$

and likewise for the integral over $(-\infty, t_{n+1}-t_n)$.

5. AUXILIARY RESULTS

Now we collect all the lemmas which are used in the proofs of the theorems.

LEMMA 1 ([2], Lemma 4). *For all $1 < \alpha < 2$, $\alpha < r < 2$ and $z \in \boldsymbol{R}$, the following inequalities are true:*

(i) $\big||1+z|^{\alpha}-1\big| \leqslant \alpha|z|^{\alpha}+\alpha|z|$,

(ii) $0 \leqslant |1+z|^{\alpha}+|1-z|^{\alpha}-2 \leqslant c_1|z|^r$,

where c_1 is a positive constant depending only on α.

The following property of multivariate SαS characteristic functions is used.

LEMMA 2 ([9], Lemma 2.1). *If $X_1, \ldots, X_n$ are linearly independent, i.e., for all $t^{(n)} \in \boldsymbol{R}^n$,*

$$\Gamma\{s^{(n)} \in S_n:\ \langle t^{(n)}, s^{(n)}\rangle \neq 0\} > 0,$$

then for all $t^{(n+1)} = (t^{(n)}, t_{n+1}) \in \boldsymbol{R}^{n+1}$,

$$\phi_{X^{(n+1)}}(t^{(n+1)}) \leqslant \exp\{-c_2\,\|t^{(n)}\|^{\alpha}\} \exp\{c_3|t_{n+1}|^{\alpha}\}$$

for some positive constants c_2, c_3; and $\int_{\boldsymbol{R}^n} |\phi_{X^{n+1}}(t^{(n)}, t_{n+1})|\,dt^{(n)} < \infty$ for all $t_{n+1} \in \boldsymbol{R}$.

LEMMA 3. *Let $\alpha \in (1, 2]$ and*

$$I = \int_{\boldsymbol{R}^n} \exp(-c\,\|t^{(n)}\|^{\alpha}) \int_{S_{n+1}} \frac{|y_{n+1}|^r\,\Gamma(dy^{(n+1)})}{|\langle t^{(n)}, y^{(n)}\rangle|^{r-\alpha}}\,dt^{(n)},$$

where $c \in (0, \infty)$. *Then* $I < \infty$ *for* $r \in [0, \alpha]$; $I < \infty$ *for* $r \in (\alpha, \alpha+1)$ *if and only if*

$$\int_{S_{n+1}} \frac{|y_{n+1}|^r}{\|y^{(n)}\|^{r-\alpha}} \Gamma(dy^{(n+1)}) < \infty;$$

and $I = \infty$ *for* $r \in [\alpha+1, \infty)$.

Proof. Using Fubini–Tonelli's theorem we can write

$$I = \int_{S_{n+1}} |y_{n+1}|^r \left\{ \int_{R^n} \frac{\exp(-c\,\|t^{(n)}\|^{\alpha})}{|\langle t^{(n)}, y^{(n)}\rangle|^{r-\alpha}} dt^{(n)} \right\} \Gamma(dy^{(n+1)}).$$

Rotating the axes in R^n so that one of them is along $y^{(n)}$, and thus

$$\langle t^{(n)}, y^{(n)} \rangle = \|t^{(n)}\| \, \|y^{(n)}\| \cos(t^{(n)}, y^{(n)}),$$

and using polar coordinates in the inner integral over R^n, we obtain

$$I = c' \int_{S_{n+1}} \frac{|y_{n+1}|^r}{\|y^{(n)}\|^{r-\alpha}} \Gamma(dy^{(n+1)}) \int_0^{\infty} \exp(-c\varrho^{\alpha}) \varrho^{n-1-r+\alpha} d\varrho \int_0^{\pi} \frac{(\sin\theta)^{n-2}}{|\cos\theta|^{r-\alpha}} d\theta.$$

The ϱ integral is finite if and only if $n-1-r+\alpha > -1$, and likewise the θ integral is finite if and only if $r-\alpha < 1$. Also the integral over S_{n+1} is automatically finite when $0 \leqslant r \leqslant \alpha$. The result then follows. ■

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Department of Statistics
University of North Carolina
Chapel Hill, NC 27599-3260, U.S.A.

Department of Management and Systems
College of Business and Economics
Washington State University
Pullman, WA 99164-4726, U.S.A.

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