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ON THE KOLMOGOROV QUASIMARTINGALE PROPERTY

BY

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Abstract. Let (X_k) be a sequence of real-valued random variables (r.v.), which are centered, square integrable and independent. A well-known result, due to Kolmogorov, states that if

$$\text{(i)} \qquad \sum_{k\geq 1} \frac{\mathrm{E}(X_k^2)}{k^2} < +\infty,$$

then (S_n/n) converges almost surely (a.s.) to 0, where $S_n = X_1 + \ldots + X_n$.

This paper is devoted to the interpretation of condition (i). For instance, it is shown that if the r.v. X_k are weighted Rademacher r.v., then (i) is equivalent to the fact that $((S_n/n)^2, \mathcal{G}_n)$ is a quasimartingale ($\mathcal{G}_n$ being the natural filtration associated with the sequence (X_n)).

The problem of the interpretation of (i) for Banach space valued r.v. X_k is also studied.

0. Kolmogorov's theorem is the most famous strong law of large numbers (SLLN) for a sequence of not identically distributed r.v., it is stated as follows:

THEOREM 0.1 (Kolmogorov [7]). *If (X_k) is a sequence of real-valued random variables (r.v.), which are independent, centered, square integrable and such that*

$$(0.1) \qquad \sum_{k\geq 1} \frac{\mathrm{E}(X_k^2)}{k^2} < +\infty,$$

then (X_k) satisfies the SLLN:

$$(0.2) \qquad \frac{S_n}{n} = \frac{X_1 + \ldots + X_n}{n} \to 0 \ a.s.$$

Hypothesis (0.1) appears as a tool which measures how fast the sequence (S_n/n) converges to 0; this can – for instance – be seen by applying the Hajek–Rényi inequality [4] for bounding

$$\theta_k(t) = P\left(\sup_{n\geq k}\left|\frac{S_n}{n}\right| > t\right).$$

From this point of view, (0.1) seems only to be a — rather technical and rough — bound for $\theta_k(t)$. A more satisfactory interpretation of assumption (0.1), as, say, an intrinsic property of the sequence (S_n/n), would be welcome!

Intuitively, as (S_n/n) is derived by Kronecker's transform from the martingale $(S_n, \mathscr{G}_n)$, where $\mathscr{G}_n = \sigma(X_1, \ldots, X_n)$, the expected "intrinsic property" would be that (S_n/n) has some "generalized martingale" property. We are going to try to guess which one, using a few informal lines and naive discussion.

Suppose that the assumptions of Theorem 0.1 are fulfilled. For every integer n, we denote by $\mathscr{G}_n$ the σ-field generated by $X_1, \ldots, X_n$. First observe that $(S_n/n, \mathscr{G}_n)$ is not a martingale. The first classical less restrictive "generalized martingale" property is the quasimartingale property (see, e.g., [2], p. 323). We recall briefly its definition:

Definition 0.2. Assume that (X_n) is a sequence of integrable r.v., and $(\mathscr{H}_n)$ is an increasing sequence of σ-fields. Then $(X_n, \mathscr{H}_n)$ is a *quasimartingale* if the following hold:

(i) $\forall n \in N^*$, X_n is $\mathscr{H}_n$-measurable;

(ii) $\sup \mathrm{E}|X_n| < +\infty$;

(iii) $\sum_{n \geq 1} \mathrm{E}|\mathrm{E}(X_{n+1} | \mathscr{H}_n) - X_n| < +\infty$.

Remark. Notice that if the r.v. X_n are positive, then property (ii) follows from (iii). A very small amount of information on quasimartingales we will need in the sequel can be found in [1] or [2].

If $(S_n/n, \mathscr{G}_n)$ were a quasimartingale, we would have

$$\sum_{n \geq 1} n^{-2} \mathrm{E}|S_n| < +\infty, \tag{0.3}$$

which is more restrictive than (0.2). However, an easy computation shows that

$$\forall n, \quad \sum_{1 \leq k \leq n} \frac{1}{k^2} \mathrm{E}|S_k| \leq 2\sqrt{1+\ln n} \sqrt{\sum_{1 \leq k \leq n} \frac{\mathrm{E}(X_k^2)}{k^2}}, \tag{0.4}$$

and so (0.3) seems closer to (0.1) than we would think at first glance!

This remark makes reasonable the hope that for some $p > 1$ the sequence $(|S_n/n|^p)$ — which for large n is small with large probability — will have a good "generalized martingale" property.

From this naive discussion we will now get an intrinsic explanation of assumption (0.1), the explanation which will be complete for scalar valued r.v. and partial in the infinite-dimensional setting.

Before stating the main definition we need some notation.

We denote by $(B, \| \ \|)$ a real separable Banach space, equipped with its Borel σ-field $\mathscr{B}$. In the sequel, (ε_k) always stands for a sequence of independent Rademacher r.v. ($P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = \frac{1}{2}$). For every integer n, by $\mathscr{F}_n$ we denote the σ-field generated by $\varepsilon_1, \ldots, \varepsilon_n$.

DEFINITION 0.3. (1) We say that the Banach space $(B, \| \ \|)$ has the *property* K1 if, for every sequence (x_n) of elements of B such that

$$\sum_{k\geqslant 1}\frac{\|x_k\|^2}{k^2} < +\infty, \tag{0.5}$$

$(\|S_n\|^2/n^2, \mathscr{F}_n)$ is a quasimartingale, where $S_n = \sum_{1\leqslant k\leqslant n}\varepsilon_k x_k$.

(2) We say that the Banach space $(B, \| \ \|)$ has the *property* K2 if for every sequence (x_n) of elements of B such that $(\|S_n/n\|^2, \mathscr{F}_n)$ is a quasimartingale (0.5) holds true.

(3) A Banach space $(B, \| \ \|)$ having both the properties K1 and K2 is said to have *Kolmogorov's quasimartingale property* (in short, Kqm-property).

The present paper is devoted to the study of these three properties, which shed a new light on the nature of hypothesis (0.1). In Section 1, it is shown that the real line has the Kqm-property. In Sections 2 and 3, the case of infinite-dimensional space valued r.v. is considered; for instance, it is shown that a Banach space has the property K1 if and only if it is of type 2 and that an infinite-dimensional Hilbert space does not have the Kqm-property.

Finally, in the Appendix, an exponential lower bound (which will be a crucial tool in Section 1), due to Ledoux and Talagrand [10] and Montgomery-Smith [11], is recalled.

1. The real line has the Kqm-property. In this section we will give a complete explanation of Theorem 0.1 in the case of weighted Rademacher r.v.:

THEOREM 1.1. *The space* $\boldsymbol{R}$ *of the real numbers has the* Kqm-*property.*

Proof. Let us start with the easy part:

LEMMA 1.2. *The space* $\boldsymbol{R}$ *has the property* K1.

For showing that if $\sum_{k\geqslant 1} x_k^2/k^2 < +\infty$, then $(|S_n/n|^2, \mathscr{F}_n)$ is a quasimartingale, we have to verify that

$$\sum_{n\geqslant 1} \mathrm{E}\left|\mathrm{E}\left(\left|\frac{S_{n+1}}{n+1}\right|^2 \mid \mathscr{F}_n\right) - \frac{S_n^2}{n^2}\right| < +\infty. \tag{1.1}$$

It is easy to see that

$$\left|\frac{S_n}{n}\right|^2 - \mathrm{E}\left\{\left(\frac{S_{n+1}}{n+1}\right)^2 \mid \mathscr{F}_n\right\} = \frac{2n+1}{n^2(n+1)^2}S_n^2 - \frac{x_{n+1}^2}{(n+1)^2}.$$

Thus, by the assumption made on the sequence (x_k), it remains only to show

$$\sum_{n\geqslant 1} n^{-3}\,\mathrm{E}(S_n^2) < +\infty. \tag{1.2}$$

An easy computation gives

$$\sum_{n\geqslant 1} n^{-3}\,\mathrm{E}(S_n^2) \leqslant C_1 \sum_{k\geqslant 1} x_k^2/k^2 < +\infty,$$

and this completes the proof of Lemma 1.2.

For the proof of Theorem 1.1, it remains to show the following more interesting lemma:

LEMMA 1.3. *The space* $\boldsymbol{R}$ *has the property* K2.

Let (x_k) be a sequence of real numbers such that $((S_n/n)^2, \mathscr{F}_n)$ is a quasimartingale. The first technical tool that will be needed later is the following

LEMMA 1.4. *There exist a sequence* (δ_k) *of positive numbers and a positive constant* C_2 *such that*

(i) $\sum_{k\geqslant 1}\delta_k < +\infty$;

(ii) $\forall k,\ x_k^2/k^2 \leqslant C_2 \sup(1/k,\ \delta_k)$.

Proof of Lemma 1.4. For every integer n, we denote by μ_n a median of

$$\frac{(2n+1)\,S_n^2}{n^2\,(n+1)^2}.$$

By the quasimartingale convergence theorem (see [1], § VI.38), the sequence $((S_n/n)^2)$ converges a.s. to a random variable Z, which, according to Kolmogorov's 0-1 law, has to be a constant. Consequently, there exists a positive constant C_3 such that, for all n, $\mu_n \leqslant C_3/n$. By assumption, we have

$$\sum_{n\geqslant 1} \mathrm{E}\left|\frac{(2n+1)\,S_n^2}{n^2\,(n+1)^2} - \frac{x_{n+1}^2}{(n+1)^2}\right| < +\infty; \tag{1.3}$$

so, by a well-known property of the median,

$$\sum_{n\geqslant 1} \mathrm{E}\left|\frac{(2n+1)\,S_n^2}{n^2\,(n+1)^2} - \mu_n\right| < +\infty,$$

and also

$$\sum_{n\geqslant 1} \left|\frac{x_{n+1}^2}{(n+1)^2} - \mu_n\right| < +\infty. \tag{1.4}$$

Thus Lemma 1.4 follows immediately from property (1.4).

Let us define

$$t = [2\sqrt{2}\,C+1],$$

where, as usual, [ ] denotes the integer part of a real number, and C is the constant involved in Theorem 4.2 below. Denote by M the integer t^2.

The next step in the proof of Lemma 1.3 is the following

LEMMA 1.5. *For every integer* $n > n_0 = \sup(M, 3)$ *we define the following sets of integers*:

$$U(n) = \{1, \ldots, n\},$$

$$U_1(n) = U(n) \cap \{k\colon \sup(1/k, \delta_k) = 1/k\}, \qquad U_2(n) = U(n) \setminus U_1(n).$$

Denote by $\alpha_1(n), \ldots, \alpha_M(n)$ *(respectively,* $\beta_1(n), \ldots, \beta_M(n)$*) the non-increasing rearrangement of the sequence* $(|x_k|, k \in U_1(n))$ *(respectively,* $(|x_k|, k \in U_2(n))$*) stopped at the index* M. *Then the following conditions hold*:

(i) $\sum_{n \geqslant n_0} n^{-3}(\alpha_1^2(n) + \ldots + \alpha_M^2(n)) < +\infty$;

(ii) $\sum_{n \geqslant n_0} n^{-3}(\beta_1^2(n) + \ldots + \beta_M^2(n)) < +\infty$.

Proof of Lemma 1.5. Let us first prove (i). By the definition of $U_1(n)$, we have

$$n^{-3}(\alpha_1^2(n) + \ldots + \alpha_M^2(n)) \leqslant C_4/n^2,$$

so (i) holds.

To see (ii) observe that

$$\sum_{n \geqslant n_0} n^{-3}(\beta_1^2(n) + \ldots + \beta_M^2(n)) \leqslant C_5 \sum_{n \geqslant n_0} n^{-3} \sum_{k \in U(n)} k^2 \delta_k.$$

Property (ii) then follows easily from the inequalities

$$\sum_{n \geqslant n_0} n^{-3} \sum_{1 \leqslant k \leqslant n} k^2 \delta_k \leqslant \sum_{k \geqslant 1} k^2 \delta_k \sum_{n \geqslant k} n^{-3} \leqslant C_6 \sum_{k \geqslant 1} \delta_k < +\infty.$$

From Lemma 1.5 we get

LEMMA 1.6. *For every* n *we denote by* $z_1(n), \ldots, z_n(n)$ *the non-increasing rearrangement of the sequence* $(|x_1|, \ldots, |x_n|)$. *Then for every* $n \geqslant n_0$ *we have*

$$\frac{2n+1}{n^2(n+1)^2} \sum_{1 \leqslant k \leqslant M} z_k^2(n) \leqslant u_n,$$

where u_n *is the general term of a convergent series.*

By the application of the Ledoux and Talagrand and Montgomery-Smith exponential lower bound (see the Appendix), we get

$$P\left(\frac{\sqrt{2n+1}}{n(n+1)} \sum_{1 \leqslant k \leqslant n} \varepsilon_k x_k > \frac{1}{C} K_{1,2}(x, t)\right) \geqslant \frac{1}{C} \exp(-Ct^2). \tag{1.5}$$

From Holmstedt's result (4.2) recalled in the Appendix and from Lemma 1.6 it follows that

$$\frac{1}{C} K_{1,2}(x, t) \geqslant \frac{t}{C}\left(\sqrt{\frac{2n+1}{n^2(n+1)^2} \sum_{1 \leqslant k \leqslant n} x_k^2} - \sqrt{u_n}\right).$$

Now we consider two cases.

Case 1. We have

$$\frac{t}{C}\left(\sqrt{\frac{2n+1}{n^2(n+1)^2}\sum_{1\leqslant k\leqslant n} x_k^2}-\sqrt{u_n}\right)\geqslant\sqrt{\frac{2x_{n+1}^2}{(n+1)^2}}. \tag{1.6}$$

Then it follows from (1.5) that

$$P\left(\sqrt{\frac{2n+1}{n(n+1)}}\left|\sum_{1\leqslant k\leqslant n}\varepsilon_k x_k\right|>\sqrt{\frac{2x_{n+1}^2}{(n+1)^2}}\right)\geqslant\frac{1}{C}\exp(-Ct^2).$$

Denoting by I the set of integers n for which (1.6) holds, we get from (1.3) the relation

$$\sum_{n\in I}\frac{x_{n+1}^2}{(n+1)^2}<+\infty.$$

Case 2. First observe that, by the choice made for t, for each $n\in I^c$ we have

$$\frac{2n+1}{n^2(n+1)^2}\sum_{1\leqslant k\leqslant n} x_k^2\leqslant 2u_n+\frac{x_{n+1}^2}{4(n+1)^2}. \tag{1.7}$$

Now denote by A the set of elements of I^c for which

$$u_n\leqslant\frac{x_{n+1}^2}{8(n+1)^2},$$

and by B the set $I^c\backslash A$. It follows immediately from (1.3) and (1.7) that

$$\sum_{n\in A}\frac{x_{n+1}^2}{(n+1)^2}<+\infty.$$

Finally, u_n being the general term of a convergent series, we also obtain

$$\sum_{n\in B}\frac{x_{n+1}^2}{(n+1)^2}<+\infty.$$

This completes the proof of Lemma 1.3 and, consequently, the real line has the Kqm-property.

Several authors have extended Theorem 0.1 to r.v. taking their values in a real separable Banach space $(B, \|\ \|)$ equipped with its Borel σ-field $\mathscr{B}$ (see [5], [8] and [10], Chapter 7 and the references given therein). Their results incite to study the connection between the geometric properties of a Banach space $(B, \|\ \|)$ and the properties K1, K2 or Kqm-property! The sequel of this paper is devoted to that connection.

2. The type 2 property and the property K1. In this section we will study the connection between the property K1 and the type 2 property. Let us recall the definition of the type 2 (and also that of the cotype 2, which will be used in the next section):

DEFINITION 2.1. Let $(B, \| \ \|)$ be a real separable Banach space and let, as above, (ε_k) be a sequence of independent Rademacher r.v.

(1) The space $(B, \| \ \|)$ is of *type* 2 if there exists a constant $C_8 > 0$ such that

$$\forall n \in N^*, \ \forall (x_1, \ldots, x_n) \in B^n, \ \mathrm{E}\Big\| \sum_{1 \leqslant k \leqslant n} \varepsilon_k x_k \Big\|^2 \leqslant C_8 \sum_{1 \leqslant k \leqslant n} \|x_k\|^2.$$

(2) The space $(B, \| \ \|)$ is of *cotype* 2 if there exists a constant $C_9 > 0$ such that

$$\forall n \in N^*, \ \forall (x_1, \ldots, x_n) \in B^n, \ \mathrm{E}\Big\| \sum_{1 \leqslant k \leqslant n} \varepsilon_k x_k \Big\|^2 \geqslant C_9 \sum_{1 \leqslant k \leqslant n} \|x_k\|^2.$$

We refer the reader to [10] (Chapter 9) for further information on type and cotype; we only recall the Hoffmann-Jørgensen and Pisier theorem [5], on which the property K1 will shed some new light:

THEOREM 2.2. *Let $(B, \| \ \|)$ be a real separable Banach space and let (ε_k) be a sequence of independent Rademacher r.v. The following two properties are equivalent*:

(1) *$(B, \| \ \|)$ is of type* 2.

(2) *For every sequence (x_k) of elements of B such that*

$$\sum_{k \geqslant 1} \frac{\|x_k\|^2}{k^2} < +\infty, \tag{2.1}$$

the sequence $S_n/n = \sum_{1 \leqslant k \leqslant n} \varepsilon_k x_k/n$ converges a.s. to 0.

This famous result can be precised in the following way:

THEOREM 2.3. *For a real separable Banach space $(B, \| \ \|)$, the following two properties are equivalent*:

(1) *$(B, \| \ \|)$ has the property* K1.

(2) *$(B, \| \ \|)$ is of type* 2.

(2)$\Rightarrow$(1). Suppose that (x_k) is a sequence of elements of B such that (2.1) holds. As above, we denote by S_n the sum $\sum_{1 \leqslant k \leqslant n} \varepsilon_k x_k$, and by $\mathscr{F}_n$ the σ-field generated by $(\varepsilon_1, \ldots, \varepsilon_n)$. Then, by the conditional Jensen inequality,

$$\mathrm{E}(\|S_{n+1}\|^2 \mid \mathscr{F}_n) \geqslant \|S_n\|^2.$$

Consequently,

$$\left|\mathrm{E}\left\{\frac{\|S_{n+1}\|^2}{(n+1)^2}\mid \mathscr{F}_n\right\}-\frac{\|S_n\|^2}{n^2}\right| \leqslant \frac{2n+1}{n^2(n+1)^2}\|S_n\|^2+\left|\mathrm{E}\left\{\frac{\|S_{n+1}\|^2-\|S_n\|^2}{(n+1)^2}\mid \mathscr{F}_n\right\}\right|$$

$$= \frac{2n+1}{n^2(n+1)^2}\|S_n\|^2+\frac{\mathrm{E}(\|S_{n+1}\|^2\mid \mathscr{F}_n)-\|S_n\|^2}{(n+1)^2}.$$

Therefore, for every fixed integer N we have

$$\sum_{1\leqslant n\leqslant N} \mathrm{E}\left|\mathrm{E}\left(\frac{\|S_{n+1}\|^2}{(n+1)^2}\mid \mathscr{F}_n\right)-\frac{\|S^2\|}{n^2}\right| \leqslant \sum_{1\leqslant n\leqslant N}\frac{2(2n+1)}{n^2(n+1)^2}\mathrm{E}\,\|S_n\|^2+\frac{\mathrm{E}\,\|S_{N+1}\|^2}{(N+1)^2}.$$

Applying now the type 2 property, we obtain easily

$$\sum_{n\geqslant 1} \mathrm{E}\left|\mathrm{E}\left(\frac{\|S_{n+1}\|^2}{(n+1)^2}\mid \mathscr{F}_n\right)-\frac{\|S_n\|^2}{n^2}\right| \leqslant C_{10}\sum_{k\geqslant 1}\frac{\|x_k\|^2}{k^2},$$

which completes the proof of the implication (2)$\Rightarrow$(1).

(1)$\Rightarrow$(2). Let (x_k) be a sequence of elements of B such that (2.1) holds. By the property K1, $(\|S_n/n\|^2, \mathscr{F}_n)$ is a quasimartingale; therefore the sequence $(\|S_n/n\|^2)$ converges a.s. to a random variable Z (see [1], § VI.38). It follows easily from (2.1) that there exists a non-decreasing sequence of strictly positive numbers (α_k), with $\lim_{k\to+\infty}\alpha_k = +\infty$ and such that if we put

$$y_1 = x_1,$$

$$\forall n \geqslant 0,\quad \forall k \in I(n) = \{2^n+1, \ldots, 2^{n+1}\},\ y_k = \alpha_n x_k,$$

then

$$\sum_{k\geqslant 1}\frac{\|y_k\|^2}{k^2} < +\infty. \tag{2.2}$$

Here again the sequence $(\|T_n/n\|^2, \mathscr{F}_n)-T_n = \sum_{1\leqslant k\leqslant n}\varepsilon_k y_k$ is a quasimartingale, and so there exists a constant $C_{11} > 0$ such that

$$\sup_k \mathrm{E}\left\|\sum_{j\in I(k)}\frac{\varepsilon_j y_j}{2^k}\right\| \leqslant C_{11}.$$

Let us notice now that, for every integer $n \geqslant 1$,

$$\mathrm{E}\left\|\frac{S_{2^n}}{2^n}\right\| \leqslant \mathrm{E}\frac{\|\varepsilon_1 x_1\|}{2^n}+\sum_{0\leqslant k\leqslant n-1}\frac{2^k C_{11}}{2^n\alpha_k}, \tag{2.3}$$

which implies

$$\lim_{n\to+\infty}\mathrm{E}\left\|\frac{S_{2^n}}{2^n}\right\| = 0,$$

and therefore $Z = 0$ a.s. So we have shown that if (2.1) holds, then the SLLN is fulfilled for the sequence $(\varepsilon_k x_k)$. By Theorem 2.2, this implies that $(B, \|\ \|)$ is of type 2.

3. The cotype 2 property and the Kqm-property. Now we will study the connection between the cotype 2 property and the Kqm-property.

THEOREM 3.1. *A real separable Banach space $(B, \|\ \|)$ having the* Kqm-*property is of cotype* 2.

Proof. Let $(B, \|\ \|)$ be a real separable Banach space having the Kqm-property, but which is not of cotype 2. Then for every integer $k \geqslant 1$ there exists a sequence $(u_1^k, \ldots, u_{n(k)}^k)$ of elements of B such that

$$\mathrm{E}\Big\|\sum_{1 \leqslant j \leqslant n(k)} \varepsilon_j u_j^k\Big\|^2 \leqslant k^{-8} \sum_{1 \leqslant j \leqslant n(k)} \|u_j^k\|^2, \tag{3.1}$$

where, as usual, the (ε_k) are independent Rademacher r.v. Since there is no loss of generality in assuming that

$$\sum_{1 \leqslant j \leqslant n(k)} \|u_j^k\|^2 = 1,$$

we work on this assumption in the sequel. Let us put

$$\forall k \geqslant 1,\ m(k) = \sup\big(n(1), \ldots, n(k-1), n(k)\big),$$

and consider the following sequence of integers:

$$c_0 = 0,$$
$$c_1 = 2m(1),$$
$$c_2 = c_1 + 2m(2),$$
$$\cdots\cdots\cdots$$
$$c_k = c_{k-1} + 2m(k).$$

We associate with the sequence (c_k) two sequences of sets of integers:

$$A_k = \{c_{k-1}+1, \ldots, c_{k-1}+m(k)\}, \qquad B_k = \{c_{k-1}+m(k)+1, \ldots, c_k\}.$$

Using the (u_j^k), (A_k) and (B_k) we define now, for every integer $k \geqslant 1$, a remarkable sequence (x_j) of elements of B in the following way:

$$\forall j \in A_k,\ x_j = 0;$$

$$\forall j = c_{k-1}+m(k)+1, \ldots, c_{k-1}+m(k)+n(k),\ x_j = c_k u^k_{(j-c_{k-1}-m(k))};$$

$$\forall j \in B_k \cap \{c_{k-1}+m(k)+n(k)+1, \ldots, c_k\},\ x_j = 0.$$

Let us examine the properties of the sequence (x_j). From the definition of the x_j we get immediately

$$\sum_{j\geq 1} \frac{\|x_j\|^2}{j^2} = +\infty. \tag{3.2}$$

Let us now consider the sequence

$$\left\|\frac{S_n}{n}\right\|^2 = \left\|\sum_{1\leq k\leq n} \frac{\varepsilon_k x_k}{n}\right\|^2,$$

and show that $(\|S_n/n\|^2, \mathscr{F}_n)$ is a quasimartingale. The same computation as in the proof of Theorem 2.3 shows that

$$w_k = \mathrm{E}\left|\mathrm{E}\left\{\frac{\|S_{k+1}\|^2}{(k+1)^2} \mid \mathscr{F}_k\right\} - \frac{\|S_k\|^2}{k^2}\right| \leq \frac{(2k+1)\,\mathrm{E}\,\|S_k\|^2}{k^2(k+1)^2} + \frac{\mathrm{E}\,\|S_{k+1}\|^2}{(k+1)^2} - \frac{\mathrm{E}\,\|S_k\|^2}{(k+1)^2}.$$

Therefore

$$\sum_{j\in A_k\cup B_k} w_j \leq C_{12} \sum_{j\in A_k\cup B_k} \left(\frac{1}{j^2} - \frac{1}{(j+1)^2}\right) \mathrm{E}\,\|S_j\|^2 + \frac{\mathrm{E}\,\|S_{c_k+1}\|^2}{(c_k+1)^2}. \tag{3.3}$$

Observe that $c_k \leq 2(c_{k-1} + m(k) + 1)$. Consequently, by (3.3), we have

$$\begin{aligned}\sum_{j\in A_k\cup B_k} w_j &\leq C_{13} \frac{\mathrm{E}\,\|S_{c(k-1)}\|^2}{c_{k-1}^2} + C_{14} \frac{\mathrm{E}\,\|S_{c_k+1}\|^2}{(c_k+1)^2} \\ &\leq C_{13} \frac{\mathrm{E}\,\|S_{c(k-1)}\|^2}{c_{k-1}^2} + C_{15} \frac{\mathrm{E}\,\|S_{c_k}\|^2}{c_k^2}.\end{aligned}$$

Thus, showing that $(\|S_n/n\|^2, \mathscr{F}_n)$ is a quasimartingale reduces to proving the following

LEMMA 3.2. $\sum_{k\geq 1} (\mathrm{E}\,\|S_{c_k}\|^2)/c_k^2 < +\infty$.

Since B having the property K1 is of type 2 (see Theorem 2.3), for every integer $k \geq 1$ we obtain

$$\frac{\mathrm{E}\,\|S_{c_k}\|^2}{c_k^2} \leq C_{16} \sum_{1\leq j\leq k} \frac{\mathrm{E}\,\|\sum_{s\in B_j} \varepsilon_s x_s\|^2}{c_k^2} \leq C_{16} \sum_{1\leq j\leq k} \frac{c_j^2}{j^8 c_k^2}$$

(this follows from (3.1)). Notice that, by the definition of $m(k)$, if $j \leq k^{1/4}$, then $c_k \geq (k^{3/4} - 1)\, c_j$. Consequently,

$$\forall k \geq 2, \quad \sum_{1\leq j\leq k} \frac{c_j^2}{j^8 c_k^2} \leq \frac{C_{17}}{k^{3/2}} + \sum_{[k^{1/4}]\leq j\leq k} \frac{1}{j^8} \leq \frac{C_{18}}{k^{3/2}}.$$

The last inequality implies Lemma 3.2. Hence $(\|S_n/n\|^2, \mathscr{F}_n)$ is a quasimartingale; it follows from (3.2) that B has not the property K2. Therefore our hypothesis on the geometry of B is wrong, and so $(B, \|\ \|)$ must be of cotype 2.

A famous result due to Kwapień [9] states that if a Banach space $(B, \|\ \|)$ is both of type and cotype 2, then it is isomorphic to a Hilbert space. According to Theorems 2.3 and 3.1, if there exist spaces other than the real line having the Kqm-property, these spaces are necessarily isomorphic to Hilbert spaces. The following result makes more precise the relationship between being a Hilbert space and having the Kqm-property:

THEOREM 3.3. *An infinite-dimensional Hilbert space does not have the* Kqm-*property.*

Proof. Let $(H, \langle , \rangle)$ be a real separable infinite-dimensional Hilbert space. Being of type 2, it has the property K1. Giving a counterexample we will check that the space does not have the property K2.

Let (e_j) denote the basis of H. Define

$$\forall k = 1, 2, 3,\ x_k = 0; \qquad \forall k \geqslant 4,\ x_k = \sqrt{\frac{k}{\ln k}}\, e_k.$$

As above, for $n \geqslant 4$ we put $S_n = \sum_{1 \leqslant k \leqslant n} \varepsilon_k x_k$. Let us check that $(\|S_n/n\|^2; \mathscr{F}_n)$ is a quasimartingale. Write

$$u_n = \frac{\|S_n\|^2}{n^2} - \mathrm{E}\left\{\frac{\|S_{n+1}\|^2}{(n+1)^2} \,|\, \mathscr{F}_n\right\} = \frac{1}{n^2} \sum_{4 \leqslant k \leqslant n} \frac{k}{\ln k} - \frac{1}{(n+1)^2} \sum_{4 \leqslant k \leqslant n+1} \frac{k}{\ln k}$$
$$= \frac{2n+1}{n^2 (n+1)^2} \sum_{4 \leqslant k \leqslant n} \frac{k}{\ln k} - \frac{1}{(n+1)\ln(n+1)} = v_n - \frac{1}{(n+1)\ln(n+1)}.$$

An obvious comparison between an integral and a series shows that

$$\frac{2n+1}{n^2 (n+1)^2} \int_3^n \frac{x}{\ln x} dx \leqslant v_n \leqslant \frac{2n+1}{n^2 (n+1)^2} \int_4^{n+1} \frac{x}{\ln x} dx. \tag{3.4}$$

From (3.4) we deduce immediately that there exist two positive constants C_{19} and C_{20} such that

$$\frac{2n+1}{2(n+1)^2 \ln n} + \frac{C_{19}}{n(\ln n)^2} \leqslant v_n \leqslant \frac{2n+1}{2n^2 \ln(n+1)} + \frac{C_{20}}{n(\ln n)^2}.$$

Consequently,

$$\frac{1}{n+1}\left\{\frac{1}{\ln n} - \frac{1}{\ln(n+1)}\right\} - \frac{1}{2(n+1)^2 \ln n} + \frac{C_{19}}{n(\ln n)^2} \leqslant u_n$$
$$\leqslant \frac{1}{\ln(n+1)}\left\{\frac{1}{n} - \frac{1}{n+1}\right\} + \frac{1}{2n^2 \ln(n+1)} + \frac{C_{20}}{n(\ln n)^2}$$

and, finally,

$$|u_n| \leqslant \frac{C_{21}}{n(\ln n)^2}.$$

Therefore

$$\sum_{n\geqslant 4} \mathrm{E}\left|\frac{\|S_n\|^2}{n^2} - \mathrm{E}\left\{\frac{\|S_{n+1}\|^2}{(n+1)^2} \mid \mathscr{F}_n\right\}\right| = \sum_{n\geqslant 4} |u_n| < +\infty.$$

Thus $(\|S_n/n\|^2, \mathscr{F}_n)$ is a quasimartingale. But

$$\sum_{k\geqslant 4} \frac{\|x_k\|^2}{k^2} = +\infty,$$

so the Hilbert space $(H, \langle , \rangle)$ does not have the property K2; this completes the proof of Theorem 3.3.

4. Appendix: A lower exponential bound for the tail of the distribution of a weighted Rademacher sum. In the proof of Theorem 1.1 we used a very powerful exponential bound for the tail of the distribution of a weighted Rademacher sum. This bound is due independently to Ledoux and Talagrand ([10], Lemma 4.9) and Montgomery-Smith [11]. We will give the statement of this result in Montgomery-Smith's language.

First we have to recall some definitions and properties related to the l_2-space.

Let $t > 0$ be given. We will consider the following norm $K_{1,2}(\cdot, t)$ on l_2, associated with t, arising in the theory of interpolation of Banach spaces:

(4.1) $\forall x \in l_2,\ K_{1,2}(x, t) = \inf\{\|x'\|_1 + t\,\|x''\|_2\colon x' \in l_1,\ x'' \in l_2\colon x' + x'' = x\}$

(see, e.g., [6]). Holmstedt ([6], Theorem 4.1) proved the following

PROPOSITION 4.1. *There exists a universal constant $c > 0$ such that for all $x \in l_2$:*

(4.2) $$\frac{1}{c} K_{1,2}(x, t) \leqslant \sum_{1\leqslant k\leqslant [t^2]} x_k^* + t\sqrt{\sum_{k\geqslant [t^2]+1} (x_k^*)^2} \leqslant K_{1,2}(x, t),$$

where [ ] *stands for the integer part of a real number, and (x_k^*) denotes the non-increasing rearrangement of the sequence $(|x_k|)$.*

The announced exponential lower bound is as follows:

THEOREM 4.2. *For every element $x = (x_k)$ of l_2, we define $X(x) = \sum_{k\geqslant 1} \varepsilon_k x_k$, where, as usual, (ε_k) denotes a sequence of independent Rademacher r.v. Then there exists a constant $C > 0$ such that*

(4.3) $\forall x \in l_2,\ \forall t > 0,\ P\big(X(x) > C^{-1} K_{1,2}(x, t)\big) \geqslant C^{-1} \exp(-Ct^2).$

5. Some concluding remarks and some problems. As a conclusion to this paper, we will make some comments suggested by the above results.

1. The first natural question which raises is the following: Does Theorem 1.1 extend to $\boldsymbol{R}^n$? A positive answer to this question would – with the help of Theorem 3.3 – give a characterization of finite-dimensional spaces.

2. A second natural question, connected with Theorem 3.1, is the one of the relation between the property K2 and the cotype 2 property. Intuitively, one expects that Godbole's characterization of cotype q spaces in terms of the SLLN for symmetrically distributed r.v. [3] would play a role for answering this question as the Hoffmann–Jørgensen and Pisier theorem does in the proof of Theorem 2.3.

As the convergence of the Kolmogorov series

$$\sum_{k\geqslant 1}\frac{\|x_k\|^2}{k^2}<+\infty$$

is more restrictive than Godbole's assumption

$$\frac{1}{n^2}\Big(\sum_{1\leqslant k\leqslant n}\|x_k\|^2\Big)\to 0,$$

it is probable that the property K2 does not hold in every cotype 2 space.

3. From the Hoffmann-Jørgensen and Pisier theorem we know also that $(B,\|\ \|)$ is of type $p(1<p<2)$ if and only if the following implication holds:

$$\forall (x_k)\in B^N,\ \sum_{k\geqslant 1}\frac{\|x_k\|^p}{k^p}<+\infty\Rightarrow\frac{S_n}{n}=\frac{\sum_{1\leqslant k\leqslant n}\varepsilon_k x_k}{n}\to 0 \text{ a.s.} \tag{5.1}$$

Therefore, an idea similar to the one developed in this paper is to compare (5.1) and the fact that $(\|S_n/n\|^p, \mathscr{F}_n)$ is a quasimartingale. By the same proof as for Theorem 2.3 we get

THEOREM 5.1. *Let $(B,\|\ \|)$ be a real separable Banach space, and $p\in\,]1,2[$. Then the following are equivalent:*

(i) $\forall (x_k)\in B^N$, $\sum_{k\geqslant 1}\|x_k\|^p/k^p<+\infty\Rightarrow(\|S_n/n\|^p,\mathscr{F}_n)$ *is a quasimartingale.*
(ii) B *is of type* p.

A natural question, which relates Theorem 1.1 to Theorem 5.1 is the following one: Is the implication converse to (i) of Theorem 5.1 also true (of course, in the scalar setting)? The answer to this question is negative. This can be seen as follows:

Using the above notation, we get immediately

$$\sum_{n\geqslant 1}\frac{\mathrm{E}\,|S_n|^p}{n^{p+1}}<+\infty\Rightarrow\left(\left|\frac{S_n}{n}\right|^p,\mathscr{F}_n\right)\ \text{is a quasimartingale,}$$

which, by Khinchin's inequalities (see [10], Lemma 4.1), is equivalent to

$$\sum_{n\geqslant 1}\frac{\left(\sum_{1\leqslant k\leqslant n}x_k^2\right)^{p/2}}{n^{p+1}}<+\infty. \tag{5.2}$$

It is easy to construct sequences (x_k) of real numbers such that the series having the general term $k^{-p}|x_k|^p$ diverges, but for which (5.2) holds. Thus Theorem 1.1 has no analogue for $p\neq 2$.

REFERENCES

[1] C. Dellacherie and P. A. Meyer, *Probabilités et potentiel*, chap. V–VIII, *Théorie des martingales*, Hermann, Paris 1980.

[2] L. Egghe, *Stopping time techniques for analysts and probabilists*, Cambridge University Press, Cambridge 1984.

[3] A. P. Godbole, *Strong laws of large numbers and laws of the iterated logarithm in Banach spaces*, Ph. D. Thesis of the Michigan State University, 1984.

[4] J. Hajek and A. Rényi, *Generalization of an inequality of Kolmogorov*, Acta. Math. Acad. Sci. Hungar. 6 (1955), pp. 281–283.

[5] J. Hoffmann-Jørgensen and G. Pisier, *The law of large numbers and the central limit theorem in Banach spaces*, Ann. Probab. 4 (1976), pp. 587–599.

[6] T. Holmstedt, *Interpolation of quasi-normed spaces*, Math. Scand. 26 (1970), pp. 177–199.

[7] A. Kolmogorov, *Sur la loi forte des grands nombres*, C.R. Acad. Sci. Paris 191 (1930), pp. 910–912.

[8] J. Kuelbs and J. Zinn, *Some stability results for vector valued random variables*, Ann. Probab. 7 (1979), pp. 75–84.

[9] S. Kwapień, *Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients*, Studia Math. 44 (1972), pp. 583–595.

[10] M. Ledoux and M. Talagrand, *Probability in Banach spaces*, Springer, Berlin 1991.

[11] S. J. Montgomery-Smith, *The distribution of Rademacher sums*, Proc. Amer. Math. Soc. 109 (1990), pp. 517–522.

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