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ON THE ALMOST SURE CENTRAL LIMIT THEOREM FOR ASSOCIATED RANDOM VARIABLES

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Abstract. The aim of this note is to prove the strong version of the CLT for associated sequences without any strong approximation theorems. In the proofs we only apply the weighted convergence result for averages of associated random variables and the standard CLT.

1. Introduction. Let $(X_n)_{n\in N}$ be a sequence of random variables (r.v.'s) defined on some probability space $(\Omega, \mathscr{F}, P)$. In the following we assume $EX_n = 0$, $EX_n^2 < \infty$ for $n \in N$ and put $S_n = \sum_{k=1}^{n} X_k$, $\sigma_n^2 = ES_n^2$. The almost sure central limit theorem states that under appropriate conditions there is a P-null set $N \subset \Omega$ such that, for all $\omega \in N^c$,

$$(1) \qquad (\log n)^{-1} \sum_{k=1}^{n} k^{-1} I_A\left(\sigma_n^{-1} S_k(\omega)\right) \to (2\pi)^{-1/2} \int_A \exp(-u^2/2)\, du$$

for all Borel sets $A \subset \boldsymbol{R}$ with $\lambda(\partial A) = 0$. We shall prove this theorem in the following form:

$$(2) \qquad P\left[\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I_{(-\infty,u)}(S_k/\sigma_k) = \Phi(u)\right] = 1 \quad \text{for all } u \in \boldsymbol{R},$$

where $\Phi(u)$ denotes here and in the sequel the standard normal distribution. Many recent papers deal with the almost sure CLT for independent r.v.'s. The i.i.d. case was considered in [13], [9], [1], [7]. Sequences of r.v.'s which are not identically distributed were studied in [12]. There are only a few results of this type for weakly dependent sequences (cf. [11], [8]). Our goal is to prove (2) for associated sequences as well as for certain classes of independent r.v.'s.

Let us recall (cf. [6]) that $(X_n)_{n\in N}$ is a sequence of associated random variables if for every finite subcollection $X_{n_1}, X_{n_2}, \ldots, X_{n_k}$ and coordinatewise nondecreasing functions $f, g\colon \boldsymbol{R}^k \to \boldsymbol{R}$ the inequality

$$\operatorname{Cov}\left(f(X_{n_1}, X_{n_2}, \ldots, X_{n_k}), g(X_{n_1}, X_{n_2}, \ldots, X_{n_k})\right) \geqslant 0$$

holds, whenever this covariance is defined. Associated processes play a very important role in mathematical physics and statistics. Many recent papers have been concerned with limit theorems for such processes (see for example [2]–[5], [10], [11], and references therein). In our considerations we shall need the following coefficient:

$$u(n) = \sup_{k\in N} \sum_{j:|j-k|\geq n} \operatorname{Cov}(X_j, X_k), \qquad n\in N\cup\{0\}.$$

2. Results.

THEOREM 1. *Let $(X_n)_{n\in N}$ be a sequence of associated zero mean r.v.'s with finite second moments. Assume that*

$$(3) \qquad C_1 \leqslant \sigma_n^2/n \leqslant C_2 \quad \textit{for some constants } C_1, C_2 > 0 \textit{ and all } n\in N,$$

$$(4) \qquad \lim_{n\to\infty} \sigma_n^{-2} \sum_{k=1}^{n} EX_k^2 I[|X_k| \geqslant \varepsilon\sigma_n] = 0 \quad \textit{for every } \varepsilon > 0,$$

$$(5) \qquad u(n) = O(n^{-\alpha}) \quad \textit{for some } \alpha > 1.$$

Then (2) *holds.*

From Theorem 1 we get the following corollary:

COROLLARY 1. *Let $(X_n)_{n\in N}$ be a sequence of associated zero mean r.v.'s. Assume that* (5) *is satisfied and*

$$(6) \qquad \sup_{n\in N} E|X_n|^3 < \infty,$$

$$(7) \qquad \inf_{n\in N} \sigma_n^2/n > 0.$$

Then (2) *holds.*

Proof. It is easy to see that the Lindeberg condition (4) holds, so it suffices to verify (3). The left bound follows from (7), while the right one is a consequence of Theorem 1 of [3], where we take $r = 2+2\alpha/(2\alpha+3)$ and $\delta = 3/(2\alpha+3)$.

From Theorem 1 we also get the well-known almost sure CLT for i.i.d. r.v.'s (cf. [9] and [13]).

COROLLARY 2. *Let $(X_n)_{n\in N}$ be a sequence of i.i.d. r.v.'s. Assume that $EX_1 = 0$, $EX_1^2 = 1$. Then for all $u\in R$*

$$P\left[\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I_{(-\infty,u)}(S_k/\sqrt{k}) = \Phi(u)\right] = 1.$$

For independent r.v.'s with nonidentical distributions we shall prove two theorems.

THEOREM 2. *Let $(X_n)_{n\in N}$ be a sequence of independent, zero mean r.v.'s with finite second moments. Assume that* (4) *is satisfied and*

$$\lim_{n\to\infty} \sigma_n^2/n^\beta = C \quad \text{for some } C>0 \text{ and } \beta>0. \tag{8}$$

Then (2) *holds.*

THEOREM 3. *Let $(X_n)_{n\in N}$ be a sequence of independent, zero mean r.v.'s with finite second moments. Assume that* (4) *is satisfied and*

$$\lim_{n\to\infty} \sigma_n^2/\log^\beta n = C \quad \text{for some } C>0 \text{ and } \beta>0. \tag{9}$$

Then

$$P\left[\lim_{n\to\infty}\frac{1}{\log\log n}\sum_{k=2}^{n}\frac{1}{k\log k}I_{(-\infty,u)}(S_k/\sigma_k)=\Phi(u)\right]=1 \quad \text{for all } u\in \boldsymbol{R}. \tag{10}$$

3. Auxiliary lemmas and proofs. Let $(a_n)_{n\in N}$ be a sequence of positive numbers. We set $b_n=\sum_{k=1}^{n}a_k$ and assume that

$$a_n/b_n\to 0 \quad \text{and} \quad b_n\to\infty \quad \text{as } n\to\infty. \tag{11}$$

Let us also define $S_n^*=\sum_{k=1}^{n}a_kX_k$. We have the following weighted convergence result ([10], Theorem 1).

LEMMA 1. *Let $(X_n)_{n\in N}$ be a sequence of associated r.v.'s with finite second moments and $(a_n)_{n\in N}$ be a sequence satisfying* (11). *Assume that*

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j}a_ia_j\operatorname{Cov}(X_i,X_j)/b_j^2<\infty.$$

Then, as $n\to\infty$, $(S_n^*-ES_n^*)/b_n\to 0$ almost surely.

From Lemma 3.1 (ii) of [4] we get the following

LEMMA 2. *Let X, Y be square integrable associated r.v.'s. If $h\colon \boldsymbol{R}\to\boldsymbol{R}$ is a bounded differentiable function with bounded derivative, then*

$$|\operatorname{Cov}(h(X),h(Y))|\leqslant \|h'\|_\infty^2\operatorname{Cov}(X,Y).$$

Proof of Theorem 1. Let $\varepsilon>0$ and let $u\in\boldsymbol{R}$ be given. Let $g(t)$ be a function such that $g(t)=1$ for $t\in(-\infty,0]$, and $g(t)=0$ for $t\in[1,+\infty)$. Moreover, let g be nonincreasing, differentiable and let $\|g'\|_\infty\leqslant 2$. Let us define

$$g_{\varepsilon,u}=g\big((t-u)/\varepsilon\big) \quad \text{and} \quad g_{-\varepsilon,u}=g\big((t-u+\varepsilon)/\varepsilon\big).$$

Then

$$g_{-\varepsilon,u}(t)\leqslant I_{(-\infty,u)}(t)\leqslant g_{\varepsilon,u}(t) \quad \text{and} \quad \|g'_{-\varepsilon,u}\|_\infty=\|g'_{\varepsilon,u}\|_\infty\leqslant 2/\varepsilon.$$

Random variables $g_{\varepsilon,u}(S_k/\sigma_k)$ are associated as nonincreasing functions of associated r.v.'s. By Lemma 2 and conditions (3) and (5), for $i<j$ we get

$$(12)\qquad \mathrm{Cov}\big(g_{\varepsilon,u}(S_i/\sigma_i),\, g_{\varepsilon,u}(S_j/\sigma_j)\big) \leqslant \|g'_{\varepsilon,u}\|_\infty^2 \,\mathrm{Cov}(S_i/\sigma_i,\, S_j/\sigma_j)$$

$$\leqslant 4/\varepsilon^2\left(\frac{\sigma_i^2+\mathrm{Cov}(S_i,\, S_j-S_i)}{\sigma_i\sigma_j}\right) \leqslant 4/\varepsilon^2\left(\frac{\sigma_i^2+u(1)+\ldots+u(j-i)}{\sigma_i\sigma_j}\right)$$

$$\leqslant 4/\varepsilon^2\left(\frac{\sigma_i^2+\sum_{k=1}^{\infty}u(k)}{\sigma_i\sigma_j}\right) \leqslant C_1\frac{\sigma_i}{\sigma_j} \leqslant C_2\sqrt{\frac{i}{j}}$$

for some constants C_1, C_2 which depend only on ε. Let us note that this estimate holds also for $i=j$. Thus we get

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j}\frac{1}{ij}\mathrm{Cov}\big(g_{\varepsilon,u}(S_i/\sigma_i),\, g_{\varepsilon,u}(S_j/\sigma_j)\big)/\log^2 j$$

$$\leqslant C_2\sum_{j=1}^{\infty}\sum_{i=1}^{j}\frac{1}{ij}\frac{\sqrt{i}}{\sqrt{j}}/\log^2 j \leqslant C_3\sum_{j=1}^{\infty}\frac{1}{j\log^2 j} < +\infty.$$

Therefore, by Lemma 1, we have

$$(13)\qquad \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\big(g_{\varepsilon,u}(S_k/\sigma_k)-Eg_{\varepsilon,u}(S_k/\sigma_k)\big)\to 0$$

almost surely as $n\to\infty$. Now observe that

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\big(I_{(-\infty,u)}(S_k/\sigma_k)-\Phi(u)\big) \leqslant I_1+I_2+I_3+I_4+I_5,$$

where

$$I_1=\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\big(g_{\varepsilon,u}(S_k/\sigma_k)-Eg_{\varepsilon,u}(S_k/\sigma_k)\big),$$

$$I_2=\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\big(P(S_k/\sigma_k\leqslant u)-\Phi(u)\big),$$

$$I_3=\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\big(P(S_k/\sigma_k\leqslant u+\varepsilon)-\Phi(u+\varepsilon)\big),$$

$$I_4=\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\big(\Phi(u-\varepsilon)-P(S_k/\sigma_k\leqslant u-\varepsilon)\big),$$

$$I_5=\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\big(\Phi(u+\varepsilon)-\Phi(u-\varepsilon)\big).$$

$I_1 \to 0$ almost surely by (13), $I_2, I_3, I_4 \to 0$ by the CLT for associated sequences (cf. [5], Theorem 3) and $I_5 \leqslant \varepsilon$ by standard estimation of the normal distribution. Thus

$$\limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left(I_{(-\infty,u)}(S_k/\sigma_k) - \Phi(u)\right) \leqslant \varepsilon \text{ almost surely.}$$

Similarly we prove that

$$\liminf_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left(I_{(-\infty,u)}(S_k/\sigma_k) - \Phi(u)\right) \geqslant -\varepsilon \text{ almost surely.}$$

Therefore

$$P\left[\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left(I_{(-\infty,u)}(S_k/\sigma_k) - \Phi(u)\right) = 0\right]$$

$$= P\left[\bigcap_{j=1}^{\infty} \left(\left|\limsup_{n\to\infty}(\ldots) - \liminf_{n\to\infty}(\ldots)\right| \leqslant \frac{1}{j}\right)\right]$$

$$= \lim_{j\to\infty} P\left[\left|\limsup_{n\to\infty}(\ldots) - \liminf_{n\to\infty}(\ldots)\right| \leqslant \frac{1}{j}\right] = 1.$$

Proof of Theorem 2. The proof is similar to the proof of Theorem 1, so we omit details.

Proof of Theorem 3. As in the proof of Theorem 1 we see that under given conditions we obtain

$$\operatorname{Cov}\left(g_{\varepsilon,u}(S_i/\sigma_i),\, g_{\varepsilon,u}(S_j/\sigma_j)\right) \leqslant C \left(\frac{\log i}{\log j}\right)^{\beta/2}$$

and, consequently,

$$\sum_{j=2}^{\infty} \sum_{i=2}^{j} \frac{1}{ij \log i \log j} \left(\frac{\log i}{\log j}\right)^{\beta/2} /(\log\log j)^2 \leqslant C \sum_{j=2}^{\infty} \frac{1}{j \log j (\log\log j)^2} < \infty.$$

Moreover, the CLT holds, and so the proof may be completed along the lines of the proof of Theorem 1.

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