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INFINITE HORIZON REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS AND APPLICATIONS IN MIXED CONTROL AND GAME PROBLEMS

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Abstract. We prove existence and uniqueness results of the solution for infinite horizon reflected backward stochastic differential equations with one or two barriers. We also apply these results to get the existence of optimal control strategy for the mixed control problem and a saddle-point strategy for the mixed game problem when, in both situations, the horizon is infinite.

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1. Introduction. Nonlinear backward stochastic differential equations (BSDE's in short) have been independently introduced by Pardoux and Peng [18] and Duffie and Epstein [7]. It has already been discovered by Peng [20] that, coupled with a forward SDE, such BSDE's give a probabilistic interpretation for a large kind of second order quasilinear partial differential equations (PDE's). Then Pardoux and Peng [19] obtained an existence result of the viscosity solution for this kind of PDE systems. These results generalize the well-known Feynman–Kac formula to the nonlinear case. El-Karoui et al. [13] gave some important properties such as a comparison theorem and applications in optimal control and financial mathematics. Using results on BSDE's, Hamadène and Lepeltier ([14] and [15]) obtained the existence of a saddle-point strategy under the Isaacs condition for the zero-sum differential game problem and the existence of an optimal strategy for the optimal stochastic control problem.

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Then El-Karoui et al. [12] studied the reflected BSDE with one barrier. The solution of the reflected BSDE is forced to stay above one given continuous stochastic process which is called *obstacle*. For this purpose they introduced one increasing process to push the solution upwards and also required the push power to be minimum. They got the existence and uniqueness of the solution for this kind of reflected BSDE and also studied its relation with the obstacle problem for nonlinear parabolic PDE's within the Markov framework. Using two different methods, the Snell envelope theory connected with fixed point principle and the penalization method, Cvitanic and Karatzas [5] extended the result to reflected BSDE's with two barriers, called *upper and lower barriers*, which are two given continuous processes.

Recently Hamadène and Lepeltier [16] generalized the results of El-Karoui et al. [12] to one barrier which is right continuous and left upper semicontinuous. They used this model to solve the mixed optimal stochastic control problem when the terminal reward is only right continuous and left upper semicontinuous. In this kind of mixed control problem, the controller has two actions, one is of control and the other is of stopping his control strategy in view to maximize his payoff. Also in this paper Hamadène and Lepeltier generalized the result of Cvitanic and Karatzas [5] to reflected BSDE's with two barriers to processes S (lower barrier) and $-U$ (U is upper barrier) merely right continuous and left upper semicontinuous. They also used this result to obtain a saddle-point strategy for the mixed game problem, which means that two players have two actions, control and stopping their strategies in view to minimize (respectively, maximize) the payoff, when the Isaacs assumption is fulfilled and the terminal payoffs S and U satisfy the above condition. The first result gives another very simple method, different from that of El-Karoui [11], who used martingale methods to get the existence of an optimal mixed control. The second result about the mixed stochastic game is to our knowledge new.

We notice that the above results on reflected BSDE's, mixed control and game problems are all with finite time horizon. So our problem is how to generalize the reflected BSDE's to an infinite horizon.

First we need to review some results on infinite horizon BSDE's. Peng [20] obtained an existence and uniqueness result under some monotone conditions. But the solution is in a special kind of square integrable space. Chen [4] gave an existence and uniqueness result under a kind of Lipschitz condition suitable for infinite horizon BSDE's. In Section 2, we give this result as preliminary. We also prove the corresponding comparison theorem in that section.

In Section 3, we study infinite horizon reflected BSDE's with one barrier. Using the Snell envelope theory connected with the contraction method, we obtain the existence and uniqueness result. Then we use this result to deal with the mixed control problem with infinite horizon in Section 4. We obtain the existence of an optimal strategy for the controller.

In Section 5, we study the double barrier reflected BSDE with infinite horizon. We also use the Snell envelope theory connected with the contraction method to solve our problem. Under some additional assumptions on the barriers, we obtain the existence and uniqueness result.

At last, we use the result on double barrier reflected BSDE's with infinite horizon to study the mixed game problem in Section 6. When the Isaacs assumption on the Hamiltonian is satisfied, we obtain a saddle-point strategy for the two players.

2. Preliminary: Infinite horizon BSDE's. In this section, let us first give some preliminary results about infinite horizon BSDE's.

Let $(B_t)_{t\geqslant 0}$ be a standard m-dimensional Brownian motion, defined on a probability space $(\Omega, \mathscr{F}, P)$; let $(\mathscr{F}_t)_{t\geqslant 0}$ be the natural filtration of B_t, where $\mathscr{F}_0$ contains all P-null sets of $\mathscr{F}$, and $\mathscr{F}_\infty = \bigvee_{t\geqslant 0}\mathscr{F}_t$. We introduce the following notation:

$$\mathscr{S}^2 = \{v_t,\ 0\leqslant t\leqslant \infty, \text{ is an } \mathscr{F}_t\text{-adapted process such that } E[\sup_{0\leqslant t\leqslant\infty}|v_t|^2] < \infty\},$$

$$\mathscr{H}^2 = \{v_t,\ 0\leqslant t< \infty, \text{ is an } \mathscr{F}_t\text{-adapted process such that } E[\int_0^\infty |v_t|^2\,dt] < \infty\},$$

$$L^2 = \{\xi,\ \xi \text{ is an } \mathscr{F}_\infty\text{-measurable random variable such that } E|\xi|^2<\infty\}.$$

We consider the infinite horizon BSDE

$$Y_t = \xi + \int_t^\infty f(s, Y_s, Z_s)\,ds - \int_t^\infty Z_s\,dB_s, \qquad t\in[0, \infty], \tag{2.1}$$

where $\xi\in L^2$ and f is a map from $\Omega\times[0, \infty)\times \boldsymbol{R}\times\boldsymbol{R}^m$ onto $\boldsymbol{R}$ which satisfies the following:

(H2.1) For all $(y, z)\in\boldsymbol{R}^{1+d}$, $f(\cdot, y, z)$ is progressively measurable and

$$E\Big(\int_0^\infty |f(s, 0, 0)|\,ds\Big)^2 < \infty.$$

(H2.2) There exist two positive deterministic functions $u_1(t)$ and $u_2(t)$ such that, for all $(y_i, z_i)\in\boldsymbol{R}^{1+d}$, $i=1, 2$,

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leqslant u_1(t)|y_1-y_2| + u_2(t)|z_1-z_2|, \qquad t\in[0, \infty),$$

and $\int_0^\infty u_1(t)\,dt<\infty$, $\int_0^\infty u_2^2(t)\,dt<\infty$.

Then we have

THEOREM 2.1 (Chen [4]). *There exists a unique solution $(y, z)\in\mathscr{S}^2\times\mathscr{H}^2$ satisfying the BSDE* (2.1).

Now, if we consider the following two BSDE's:

(2.2) $$Y_t^i = \xi^i + \int_t^\infty f^i(s, Y_s^i, Z_s^i)\,ds - \int_t^\infty Z_s^i\,dB_s, \qquad t \geqslant 0,\ i = 1, 2,$$

where $\xi^i \in L^2$, f^i satisfy (H2.1) and (H2.2), by Theorem 2.1 there exist (y^i, z^i) which satisfy BSDE's (2.2), respectively. Further, if

(H2.3) $\xi^1 \geqslant \xi^2$ and $f^1(s, y_s^2, z_s^2) \geqslant f^2(s, y_s^2, z_s^2)$ a.s. for all $s \geqslant 0$,

then we have also a comparison theorem between the solutions of the infinite reflected BSDE's, that is:

THEOREM 2.2. *For all* $t \geqslant 0$, $y_t^1 \geqslant y_t^2$ *P-a.s.*

Proof. For the notational convenience, we assume that $d = 1$ and set $\hat{y} = (y^1 - y^2)$, $\hat{z} = (z^1 - z^2)$. Then $(\hat{y}, \hat{z})$ satisfies

$$\hat{y}_t = \xi^1 - \xi^2 + \int_t^\infty [\beta_1(s)\hat{y}_s + \beta_2(s)\hat{z}_s + f^1(s, y_s^2, z_s^2) - f^2(s, y_s^2, z_s^2)]\,ds - \int_t^\infty \hat{z}_s\,dB_s,$$

$$\beta_1(s) = \begin{cases} \dfrac{f^1(s, y_s^1, z_s^1) - f(s, y_s^2, z_s^1)}{y_s^1 - y_s^2} & \text{if } \hat{y}_s \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta_2(s) = \begin{cases} \dfrac{f^1(s, y_s^2, z_s^1) - f^1(s, y_s^2, z_s^2)}{z_s^1 - z_s^2} & \text{if } \hat{z}_s \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

From (H2.2) it is easily seen that $|\beta_1(s)| \leqslant u_1(s)$ and $|\beta_2(s)| \leqslant u_2(s)$ a.s. Introduce the process x_s, $0 \leqslant t \leqslant s \leqslant \infty$, which satisfies

$$dx_s = \beta_1(s)x_s\,ds + \beta_2(s)x_s\,dB_s, \qquad x_t = 1.$$

Since $x_s = \exp\left[\int_t^s (\beta_1(r) - \frac{1}{2}\beta_2^2(r))\,dr + \int_t^s \beta_2(r)\,dB_r\right]$, we have

$$\hat{y}_t = E^{\mathcal{F}_t}\left[(\xi^1 - \xi^2)x_\infty + \int_t^\infty (f^1(s, y_s^2, z_s^2) - f^2(s, y_s^2, z_s^2))x_s\,ds\right] \geqslant 0. \ \blacksquare$$

If we consider the following two BSDE's:

(2.3) $$Y_t^i = \xi^i + \int_t^\infty f^i(s, Y_s^i, Z_s^i)\,ds + A_\infty^i - A_t^i - \int_t^\infty Z_s^i\,dB_s, \qquad t \geqslant 0,\ i = 1, 2,$$

with the additional assumption:

(H2.4) A_t^i, $i = 1, 2$, $t \in [0, \infty]$, are continuous increasing processes satisfying $A_0^i = 0$, $A_\infty^i \in L^2$, $A_t^1 - A_t^2$ is also an increasing process, then we have

COROLLARY 2.3. *Assume that the two BSDE's* (2.3) *satisfy* (H2.1)–(H2.4), *and* (y_t^1, z_t^1) *and* (y_t^2, z_t^2) *are their respective solutions. Then* $y_t^1 \geqslant y_t^2$ *a.s.*

3. Infinite horizon reflected BSDE with one barrier. In this section we discuss the following infinite horizon reflected BSDE with one barrier:

$$(3.1) \qquad Y_t = \xi + \int_t^\infty f(s, Y_s, Z_s)\,ds + K_\infty - K_t - \int_t^\infty Z_s\,dB_s, \qquad t \in [0, \infty],$$

which will be used in the next section to deal with the mixed control problem with infinite horizon.

Here $\xi \in L^2$, f is a map from $\Omega \times [0, \infty) \times \boldsymbol{R} \times \boldsymbol{R}^m$ onto $\boldsymbol{R}$ satisfying (H2.1) and (H2.2). We consider a barrier $\{S_t, t \geqslant 0\}$, which is a continuous progressively measurable real-valued process satisfying

(H3.1) $E[\sup_{t\geqslant 0}(S_t^+)^2]^\infty < \infty$ and $\limsup_{t \nearrow +\infty} S_t \leqslant \xi$ a.s.

Our problem is to look for a triple (Y_t, Z_t, K_t) of $\mathscr{F}_t$ progressively measurable processes taking values in $\boldsymbol{R} \times \boldsymbol{R}^m \times \boldsymbol{R}^+$, satisfying the reflected BSDE (3.1), and

(i) $Y \in \mathscr{S}^2$, $Z \in \mathscr{H}^2$, $K_\infty \in L^2$;

(ii) $Y_t \geqslant S_t$, $t \geqslant 0$;

(iii) K_t is continuous and increasing, $K_0 = 0$, and $\int_0^\infty (Y_t - S_t)\,dK_t = 0$.

Remark. In fact, if we consider the following infinite horizon BSDE:

$$Y_t^0 = \xi + \int_t^\infty f(s, Y_s^0, Z_s^0)\,ds - \int_t^\infty Z_s^0\,dB_s, \qquad t \in [0, \infty],$$

where $\xi \in L^2$, f satisfies (H2.1) and (H2.2), and (Y_t^0, Z_t^0) is the solution of the BSDE, then from Corollary 2.3 we have $Y_t \geqslant Y_t^0$, $0 \leqslant t \leqslant \infty$, where Y_t is a solution of (3.1). So we can replace S_t by $S_t \vee Y_t^0$ and, consequently, we may assume without loss of generality that $E[\sup_{t\geqslant 0} S_t^2] < \infty$.

One approach to solve reflected BSDE's with infinite horizon is to use the Snell envelope theory connected with the contraction method. For this we consider first the following reflected BSDE:

$$(3.2) \qquad Y_t = \xi + \int_t^\infty f(s)\,ds + K_\infty - K_t - \int_t^\infty Z_s\,dB_s, \qquad t \in [0, \infty],$$

where f does not depend on (y, z) and is an $\mathscr{F}_t$-progressively measurable process satisfying

(H3.2) $E\left(\int_0^\infty |f(t)|\,dt\right)^2 < \infty$.

Then we have

PROPOSITION 3.1. *Assume that* $\xi \in L^2$, *and* (H3.1) *and* (H3.2) *are satisfied. Then there exists a unique solution* (Y, Z, K) *of the reflected BSDE* (3.2) *associated with* (f, ξ, S).

Proof. Let us introduce the process $\{Y_t;\ 0 \leqslant t \leqslant \infty\}$ defined by

$$Y_t = \operatorname*{ess\,sup}_{v \in \mathcal{T}_t} E\Big[\int_t^v f(s)\,ds + S_v 1_{\{v<\infty\}} + \xi 1_{\{v=\infty\}} \mid \mathcal{F}_t\Big],$$

where $\mathcal{T}$ is the set of all $\mathcal{F}_t$-stopping times taking values in $[0, \infty]$, and $\mathcal{T}_t = \{v \in \mathcal{T},\ v \geqslant t\}$. The process $Y_t + \int_0^t f(s)\,ds$ is the value function of an optimal stopping problem with payoff

$$H_t = \int_0^t f(s)\,ds + S_t 1_{\{t<\infty\}} + \xi 1_{\{t=\infty\}}.$$

By the theory of Snell envelope (El-Karoui [11]), it is also the smallest continuous supermartingale which dominates H_t. Moreover, we have

$$|Y_t| \leqslant E\Big[|\xi| + \int_0^\infty |f(t)|\,dt + \sup_{t \geqslant 0} |S_t| \mid \mathcal{F}_t\Big].$$

Hence, by Doob's inequality,

$$E\big(\sup_{0 \leqslant t \leqslant \infty} Y_t^2\big) \leqslant CE\Big[\xi^2 + \Big(\int_0^\infty |f(t)|\,dt\Big)^2 + \sup_{t \geqslant 0} S_t^2\Big].$$

Denote by D_t the stopping time

$$D_t = \begin{cases} \inf\{t \leqslant u < \infty,\ Y_u \leqslant S_u\}, \\ \infty \quad \text{otherwise.} \end{cases}$$

Then D_t is optimal in the sense that

$$Y_t = E\Big[\int_t^{D_t} f(s)\,ds + S_{D_t} 1_{\{D_t<\infty\}} + \xi 1_{\{D_t=\infty\}} \mid \mathcal{F}_t\Big], \quad 0 \leqslant t \leqslant \infty. \tag{3.3}$$

From the Doob–Meyer decomposition of the continuous supermartingale $Y_t + \int_0^t f(s)\,ds$ there exist an adapted increasing continuous process $\{K_t\}$ $(K_0 = 0)$ and a continuous uniformly integrable martingale $\{M_t\}$ such that

$$Y_t = M_t - \int_0^t f(s)\,ds - K_t.$$

By (3.2) and (3.3), we have $E[K_{D_t} - K_t \mid \mathcal{F}_t] = 0$; hence $K_{D_t} = K_t$ or, equivalently, $\int_0^\infty (Y_t - S_t)\,dK_t = 0$. It remains to prove some integrability results.

Since $\{Y_t + \int_0^t f(s)\,ds,\ 0 \leqslant t \leqslant \infty\}$ is a square integrable supermartingale, we have $EK_\infty^2 < \infty$, i.e. $K_\infty \in L^2$ (Dellacherie and Meyer [8]). Hence the martingale

$$M_t = E[M_\infty \mid \mathcal{F}_t] = E\Big[\xi + \int_0^\infty f(s)\,ds + K_\infty \mid \mathcal{F}_t\Big]$$

is also square integrable. Finally, since $\mathcal{F}_t$ is a Brownian filtration, we obtain

$$M_t = \xi + \int_t^\infty f(s)\,ds + K_\infty - \int_t^\infty Z_s\,dB_s,$$

where $E\int_0^\infty |Z_t|^2\,dt < \infty$, i.e. $Z_t \in \mathscr{H}^2$. Therefore, the triple (Y, Z, K) satisfies the reflected BSDE (3.2) and properties (i)–(iii) above.

Let us prove uniqueness. If (Y', Z', K') is another solution of the reflected BSDE (3.2) associated with (f, ξ, S) satisfying properties (i)–(iii) above, define $\hat{Y} = Y - Y'$, $\hat{Z} = Z - Z'$, and $\hat{K} = K - K'$. Using Itô's formula to $|\hat{Y}_t|^2$,

$$|\hat{Y}_t|^2 + \int_t^\infty |\hat{Z}_t|^2\,dt = 2\int_t^\infty \hat{Y}_s\,d\hat{K}_s - 2\int_t^\infty \hat{Y}_s \hat{Z}_s\,dB_s, \tag{3.4}$$

by the integrable conditions (i)–(iii) and Burkholder–Davis–Gundy's inequality, we have

$$E|\hat{Y}_t|^2 + E\int_t^\infty |\hat{Z}_s|^2\,ds = 2\int_t^\infty \hat{Y}_s\,d\hat{K}_s \leqslant 0.$$

So $E|\hat{Y}_t|^2 = 0$ a.s. for all $t \in [0, \infty]$ and $E\int_0^\infty |\hat{Z}_s|^2\,ds = 0$. Then $|\hat{Y}_t|^2 = 0$ a.s., so $Y = Y'$ by the continuity of $\hat{Y}_t$.

Finally, it is easy to get $K_t = K_t'$ a.s. for all $t \in [0, \infty]$. ∎

Now we give the main result of this section.

THEOREM 3.2. *Assume that* (H2.1), (H2.2) *and* (H3.1) *and that* $\xi \in L^2$. *Then the one-barrier reflected BSDE* (3.1) *associated with* (f, ξ, S) *has a unique solution* (Y, Z, K).

Proof. We first prove the existence. It is divided into two steps.

Step 1. Assume $\left(\int_0^\infty u_1(s)\,ds\right)^2 + \int_0^\infty u_2^2(s)\,ds < \frac{1}{18}$.

Let $\mathscr{D}$ be the space of processes (Y, Z) with values in $\mathbf{R}^{1+m}$ such that $Y \in \mathscr{S}^2$, $Z \in \mathscr{H}^2$, and $\|(Y, Z)\|_{\mathscr{D}}^2 = \|Y\|_{\mathscr{S}^2}^2 + \|Z\|_{\mathscr{H}^2}^2$. We define a mapping Ψ from $\mathscr{D}$ onto itself as follows: for any $(U, V) \in \mathscr{D}$, $(Y, Z) = \Psi(U, V)$ is the unique element of $\mathscr{D}$ such that if we define

$$K_t = Y_t - Y_0 - \int_0^t f(s, U_s, V_s)\,ds + \int_0^t Z_s\,dB_s, \qquad 0 \leqslant t \leqslant \infty,$$

then the triple (Y, Z, K) solves the one-barrier reflected backward SDE associated with $(f(s, U_s, V_s), \xi, S)$.

Let (U', V') be another element of $\mathscr{D}$ and define $(Y', Z') = \Psi(U', V')$, $\bar{U} = U - U'$, $\bar{V} = V - V'$, $\bar{Y} = Y - Y'$, $\bar{Z} = Z - Z'$, $\bar{K} = K - K'$, and $\bar{f} = f(s, U_s, V_s) - f(s, U_s', V_s')$. We want to prove that the mapping Ψ is a contraction. From the proof of Proposition 3.1 we obtain

$$Y_t = \operatorname*{ess\,sup}_{v \in \mathscr{T}_t} E\Big[\int_t^v f(s, U_s, V_s)\,ds + S_v 1_{\{v<\infty\}} + \xi 1_{\{v=\infty\}} \mid \mathscr{F}_t\Big],$$

$$Y_t' = \operatorname*{ess\,sup}_{v \in \mathscr{T}_t} E\Big[\int_t^v f(s, U_s', V_s')\,ds + S_v 1_{\{v<\infty\}} + \xi 1_{\{v=\infty\}} \mid \mathscr{F}_t\Big].$$

Then

$$|Y_t - Y_t'| \leqslant \operatorname*{ess\,sup}_{v \in \mathscr{T}_t} E\Big[\int_t^\infty |\bar{f}(s)|\,ds \mid \mathscr{F}_t\Big] \leqslant E\Big[\int_0^\infty |\bar{f}(s)|\,ds \mid \mathscr{F}_t\Big],$$

which implies

$$E[\sup_{0\leqslant t\leqslant\infty}|\bar{Y}_t|^2]\leqslant E[\sup_{0\leqslant t\leqslant\infty}(E\int_0^\infty|\bar{f}(s)|\,ds\mid\mathscr{F}_t)^2]\leqslant 4E(\int_0^\infty|\bar{f}(s)|\,ds)^2$$

by Doob's inequality. Using Itô's formula to $|\bar{Y}_t|^2$, we get

$$\begin{aligned}|\bar{Y}_t|^2+\int_t^\infty|\bar{Z}_s|^2\,ds&=2\int_t^\infty\bar{Y}_s\bar{f}(s)\,ds+2\int_t^\infty\bar{Y}_s\,d\bar{K}_s-2\int_t^\infty\bar{Y}_s\bar{Z}_s\,dB_s\\&\leqslant 2\int_t^\infty\bar{Y}_s\bar{f}(s)\,ds-2\int_t^\infty\bar{Y}_s\bar{Z}_s\,dB_s.\end{aligned}$$

Then

$$\begin{aligned}E\int_0^\infty|\bar{Z}_s|^2\,ds&\leqslant 2E\int_0^\infty|\bar{Y}_s|\,|\bar{f}(s)|\,ds\\&\leqslant E[\sup_{0\leqslant t\leqslant\infty}|\bar{Y}_t|^2]+E(\int_0^\infty|\bar{f}(s)|\,ds)^2\leqslant 5E(\int_0^\infty|\bar{f}(s)|\,ds)^2.\end{aligned}$$

From (H2.2) we know that

$$\begin{aligned}E(\int_0^\infty|\bar{f}(s)|\,ds)^2&\leqslant E(\int_0^\infty[u_1(s)|\bar{U}_s|+u_2(s)|\bar{V}_s|]\,ds)^2\\&\leqslant 2[(\int_0^\infty u_1(s)\,ds)^2+\int_0^\infty u_2^2(s)\,ds]\,\|(\bar{U},\bar{V})\|_{\mathscr{D}}^2.\end{aligned}$$

At last, we have

$$\|(\bar{Y},\bar{Z})\|_{\mathscr{D}}^2\leqslant 18[(\int_0^\infty u_1(s)\,ds)^2+\int_0^\infty u_2^2(s)\,ds]\,\|(\bar{U},\bar{V})\|_{\mathscr{D}}^2.$$

From the inequality $(\int_0^\infty u_1(s)\,ds)^2+\int_0^\infty u_2^2(s)\,ds<\frac{1}{18}$ we infer that Ψ is a strict contraction and has a unique fixed point, which is the unique solution of the reflected BSDE (3.1).

Step 2. For the general case, there exists $T_0>0$ such that

$$[(\int_{T_0}^\infty u_1(s)\,ds)^2+\int_{T_0}^\infty u_2^2(s)\,ds]<\frac{1}{18}.$$

From Step 1 we know that the reflected BSDE

$$\bar{Y}_t=\xi+\int_t^\infty 1_{\{s\geqslant T_0\}}f(s,\bar{Y}_s,\bar{Z}_s)\,ds+\bar{K}_\infty-\bar{K}_t-\int_t^\infty\bar{Z}_s\,dB_s,\qquad t\in[0,\infty],\tag{3.5}$$

has a unique solution $(\bar{Y}_t,\bar{Z}_t,\bar{K}_t)$, satisfying the properties (i)–(iii) above. Then we consider the reflected BSDE

$$\tilde{Y}_t=\bar{Y}_{T_0}+\int_t^{T_0}f(s,\tilde{Y}_s,\tilde{Z}_s)\,ds+\tilde{K}_{T_0}-\tilde{K}_t-\int_t^{T_0}\tilde{Z}_s\,dB_s,\qquad t\in[0,T_0].\tag{3.6}$$

By the result of El-Karoui et al. [12], there exists a unique solution $(\tilde{Y}_t, \tilde{Z}_t, \tilde{K}_t)$ satisfying the reflected BSDE (3.6) and the above properties (i)–(iii) on $[0, T_0]$. Let us set

$$Y_t = \begin{cases} \tilde{Y}_t, & t \in [0, T_0], \\ \bar{Y}_t, & t \in (T_0, \infty], \end{cases} \qquad Z_t = \begin{cases} \tilde{Z}_t, & t \in [0, T_0], \\ \bar{Z}_t, & t \in (T_0, \infty], \end{cases}$$

$$K_t = \begin{cases} \tilde{K}_t, & t \in [0, T_0], \\ \tilde{K}_{T_0} + \bar{K}_t - \bar{K}_{T_0}, & t \in (T_0, \infty]. \end{cases}$$

When $t \in [T_0, \infty]$, $(\bar{Y}_t, \bar{Z}_t, \bar{K}_t)$ is the solution of (3.5), and then $(\bar{Y}_t, \bar{Z}_t, \tilde{K}_{T_0} + \bar{K}_t - \bar{K}_{T_0})$ also satisfies (3.5). Now, if $t \in [0, T_0]$, $(\tilde{Y}_t, \tilde{Z}_t, \tilde{K}_t)$ is the solution of (3.6) and $\tilde{Y}_{T_0} = \bar{Y}_{T_0}$, $\tilde{K}_{T_0} = \tilde{K}_{T_0} + \bar{K}_{T_0} - \bar{K}_{T_0}$. So Y_t and K_t are continuous, and (Y, Z, K) is a solution of the reflected BSDE (3.1).

At last, we prove the uniqueness of (3.1). Let (Y', Z', K') be another solution of the reflected BSDE (3.1) associated with (f, ξ, S). We use the same notation as in Proposition 3.1. Applying Itô's formula to $|\hat{Y}_t|^2$, we have

$$|\hat{Y}_t|^2 + \int_t^\infty |\hat{Z}_s|^2\,ds = 2\int_t^\infty \hat{Y}_s\big(f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)\big)\,ds$$
$$+ 2\int_t^\infty \hat{Y}_s\,d\hat{K}_s - 2\int_t^\infty \hat{Y}_s \hat{Z}_s\,dB_s.$$

Then

$$E|\hat{Y}_t|^2 + E\int_t^\infty |\hat{Z}_s|^2\,ds \leqslant 2E\int_t^\infty |\hat{Y}_s|\,[u_1(s)|\hat{Y}_s| + u_2(s)|\hat{Z}_s|]\,ds \tag{3.7}$$
$$\leqslant \frac{1}{2}E\int_t^\infty |\hat{Z}_s|^2\,ds + E\int_t^\infty [2u_1(s) + 2u_2^2(s)]\,|\hat{Y}_s|^2\,ds.$$

From Gronwall's lemma we obtain $E|\hat{Y}_t|^2 = 0$ for all $t \in [0, \infty]$. Then $|\hat{Y}_t|^2 = 0$ a.s., so $Y = Y'$ by the continuity of $\hat{Y}_t$. Now, going back to (3.7), we have

$$E\int_0^\infty |\hat{Z}_s|^2\,ds \leqslant 4E \sup_{\{0 \leqslant t \leqslant \infty\}} |\hat{Y}_t|^2 \int_0^\infty [u_1(s) + u_2^2(s)]\,ds,$$

so $E\int_0^\infty |\hat{Z}_s|^2\,ds = 0$. Then it is easy to get $K_t = K'_t$. ∎

4. Applications in the mixed control problem with infinite horizon. In this section, we use the result on infinite horizon reflected BSDE's with one barrier to deal with the mixed stochastic control problem.

Let $\mathscr{C}$ be the space of continuous functions from $[0, \infty)$ to $\boldsymbol{R}^m$ endowed with the uniform convergence norm; $\mathscr{P}$ is the σ-algebra of progressively measurable subsets of $[0, \infty) \times \Omega$. The $(m \times m)$-matrix $\sigma = (\sigma_{ij})_{i,j=1,m}$ satisfies the following:

(i) For any continuous and $\mathscr{P}$-measurable process ζ with values in $\boldsymbol{R}^m$, the process $(\sigma_{ij}(t, \zeta))_{t \geqslant 0}$ is $\mathscr{P}$-measurable, $1 \leqslant i, j \leqslant m$.

(ii) For any $(t, x) \in [0, \infty] \times \mathscr{C}$, $\sigma(t, x)$ is invertible and $\sigma^{-1}(t, x)$ is bounded.

(iii) For any $t \in [0, \infty)$, $x, x' \in \mathscr{C}$, $|\sigma(t, x) - \sigma(t, x')| \leqslant K|x - x'|$, $K > 0$.

Under these assumptions, the stochastic differential equation

$$dX_t = \sigma(t, X)\,dB_t, \qquad X_0 = x \in \mathbf{R}^m, \qquad t \geqslant 0,$$

has a unique solution.

Now we consider a compact metric space U and let us denote by $\mathscr{U}$ the set of all $\mathscr{P}$-measurable processes with values in U; let ϕ be a function from $[0, \infty) \times \mathscr{C} \times U$ onto $\mathbf{R}^m$ such that:

(i) $\phi(\cdot, X(\cdot), \cdot)$ is $\mathscr{P} \otimes \mathscr{B}(U)$-measurable, $\mathscr{B}(U)$ is the Borel σ-algebra on U.

(ii) For any $t \in [0, \infty)$ and $x \in \mathscr{C}$, $\phi(t, x, \cdot)$ is continuous on U.

(iii) $|\phi(t, x, u)| \leqslant c(t)$ a.s., where $c(t)$ is a deterministic function such that $\int_0^\infty c^2(t)\,dt < \infty$.

For each $u \in \mathscr{U}$, we define a probability P^u on $(\Omega, \mathscr{F})$ by

$$\frac{dP^u}{dP} = \exp\Big\{\int_0^\infty \sigma^{-1}(s, X)\,\phi(s, X, u_s)\,dB_s - \tfrac{1}{2}\int_0^\infty |\sigma^{-1}(s, X)\,\phi(s, X, u_s)|^2\,ds\Big\}.$$

Under the assumptions on σ and ϕ, according to Girsanov's theorem (Karatzas and Shreve [17] or Revuz and Yor [21]), the process

$$B^u = B_t - \int_0^t \sigma^{-1}(s, X)\,\phi(s, X, u_s)\,ds, \qquad t \geqslant 0,$$

is a Brownian motion on $(\Omega, \mathscr{F}, P^u)$, and X is a weak solution of

$$dX_t = \phi(t, X, u_t)\,dt + \sigma(t, X)\,dB^u_t, \qquad X_0 = x, \qquad t \geqslant 0.$$

Suppose that we have a system whose evolution is described by the process X, which has an effect on the wealth of a controller. On the other hand, the controller has no influence on the system. The process X may represent, for example, the price of an asset on the market, and the controller be a small shareholder or a small investor. The controller acts to protect his advantages by means of $u \in \mathscr{U}$ via the probability P^u; here $\mathscr{U}$ is the set of admissible controls. On the other hand, he has also the possibility at any time $\tau \in \mathscr{T}$ to stop controlling. The control is not free. We define the payoff

$$J(u, \tau) = E^u\Big[\int_0^\tau C(s, X, u_s)\,ds + S_\tau 1_{\{\tau < \infty\}} + \xi 1_{\{\tau = \infty\}}\Big],$$

where S and ξ are the same as in Section 3, and $C(t, X, u)$ is from $[0, \infty] \times \mathscr{C} \times U$ into $\mathbf{R}$ and satisfies the same hypotheses as ϕ. For the controller, $C(t, X, u)$ is the instantaneous reward, S and ξ are, respectively, the rewards if he decides to stop before or until infinite time. The problem is to look for an optimal strategy for the controller, i.e. a strategy $(\hat{u}, \hat{\tau})$ such that

$$J(u, \tau) \leqslant J(\hat{u}, \hat{\tau}) \quad \text{for all } (u, \tau) \in \mathscr{U} \times \mathscr{T}.$$

For $(t, x, p, u) \in [0, \infty) \times \mathscr{C} \times \mathbf{R}^m \times U$ we define the Hamiltonian associated with this mixed stochastic control problem by

$$H(t, x, p, u) = p\phi(t, x, u) + C(t, x, u)$$

and denote by $u^*(t, X, p)$ the $\mathscr{P} \otimes \mathscr{B}(\mathbf{R}^m)$-measurable process with values in U such that

$$H\big(t, X, p, u^*(t, X, p)\big) = \max_{u \in U} [p\phi(t, X, u) + C(t, X, u)].$$

According to Benes's result (Benes [2]), such a process $u^*(t, X, p)$ exists. By the assumption (iii) on ϕ, $H(t, X, p, u)$ satisfies the Lipschitz assumption (H2.2) in p. Then it is easy to see that the function $H\big(t, X, p, u^*(t, X, p)\big)$ also satisfies the Lipschitz condition (H2.2) on p.

Now we give the main result of this section.

Theorem 4.1. *Let (Y^*, Z^*, K^*) be the solution of the one-barrier infinite horizon reflected BSDE associated with $\big(H(t, X, z, u^*(t, X, z)), \xi, S\big)$, $u^* = u^*(t, X, Z_t^*)$, $t \in [0, \infty)$, and*

$$\hat{\tau} = \begin{cases} \inf\{t \in [0, \infty),\ Y_t^* \leqslant S_t\}, \\ \infty \quad \text{otherwise}. \end{cases}$$

Then $Y_0^* = J(u^*, \hat{\tau})$, and $(u^*, \hat{\tau})$ is an optimal strategy for the controller.

Proof. We consider the following one-barrier infinite horizon reflected BSDE associated with $\big(H(t, X, z, u^*(t, X, z)), \xi, S\big)$:

$$Y_t^* = \xi + \int_t^\infty H\big(s, X, Z_s^*, u^*(s, X, Z_s^*)\big) ds + K_\infty^* - K_t^* - \int_t^\infty Z_s^* \, dB_s.$$

By Theorem 3.2, this BSDE has a unique solution (Y^*, Z^*, K^*). Now, since Y_0^* is a deterministic constant, we have

$$\begin{aligned} Y_0^* = E^{u^*}[Y_0^*] &= E^{u^*}\Big[\xi + \int_0^\infty H\big(s, X, Z_s^*, u^*(s, X, Z_s^*)\big) ds + K_\infty^* - \int_0^\infty Z_s^* \, dB_s\Big] \\ &= E^{u^*}\Big[\int_0^{\hat{\tau}} H\big(s, X, Z_s^*, u^*(s, X, Z_s^*)\big) ds + K_{\hat{\tau}}^* - \int_0^{\hat{\tau}} Z_s^* \, dB_s + Y_{\hat{\tau}}^*\Big] \\ &= E^{u^*}\Big[\int_0^{\hat{\tau}} C\big(s, X, u^*(s, X, Z_s^*)\big) ds + K_{\hat{\tau}}^* - \int_0^{\hat{\tau}} Z_s^* \, dB_s^{u^*} + Y_{\hat{\tau}}^*\Big]. \end{aligned}$$

From the definition of $\hat{\tau}$ and the properties of reflected BSDE's we know that the process K_t^* does not increase between 0 and $\hat{\tau}$, and then $K_{\hat{\tau}}^* = 0$. On the other hand, using the Burkholder–Davis–Gundy's inequality and the assumptions on ϕ, we know that $\big(\int_0^t Z_s^* \, dB_s^{u^*}, t \in [0, \infty]\big)$ is a P^{u^*}-martingale, and then

$$Y_0^* = E^{u^*}\Big[\int_0^{\hat{\tau}} C\big(s, X, u^*(s, X, Z_s^*)\big) ds + Y_{\hat{\tau}}^*\Big].$$

From the equality $Y_{\hat{\tau}}^* = S_{\hat{\tau}} 1_{\{\hat{\tau} < \infty\}} + \xi 1_{\{\hat{\tau} = \infty\}}$ P^{u^*}-a.s. we get $Y_0^* = J(u^*, \hat{\tau})$.

Now, let u be an admissible control and τ be a stopping time. Since P and P^{u^*} are equivalent probabilities on $(\Omega, \mathscr{F})$, we obtain

$$\begin{aligned}
Y_0^* = E^u[Y_0^*] &= E^u\Big[\xi + \int_0^\infty H(s, X, Z_s^*, u^*(s, X, Z_s^*))\,ds + K_\infty^* - \int_0^\infty Z_s^*\,dB_s\Big] \\
&= E^u\Big[\int_0^\tau H(s, X, Z_s^*, u^*(s, X, Z_s^*))\,ds + K_\tau^* - \int_0^\tau Z_s^*\,dB_s + Y_\tau^*\Big] \\
&= E^u\Big[\int_0^\tau C(s, X, u_s)\,ds + \int_0^\tau \big(H(s, X, Z_s^*, u^*(s, X, Z_s^*)) - H(s, X, Z_s^*, u_s)\big)\,ds \\
&\qquad + K_\tau^* - \int_0^\tau Z_s^*\,dB_s^u + Y_\tau^*\Big].
\end{aligned}$$

But $Y_\tau^* \geqslant S_\tau 1_{\{\tau<\infty\}} + \xi 1_{\{\tau=\infty\}}$, $K_\tau^* \geqslant 0$,

$$H(t, X, Z_t^*, u^*(t, X, Z_t^*)) - H(t, X, Z_t^*, u_t) \geqslant 0 \quad P^u\text{-a.s.},$$

and $\left(\int_0^t Z_s^*\,dB_s^u\right)_{t\geqslant 0}$ is a P^u-martingale. Then

$$J(u^*, \hat{\tau}) = Y_0^* \geqslant E^u\Big[\int_0^\tau C(s, X, u_s)\,ds + S_\tau 1_{\{\tau<\infty\}} + \xi 1_{\{\tau=\infty\}}\Big] = J(u, \tau).$$

It follows that the control $(\hat{u}, \hat{\tau})$ is optimal. ∎

5. Double-barrier reflected BSDE's with infinite horizon. In this section we discuss the double-barrier reflected BSDE's with infinite horizon, which will be used to solve the mixed game problem in the next section.

Let f, ξ and S_t be the same as in Section 3, and U_t, $t \in [0, \infty)$, be a continuous progressively measurable process valued in $\boldsymbol{R}$ such that:

(H5.1) $\boldsymbol{E}[\sup_{t\geqslant 0}(U_t^-)^2] < \infty$ and $S_t \leqslant U_t$, $t \in [0, \infty)$, $\liminf_{t\nearrow+\infty} U_t \geqslant \xi$ a.s.

Our problem is to look for a solution (Y_t, Z_t, K_t^+, K_t^-) of the reflected BSDE with values in $\boldsymbol{R} \times \boldsymbol{R}^m \times \boldsymbol{R}^+ \times \boldsymbol{R}^+$ such that

$$(5.1)\quad Y_t = \xi + \int_t^\infty f(s, Y_s, Z_s)\,ds + K_\infty^+ - K_t^+ - (K_\infty^- - K_t^-) - \int_t^\infty Z_s\,dB_s, \qquad t \in [0, \infty].$$

Moreover:

(i) Y is continuous and $Y \in \mathscr{S}^2$, $Z \in \mathscr{H}^2$.

(ii) $S_t \leqslant Y_t \leqslant U_t$, $0 \leqslant t < \infty$.

(iii) K_t^+ and K_t^- are continuous and increasing processes satisfying $K_0^+ = K_0^- = 0$, $K_\infty^+ \in L^2$, $K_\infty^- \in L^2$ and $\int_0^\infty (Y_t - S_t)\,dK_t^+ = 0$, $\int_0^\infty (U_t - Y_t)\,dK_t^- = 0$.

For $t \geqslant 0$, we set $S_t^\xi = S_t 1_{\{t<\infty\}} + \xi 1_{\{t=\infty\}}$, $U_t^\xi = U_t 1_{\{t<\infty\}} + \xi 1_{\{t=\infty\}}$ and denote by π_c^2 the space of continuous real-valued nonnegative $\mathscr{F}_t$-supermartingales M_t such that $E[\sup_{0\leqslant t\leqslant\infty} M_t^2] < \infty$.

Similarly to the one-barrier problem in Section 3, we first consider the following reflected BSDE with two barriers:

$$(5.2)\qquad Y_t = \xi + \int_t^\infty f(s)\,ds + K_\infty^+ - K_t^+ - (K_\infty^- - K_t^-) - \int_t^\infty Z_s\,dB_s, \qquad t\in[0, \infty].$$

We assume that the following hypotheses are fulfilled.

(H5.2) There exist two supermartingales h and θ of π_c^2 such that, for all $t \geqslant 0$,

$$S_t^\xi \leqslant h_t - \theta_t + E[\xi \mid \mathcal{F}_t] \leqslant U_t^\xi \text{ a.s.}$$

(H5.3) For all $t \geqslant 0$, $S_t < U_t$ a.s.

Then we have

PROPOSITION 5.1. *Let* $\xi \in L^2$, *let* f *satisfy* (H3.2), *and assume that* (H5.1)–(H5.3) *hold true. Then there exists a unique solution* (Y, Z, K^+, K^-) *for the double-barrier reflected BSDE* (5.2) *associated with* (f, ξ, S, U).

Proof. The sketch of the proof is the same as that one of Cvitanic and Karatzas [5] in $[0, T]$.

We first prove the existence of two continuous supermartingales X^+ and X^- such that

$$(5.3)\qquad X^+ = R(X^- + \tilde{S}), \qquad X^- = R(X^+ - \tilde{U}),$$

where

$$\tilde{S} = S_t^\xi - N(t), \qquad \tilde{U} = U_t^\xi - N(t), \qquad N(t) = E\Big[\xi + \int_t^\infty f(s)\,ds \mid \mathcal{F}_t\Big],$$

and R is the Snell envelope operator, that is $R(\eta)_t = \operatorname{ess\,sup}_{v\in\mathcal{T}_t} E[\eta_v \mid \mathcal{F}_t]$, $\eta \in \mathcal{S}^2$, $\mathcal{T}_t$ is the same as in Section 3.

First we notice that $\tilde{S}$ and $-\tilde{U}$ are right continuous and left upper semi-continuous. Moreover, $\tilde{S} \in \mathcal{S}^2$ and $(-\tilde{U}) \in \mathcal{S}^2$.

For $t \in [0, \infty]$, let

$$H_t = h_t + E\Big[\int_t^\infty f^-(u)\,du \mid \mathcal{F}_t\Big], \qquad \Theta_t = \theta_t + E\Big[\int_t^\infty f^+(u)\,du \mid \mathcal{F}_t\Big],$$

where $f^+(u) = f(u) \vee 0$ and $f^-(u) = (-f(u)) \vee 0$. Then $H \in \pi_c^2$, $\Theta \in \pi_c^2$ and, for all $t \geqslant 0$, $\tilde{S}_t \leqslant H_t - \Theta_t \leqslant \tilde{U}_t$.

In the following part, we prove the existence of a solution to (5.3) by considering the iterative scheme

$$X_{n+1}^+ = R(X_n^- + \tilde{S}), \qquad X_{n+1}^- = R(X_n^+ - \tilde{U}), \qquad n \geqslant 0, \qquad \text{and} \qquad X_0^+ = X_0^- = 0.$$

(a) For all $n \geqslant 0$, X_n^+ and X_n^- are defined and

$$0 \leqslant X_n^+(t) \leqslant H_t, \qquad 0 \leqslant X_n^-(t) \leqslant \Theta_t, \qquad t \geqslant 0.$$

We do the proof by recurrence. For $n = 0$, the property holds. Suppose it also holds for some n and let us show that it still holds for $n+1$.

Since $0 \leqslant X_n(t) \leqslant \Theta_t$, we have

$$\tilde{S}_t \leqslant X_n^-(t) + \tilde{S}_t \leqslant \Theta_t + \tilde{S}_t \leqslant H_t, \qquad X_n^-(\infty) + \tilde{S}_\infty \geqslant 0.$$

So X_{n+1}^+ is defined and $0 \leqslant X_{n+1}^+(t) \leqslant H_t$.

In the same way, working with X_n^+ instead of X_n^-, we can see that X_{n+1}^- is defined and $0 \leqslant X_{n+1}^-(t) \leqslant \Theta_t$.

(b) For all $n \geqslant 0$, $X_n^+ \leqslant X_{n+1}^+$ and $X_n^- \leqslant X_{n+1}^-$.

We also do the proof by recurrence. For $n = 0$, it is obvious to get $X_1^+ \geqslant 0$ and $X_1^- \geqslant 0$. Suppose that for some n we have $X_{n-1}^+ \leqslant X_n^+$ and $X_{n-1}^- \leqslant X_n^-$. It follows that $X_{n-1}^+ - \tilde{U} \leqslant X_n^+ - \tilde{U}$ and $X_{n-1}^- + \tilde{S} \leqslant X_n^- + \tilde{S}$, which yields

$$X_n^- = R(X_{n-1}^+ - \tilde{U}) \leqslant R(X_n^+ - \tilde{U}) = X_{n+1}^-,$$
$$X_n^+ = R(X_{n-1}^- - \tilde{S}) \leqslant R(X_n^- + \tilde{S}) = X_{n+1}^+.$$

(c) For any $n \geqslant 0$, X_n^+ and X_n^- are continuous processes.

We do the proof by recurrence again. For $n = 0$, the property holds. Suppose now that for some n the processes X_n^+ and X_n^- are continuous. Since S_t and U_t are continuous, by the theory of Snell envelope (Cvitanic and Karatzas [5]), X_{n+1}^- and X_{n+1}^+ are also continuous.

(d) For all $n \geqslant 0$, $X_n^+ \in \pi_c^2$, $X_n^- \in \pi_c^2$ and $X_n^+(\infty) = 0$, $X_n^-(\infty) = 0$.

Clearly, $X_0^+ = X_0^- \equiv 0$ and $X_0^+ \in \pi_c^2$, $X_0^- \in \pi_c^2$. By recurrence we easily get the conclusion.

(e) Now, let X^+ (respectively, X^-) be the pointwise increasing limit of X_n^+ (respectively, X_n^-), i.e. for all $t \geqslant 0$,

$$X_t^+ = \lim \nearrow X_n^+(t) \qquad \text{(respectively, } X_t^- = \lim \nearrow X_n^-(t)\text{)}.$$

Then X^+ and X^- are potentials, which are nonnegative supermartingales with RCLL paths, $X^+(\infty) = X^-(\infty) = 0$ and satisfy $X^+ \in \mathscr{S}^2$, $X^- \in \mathscr{S}^2$; X^+ and X^- solve the equation (5.3).

Let us show that X^+ and X^- are continuous processes. We notice that, for $t \geqslant 0$, $S_t < U_t$. Then the set $\{\bar{S} = \underline{U}\}$ vanishes, where

$$\bar{S}_t = \limsup_{s \nearrow t} S_s, \qquad \underline{U}_t = \liminf_{s \nearrow t} U_s.$$

It follows from the result of Alario-Nazaret [1] that the processes X^+ and X^- are also upper semicontinuous. Hence $X^- + \tilde{S}$ and $X^+ - \tilde{U}$ are right continuous and left upper semicontinuous $\mathscr{F}_t$-adapted processes.

Then we need the following lemma:

LEMMA 5.2. *If η_t is a right continuous and left upper semicontinuous process and $\eta \in \mathscr{S}^2$, then its Snell envelope*

$$R_t(\eta) = \operatorname*{ess\,sup}_{v \in \mathscr{T}_t} E[\eta_v \mid \mathscr{F}_t], \qquad t \in [0, \infty],$$

is continuous when the filtration is Brownian.

Proof. By the theory of Snell envelope, R_t is the smallest RCLL supermartingale which dominates η_t. Since $\eta \in \mathscr{S}^2$, R_t is a supermartingale of class $[D]$, i.e. the set $\{R_\nu, \nu \in \mathscr{T}\}$ is uniformly integrable. So according to the Doob–Meyer decomposition, there exist a unique martingale M_t and a nondecreasing predictable process K_t ($K_0 = 0$) such that $R_t = M_t - K_t$. Moreover, the jumping times of K_t are included in $\{R^- = \underline{\eta}\}$, where R^- is the left continuous version of R and

$$\underline{\eta}_t = \limsup_{s \nearrow t} \eta_s, \qquad t \in [0, \infty].$$

Let us show that R is continuous. The martingale M is obviously continuous since $(\mathscr{F}_t)_{t \geq 0}$ is a Brownian filtration. On the other hand, if τ is a predictable stopping time, then

$$ER_{\tau-} 1_{\{\Delta K_\tau > 0\}} = E\underline{\eta}_\tau 1_{\{\Delta K_\tau > 0\}} \leqslant E[{}^p\eta_\tau 1_{\{\Delta K_\tau > 0\}}] = E\eta_\tau 1_{\{\Delta K_\tau > 0\}} \leqslant ER_\tau 1_{\{\Delta K_\tau > 0\}},$$

where ${}^p\eta$ is the predictable projection of η. The first inequality is true since η_t is left upper semicontinuous. It follows that $ER_{\tau-} = ER_\tau$ for any predictable stopping time τ since R_t is a supermartingale.

Therefore, R is a regular supermartingale (Dellacherie [7]), which implies that K is continuous, and so is R. ■

We go back to the proof of Proposition 5.1.

From Lemma 5.2 we infer that $X^- + \tilde{S}$ and $X^+ - \tilde{U}$ are right continuous and left upper semicontinuous $\mathscr{F}_t$-adapted processes, their envelopes are continuous processes, so X^+ and X^- are continuous.

We know that the process X^+ (respectively, X^-) is a continuous $\mathscr{F}_t$-supermartingale of class $[D]$ which satisfies $X^+(\infty) = 0$ (respectively, $X^-(\infty) = 0$). Hence there exists a unique continuous $\mathscr{F}_t$-adapted increasing process K^+ (respectively, K^-) such that $K_0^+ = 0$, $E(K_\infty^+)^2 < \infty$ and $X_t^+ = E[K_\infty^+ \mid \mathscr{F}_t] - K_t^+$ (respectively, $K_0^- = 0$, $E(K_\infty^-)^2 < \infty$ and $X_t^- = E[K_\infty^- \mid \mathscr{F}_t] - K_t^-$). Moreover, we have

$$\int_0^\infty (X_t^+ - X_t^- - \tilde{S}_t)\, dK_t^+ = \int_0^\infty (X_t^- - X_t^+ + \tilde{U}_t)\, dK_t^- = 0$$

(see El-Karoui [11] and Cvitanic and Karatzas [5]).

Now, let $Y_t = N_t + X_t^+ - X_t^-$ and define $Z \in \mathscr{H}^2$ by

$$E[\xi + \int_0^\infty f(s)\, ds + K_\infty^+ - K_\infty^- \mid \mathscr{F}_t] = N_0 + E[K_\infty^+ - K_\infty^-] + \int_0^t Z_s\, dB_s.$$

For all $t \geqslant 0$,

$$Y_t + \int_0^t f(s)\, ds + K_t^+ - K_t^- = E[\xi + \int_0^\infty f(s)\, ds + K_\infty^+ - K_\infty^- \mid \mathscr{F}_t] = Y_0 + \int_0^t Z_s\, dB_s.$$

Then

$$Y_t = \xi + \int_t^\infty f(s)\, ds + K_\infty^+ - K_\infty^+ - (K_\infty^- - K_t^-) - \int_t^\infty Z_s\, dB_s, \qquad t \in [0, \infty].$$

It is easy to check that (Y, Z, K^+, K^-) satisfies properties (i)–(iii) in this section. Consequently, (Y, Z, K^+, K^-) is a solution of the double-barrier reflected BSDE (5.2) associated with (f, ξ, S, U).

Let us prove the uniqueness. If $(\bar{Y}, \bar{Z}, \bar{K}^+, \bar{K}^-)$ is another solution of the reflected BSDE (5.2) associated with (f, ξ, S, U), define $\hat{Y} = Y - \bar{Y}$, $\hat{Z} = Z - \bar{Z}$, $\hat{K} = K - \bar{K}$, where $K = K^+ - K^-$, $\bar{K} = \bar{K}^+ - \bar{K}^-$. Using Itô's formula to $|\hat{Y}_t|^2$, we obtain

$$|\hat{Y}_t|^2 + \int_t^\infty |\hat{Z}_s|^2\, ds = 2\int_t^\infty \hat{Y}_s\, d\hat{K}_s - 2\int_t^\infty \hat{Y}_s \hat{Z}_s\, dB_s.$$

But $(\int_0^t \hat{Y}_s \hat{Z}_s\, dB_s)_{t\geq 0}$ is a martingale and

$$\int_t^\infty \hat{Y}_s\, d\hat{K}_s = \int_t^\infty (S_s - \bar{Y}_s)\, dK_s^+ - \int_t^\infty (U_s - \bar{Y}_s)\, dK_s^- - \int_t^\infty (Y_s - S_s)\, d\bar{K}_s^+$$
$$+ \int_t^\infty (Y_s - U_s)\, d\bar{K}_s^- \leqslant 0,$$

$$E|\hat{Y}_t|^2 + E\int_t^\infty |\hat{Z}_s|^2\, ds \leqslant 0.$$

Consequently, $E|\hat{Y}_t|^2 = 0$ a.s. for all $t \in [0, \infty]$ and $E\int_0^\infty |\hat{Z}_s|^2\, ds = 0$. Then $|\hat{Y}_t|^2 = 0$ a.s., so $Y = Y'$ by the continuity of $\hat{Y}_t$. It is easy to get $K_t = \bar{K}_t$. Finally, let us show that $K^+ = \bar{K}^+$ and $K^- = \bar{K}^-$.

For any $t \geqslant 0$, $\int_0^t (Y_s - S_s)\, dK_s = \int_0^t (Y_s - S_s)\, d\bar{K}_s$. On the other hand,

$$\int_0^t (Y_s - S_s)\, dK_s = -\int_0^t (Y_s - S_s)\, dK_s^- = -\int_0^t (U_s - S_s)\, dK_s^-.$$

In the same way we have $\int_0^t (Y_s - S_s)\, d\bar{K}_s = -\int_0^t (U_s - S_s)\, d\bar{K}_s^-$, and then

$$\int_0^t (U_s - S_s)\, dK_s^- = \int_0^t (U_s - S_s)\, d\bar{K}_s^- \quad \text{for all } t \in [0, \infty],$$

which implies $(U_t - S_t)\, dK_t^- = (U_t - S_t)\, d\bar{K}_t^-$. Consequently, $K_t^- = \bar{K}_t^-$ since $K_0^- = \bar{K}_0^- = 0$ and $S_t < U_t$ for all $t \geqslant 0$. Similarly, from the equality $\int_0^t (U_s - Y_s)\, dK_s = \int_0^t (U_s - Y_s)\, d\bar{K}_s$ we obtain $K^+ = \bar{K}^+$. So we get the uniqueness of the solution to the infinite horizon reflected BSDE (5.2) with two barriers. ∎

Now we give the main result of this section.

THEOREM 5.3. *Let $\xi \in L^2$, let f satisfy* (H2.1) *and* (H2.2), *and assume that* (H5.1)–(H5.3) *hold true. Then the double-barrier reflected BSDE* (5.1) *associated with (f, ξ, S, U) has a unique solution (Y, Z, K^+, K^-).*

Proof. We first prove the existence. It is also divided into two steps.

Step 1. Assume $[(\int_0^\infty u_1(s)\, ds)^2 + \int_0^\infty u_2^2(s)\, ds]^{1/2} < \frac{1}{70}$.

Let $\mathscr{D}$ be the space of the process (Y, Z) with values in $\boldsymbol{R}^{1+m}$ such that $Y \in \mathscr{S}^2$, $Z \in \mathscr{H}^2$ and $\|(Y, Z)\|_{\mathscr{D}}^2 = \|Y\|_{\mathscr{S}^2}^2 + \|Z\|_{\mathscr{H}^2}^2$. We define a mapping Ψ from

$\mathscr{D}$ onto itself as follows: for any $(\mathscr{U}, \mathscr{V}) \in \mathscr{D}$, $(Y, Z) = \Psi(\mathscr{U}, \mathscr{V})$ is the unique element of $\mathscr{D}$ such that (Y, Z, K^+, K^-) solves the double-barrier reflected BSDE associated with $(f(s, \mathscr{U}_s, \mathscr{V}_s), \xi, S, U)$.

Let $(\bar{\mathscr{U}}, \bar{\mathscr{V}})$ be an element of $\mathscr{D}$ and define $(\bar{Y}, \bar{Z}) = \Psi(\bar{\mathscr{U}}, \bar{\mathscr{V}})$, $\hat{\mathscr{U}} = \mathscr{U} - \bar{\mathscr{U}}$, $\hat{\mathscr{V}} = \mathscr{V} - \bar{\mathscr{V}}$, $\hat{Y} = Y - \bar{Y}$, $\hat{Z} = Z - \bar{Z}$, $\hat{K} = K - \bar{K}$, $K = K^+ - K^-$, $\bar{K} = \bar{K}^+ - \bar{K}^-$ and $\hat{f} = f(s, \mathscr{U}_s, \mathscr{V}_s) - f(s, \bar{\mathscr{U}}_s, \bar{\mathscr{V}}_s)$. We want to prove that the mapping Ψ is a contraction.

Using Itô's formula to $|\hat{Y}_t|^2$, we have

$$(5.4) \qquad |\hat{Y}_t|^2 + \int_t^\infty |\hat{Z}_s|^2 ds = 2\int_t^\infty \hat{Y}_s \hat{f}(s)\, ds + 2\int_t^\infty \hat{Y}_s\, d\hat{K}_s - 2\int_t^\infty \hat{Y}_s \hat{Z}_s\, dB_s$$
$$\leqslant 2\int_t^\infty \hat{Y}_s \hat{f}(s)\, ds - 2\int_t^\infty \hat{Y}_s \hat{Z}_s\, dB_s.$$

Then

$$E \sup_{0 \leqslant t \leqslant \infty} |\hat{Y}_t|^2 \leqslant 2E \sup_{0 \leqslant t \leqslant \infty} |\hat{Y}_t| \int_0^\infty |\hat{f}(s)|\, ds + 2E \sup_{0 \leqslant t \leqslant \infty} \Big|\int_t^\infty \hat{Y}_s \hat{Z}_s\, dB_s\Big|$$
$$\leqslant \tfrac{1}{6} E \sup_{0 \leqslant t \leqslant \infty} |\hat{Y}_t|^2 + 6E\Big(\int_0^\infty |\hat{f}(s)|\, ds\Big)^2 + 4E\Big[\int_0^\infty |\hat{Y}_s|^2 |\hat{Z}_s|^2 ds\Big]^{1/2}$$
$$\leqslant \tfrac{1}{6} E \sup_{0 \leqslant t \leqslant \infty} |\hat{Y}_t|^2 + 6E\Big(\int_0^\infty |\hat{f}(s)|\, ds\Big)^2 + \tfrac{1}{2} E\big[\sup_{0 \leqslant t \leqslant \infty} |\hat{Y}_t|^2\big] + 8E\int_0^\infty |\hat{Z}_t|^2 dt.$$

Consequently,

$$(5.5) \qquad E \sup_{0 \leqslant t \leqslant \infty} |\hat{Y}_t|^2 \leqslant 18E\Big(\int_0^\infty |\hat{f}(s)|\, ds\Big)^2 + 24E\int_0^\infty |\hat{Z}_t|^2 dt.$$

Going back to (5.4), we have

$$(5.6) \qquad E\int_0^\infty |\hat{Z}_t|^2 dt \leqslant 2E\int_0^\infty |\hat{Y}_t|\,|\hat{f}(t)|\, dt \leqslant 2E \sup_{0 \leqslant t \leqslant \infty} |\hat{Y}_t| \int_0^\infty |\hat{f}(s)|\, ds$$
$$\leqslant \tfrac{1}{48} E \sup_{0 \leqslant t \leqslant \infty} |\hat{Y}_t|^2 + 48E\Big(\int_0^\infty |\hat{f}(s)|\, ds\Big)^2.$$

From (5.5) and (5.6) we get

$$E \sup_{0 \leqslant t \leqslant \infty} |\hat{Y}_t|^2 \leqslant \tfrac{1}{2} E \sup_{0 \leqslant t \leqslant \infty} |\hat{Y}_t|^2 + (18 + 24 \times 48)\, E\Big(\int_0^\infty |\hat{f}(s)|\, ds\Big)^2,$$

so

$$E \sup_{0 \leqslant t \leqslant \infty} |\hat{Y}_t|^2 \leqslant 2340E\Big(\int_0^\infty |\hat{f}(s)|\, ds\Big)^2.$$

Going back to (5.6) again, we have

$$E\int_0^\infty |\hat{Z}_t|^2 dt \leqslant \frac{2340}{48} E \sup_{0 \leqslant t \leqslant \infty} |\hat{Y}_t|^2 + 48E\Big(\int_0^\infty |\hat{f}(s)|\, ds\Big)^2 < 100E\Big(\int_0^\infty |\hat{f}(s)|\, ds\Big)^2.$$

Similarly to Theorem 3.2, we have

$$\|(\hat{Y}, \hat{Z})\|_{\mathscr{B}}^2 < 4880\Big[\Big(\int_0^\infty u_1(s)\,ds\Big)^2 + \int_0^\infty u_2^2(s)\,ds\Big]\,\|(\hat{\mathscr{U}}, \hat{\mathscr{V}})\|_{\mathscr{B}}^2$$

$$< 70^2\Big[\Big(\int_0^\infty u_1(s)\,ds\Big)^2 + \int_0^\infty u_2^2(s)\,ds\Big]\,\|(\hat{\mathscr{U}}, \hat{\mathscr{V}})\|_{\mathscr{B}}^2.$$

From the inequality $[(\int_0^\infty u_1(s)\,ds)^2 + \int_0^\infty u_2^2(s)\,ds]^{1/2} < \frac{1}{70}$ we deduce that Ψ is a strict contraction and has a unique fixed point, which is the unique solution of the double-barrier reflected BSDE (5.1).

Step 2. For the general case, there exists $T_0 > 0$ such that

$$\Big[\Big(\int_{T_0}^\infty u_1(s)\,ds\Big)^2 + \int_{T_0}^\infty u_2^2(s)\,ds\Big]^{1/2} < \frac{1}{70}.$$

From Step 1 we know that the reflected BSDE

$$\bar{Y}_t = \xi + \int_t^\infty 1_{\{s \geq T_0\}} f(s, \bar{Y}_s, \bar{Z}_s)\,ds + \bar{K}_\infty^+ - K_t^+ - (\bar{K}_\infty^- - \bar{K}_t^-) - \int_t^\infty \bar{Z}_s\,dB_s, \quad t \in [0, \infty], \tag{5.7}$$

has a unique solution $(\bar{Y}_t, \bar{Z}_t, \bar{K}_t^+, \bar{K}_t^-)$. Then we consider the double-barrier reflected BSDE

$$\tilde{Y}_t = \bar{Y}_{T_0} + \int_t^{T_0} f(s, \tilde{Y}_s, \tilde{Z}_s)\,ds + \tilde{K}_{T_0}^+ - \tilde{K}_t^+ - (\tilde{K}_{T_0}^- - \tilde{K}_t^-) - \int_t^{T_0} \tilde{Z}_s\,dB_s, \quad t \in [0, T_0]. \tag{5.8}$$

From the result of Cvitanic and Karatzas [5] we know that there exists a unique solution $(\tilde{Y}_t, \tilde{Z}_t, \tilde{K}_t^+, \tilde{K}_t^-)$ satisfying the double-barrier reflected BSDE (5.8) and the properties (i)–(iii) in this section on $[0, T_0]$. We set

$$Y_t = \begin{cases} \tilde{Y}_t, & t \in [0, T_0], \\ \bar{Y}_t, & t \in (T_0, \infty], \end{cases} \qquad Z_t = \begin{cases} \tilde{Z}_t, & t \in [0, T_0], \\ \bar{Z}_t, & t \in (T_0, \infty], \end{cases}$$

$$K_t^+ = \begin{cases} \tilde{K}_t^+, & t \in [0, T_0], \\ \tilde{K}_{T_0}^+ + \bar{K}_t^+ - \bar{K}_{T_0}^+, & t \in (T_0, \infty], \end{cases}$$

$$K_t^- = \begin{cases} \tilde{K}_t^-, & t \in [0, T_0], \\ \tilde{K}_{T_0}^- + \bar{K}_t^- - \bar{K}_{T_0}^-, & t \in (T_0, \infty]. \end{cases}$$

Then it is easy to check that (Y, Z, K^+, K^-) is a solution of the double-barrier reflected BSDE (5.1).

At last, we prove the uniqueness of the solution of (5.1). Let $(\bar{Y}, \bar{Z}, \bar{K}^+, \bar{K}^-)$ be another solution of the double-barrier reflected BSDE (5.1) associated with (f, ξ, S, U). We use the same notation as in Proposition 5.1; Itô's formula gives

$$|\hat{Y}_t|^2+\int_t^\infty |\hat{Z}_s|^2\,ds = 2\int_t^\infty \hat{Y}_s\big(f(s, Y_s, Z_s)-f(s, \bar{Y}_s, \bar{Z}_s)\big)\,ds$$
$$+2\int_t^\infty \hat{Y}_s\,d\hat{K}_s-2\int_t^\infty \hat{Y}_s\hat{Z}_s\,dB_s.$$

Then, taking expectation we get

$$(5.9)\qquad E|\hat{Y}_t|^2+E\int_t^\infty |\hat{Z}_s|^2\,ds \leqslant 2E\int_t^\infty |\hat{Y}_s|\,[u_1(s)|\hat{Y}_s|+u_2(s)|\hat{Z}_s|]\,ds$$
$$\leqslant \frac{1}{2}E\int_t^\infty |\hat{Z}_s|^2\,ds+E\int_t^\infty [2u_1(s)+2u_2^2(s)]\,|\hat{Y}_s|^2\,ds.$$

From Gronwall's lemma we obtain $E|\hat{Y}_t|^2 = 0$ for all $t\in[0, \infty]$. Then $|\hat{Y}_t|^2 = 0$ a.s., so $Y = Y'$ by the continuity of $\hat{Y}$. Going back to (5.9), we have

$$E\int_0^\infty |\hat{Z}_s|^2\,ds \leqslant 4E\sup_{\{0\leqslant t\leqslant\infty\}}|\hat{Y}_t|^2\int_0^\infty [u_1(s)+u_2^2(s)]\,ds,$$

so $E\int_0^\infty |\hat{Z}_s|^2\,ds = 0$. Then it is easy to get $K_t = \bar{K}_t$. In the same way as in the proof of Proposition 5.1, we can get $K_t^+ = \bar{K}_t^+$ and $K_t^- = \bar{K}_t^-$. ■

6. Applications in the mixed game problem with infinite horizon. Like for the mixed control problem, we now use the double-barrier reflected BSDE with infinite horizon to deal with a stochastic mixed differential game problem.

Let $\mathscr{C}$, σ, X, U and $\mathscr{U}$ be the same as in Section 4, let V be another compact metric space, and $\mathscr{V}$ be the space of $\mathscr{P}$-measurable processes with values in V.

Let φ be a function from $[0, \infty)\times\mathscr{C}\times U\times V$ into $\boldsymbol{R}^m$ such that:

(i) $\varphi(\cdot, X(\cdot), \cdot, \cdot)$ is $\mathscr{P}\otimes\mathscr{B}(U\times V)$-measurable, $\mathscr{B}(U\times V)$ is the Borel σ-algebra on $U\times V$.

(ii) For any $t\in[0, \infty)$ and $x\in\mathscr{C}$, $\varphi(t, x, \cdot, \cdot)$ is continuous on $U\times V$.

(iii) $|\varphi(t, x, u, v)| \leqslant c(t)$ P-a.s., where $c(t)$ is deterministic and $\int_0^\infty c^2(t)\,dt < \infty$.

For any $(u, v)\in\mathscr{U}\times\mathscr{V}$, we define a probability $P^{(u,v)}$ on $(\Omega, \mathscr{F})$ by

$$\frac{dP^{(u,v)}}{dP} = \exp\Big\{\int_0^\infty \sigma^{-1}(s, X)\,\varphi(s, X, u_s, v_s)\,dB_s$$
$$-\tfrac{1}{2}\int_0^\infty |\sigma^{-1}(s, X)\,\varphi(s, X, u_s, v_s)|^2\,ds\Big\}.$$

According to Girsanov's theorem, the process

$$B_t^{(u,v)} = B_t - \int_0^t \sigma^{-1}(s, X)\,\varphi(s, X, u_s, v_s)\,ds, \qquad t \geqslant 0,$$

is a Brownian motion on $(\Omega, \mathscr{F}, P^{(u,v)})$ and X is a weak solution of

$$dX_t = \varphi(t, X, u_s, v_s)\,ds + \sigma(t, X)\,dB_t^{(u,v)}, \qquad X_0 = x, \qquad t \in [0, \infty).$$

Suppose now that we have a system whose evolution is described by X, which has an effect on the payoffs of two players J_1 and J_2. For their part the controllers have no influence on the system and they act such as to protect their advantages, which are antagonistic, by means of $u \in \mathscr{U}$ for J_1 and $v \in \mathscr{V}$ for J_2 via the probability $P^{(u,v)}$. The pair $(u, v) \in \mathscr{U} \times \mathscr{V}$ is called an *admissible control* for the game. On the other hand, the two players have also the possibility to stop the game at σ for J_1 and τ for J_2, where σ and τ are elements of $\mathscr{T}$. So the controlling actions are not free and the payoff corresponding to the actions of J_1 and J_2 is defined by

$$J(u, \sigma; v, \tau) = E^{(u,v)}\Big[\int_0^{\tau \wedge \sigma} C(s, X, u_s, v_s)\,ds$$
$$+ S_\tau 1_{\{\tau \leqslant \sigma, \sigma < \infty\}} + U_\sigma 1_{\{\sigma < \tau\}} + \xi 1_{\{\tau = \sigma = \infty\}}\Big],$$

where S_t, U_t and ξ are the same as in Section 5, $C(t, x, u, v)$ is a function from $[0, \infty) \times \mathscr{C} \times U \times V$ onto $\boldsymbol{R}$ which satisfies the same hypotheses as φ. The action of J_1 (respectively, J_2) is to minimize (respectively, maximize) the payoff $J(u, \sigma; v, \tau)$. We can understand the reward and cost for the two players as follows:

(i) $C(t, X, u, v)$ is the instantaneous reward (respectively, cost) for J_2 (respectively, J_1).

(ii) U_σ is the terminal cost (respectively, reward) for J_1 (respectively, J_2) if J_1 decides to stop first the game.

(iii) S_τ is the terminal reward (respectively, cost) for J_2 (respectively, J_1) if J_2 decides to stop first the game.

Our problem is to look for a saddle-point strategy for the two players, i.e. a strategy $(\hat{u}, \hat{\sigma}; \hat{v}, \hat{\tau})$ such that

$$J(\hat{u}, \hat{\sigma}; v, \tau) \leqslant J(\hat{u}, \hat{\sigma}; \hat{v}, \hat{\tau}) \leqslant J(u, \sigma; \hat{v}, \hat{\tau})$$

for any $(u, \sigma) \in \mathscr{U} \times \mathscr{T}$ and $(v, \tau) \in \mathscr{V} \times \mathscr{T}$.

For $(t, x, p, u, v) \in [0\ \infty) \times \mathscr{C} \times \boldsymbol{R}^m \times U \times V$ we define the Hamiltonian by

$$H(t, x, p, u, v) = p\varphi(t, x, u, v) + C(t, x, u, v)$$

and we suppose that the following assumption holds:

$$\inf_{u \in U} \sup_{v \in V} H(t, X, p, u, v) = \sup_{v \in V} \inf_{u \in U} H(t, X, p, u, v). \tag{6.1}$$

Under the above condition, which is called *Isaacs's condition* (Elliott [10], Bensoussan and Lions [3], Davis and Elliott [6], Hamadène and Lepeltier [14], [15]), by the Benes theorem (Benes [2]), there exists a pair of $\mathscr{P}\otimes\mathscr{B}(\boldsymbol{R}^m)$-measurable functions $u^*(t, X, p)$ and $v^*(t, X, p)$ with values, respectively, in U and V such that for any $(t, p)\in[0, \infty)\times\boldsymbol{R}^m$, $u\in U$ and $v\in V$,

$$H\big(t, X, p, u^*(t, X, p), v^*(t, X, p)\big) = \inf_{u\in U}\sup_{v\in V} H(t, X, p, u, v) \tag{6.2}$$

$$= \sup_{v\in V}\inf_{u\in U} H(t, X, p, u, v)$$

and

$$H\big(t, X, p, u^*(t, X, p), v\big) \leqslant H\big(t, X, p, u^*(t, X, p), v^*(t, X, p)\big) \tag{6.3}$$

$$\leqslant H\big(t, X, p, u, v^*(t, X, p)\big)$$

for all $(u, v)\in U\times V$. Under the assumption (iii) of φ, $H(t, X, p, u, v)$ satisfies the Lipschitz condition (H2.2) in p. Then it is easily deduced from (6.2) that the function $H\big(t, X, p, u^*(t, X, p), v^*(t, X, p)\big)$ also satisfies the Lipschitz condition (H2.2) in p.

We now give the main result of this section.

THEOREM 6.1. *Assume* (H5.1)–(H5.3) *and Isaacs's condition* (6.1) *are satisfied; let* $(Y^*, Z^*, K^{*+}, K^{*-})$ *be the solution of the double-barrier reflected BSDE with infinite horizon associated with* $\big(H\big(t, X, Z, u^*(t, X, Z), v^*(t, X, Z)\big), \xi, S, U\big)$, $u^* = u^*(t, X, Z_t^*)$, $v^* = v^*(t, X, Z_t^*)$, $t\in[0, \infty)$, *and*

$$\hat{\tau} = \begin{cases}\inf\{t\in[0, \infty),\ Y_t^* \leqslant S_t\}, \\ \infty \quad \textit{otherwise},\end{cases} \qquad \hat{\sigma} = \begin{cases}\inf\{t\in[0, \infty),\ Y_t^* \geqslant U_t\}, \\ \infty \quad \textit{otherwise}.\end{cases}$$

Then $Y_0^* = J(u^*, \hat{\sigma}; v^*, \hat{\tau})$ *and* $(u^*, \hat{\sigma}; v^*, \hat{\tau})$ *is a saddle-point strategy for the mixed stochastic game problem with infinite horizon.*

Proof. We consider the following double-barrier infinite horizon reflected BSDE associated with $\big(H\big(t, X, Z, u^*(t, X, Z), v^*(t, X, Z)\big), \xi, S, U\big)$:

$$Y_t^* = \xi + \int_t^\infty H\big(s, X, Z_s^*, u^*(s, X, Z_s^*), v^*(s, X, Z_s^*)\big)\,ds$$

$$+K_\infty^{*+} - K_t^{*+} - (K_\infty^{*-} - K_t^{*-}) - \int_t^\infty Z_s^*\,dB_s.$$

By Theorem 5.3, there exists a unique solution $(Y_t^*, Z_t^*, K_t^{*+}, K_t^{*-})$ satisfying the properties (i)–(iii) in Section 5 and we know that Y_0^* is a deterministic constant. Then

$$Y_0^* = E^{(u^*,v^*)}[Y_0^*]$$

$$= E^{(u^*,v^*)}\Big[\xi + \int_0^\infty H\big(s, X, Z_s^*, u^*(s, X, Z_s^*), v^*(s, X, Z_s^*)\big)\,ds$$

$$+K_\infty^{*+}-K_\infty^{*-}-\int_0^\infty Z_s^*\,dB_s]$$

$$=E^{(u^*,v^*)}\Big[\int_0^{\hat\tau\wedge\hat\sigma} H\big(s,X,Z_s^*,u^*(s,X,Z_s^*),v^*(s,X,Z_s^*)\big)\,ds$$

$$+K_{\hat\tau\wedge\hat\sigma}^{*+}-K_{\hat\tau\wedge\hat\sigma}^{*-}-\int_0^{\hat\tau\wedge\hat\sigma} Z_s^*\,dB_s+Y_{\hat\tau\wedge\hat\sigma}^*\Big]$$

$$=E^{(u^*,v^*)}\Big[\int_0^{\hat\tau\wedge\hat\sigma} C\big(s,X,u^*(s,X,Z_s^*),v^*(s,X,Z_s^*)\big)\,ds$$

$$+K_{\hat\tau\wedge\hat\sigma}^{*+}-K_{\hat\tau\wedge\hat\sigma}^{*-}-\int_0^{\hat\tau\wedge\hat\sigma} Z_s^*\,dB_s^{(u^*,v^*)}+Y_{\hat\tau\wedge\hat\sigma}^*\Big].$$

We know that the processes K^{*+} and K^{*-} increase only when $Y_t^*=S_t$ and $Y_t^*=U_t$, respectively. Therefore, they do not increase between 0 and $\hat\tau\wedge\hat\sigma$, and then $K_{\hat\tau\wedge\hat\sigma}^{*+}=K_{\hat\tau\wedge\hat\sigma}^{*-}=0$. On the other hand, using the Burkholder–Davis–Gundy inequality and the assumptions on φ, we deduce that the process $\big(\int_0^t Z_s^*\,dB_s^{(u^*,v^*)}\big)_{t\geq 0}$ is a $P^{(u^*,v^*)}$-martingale, and then

$$Y_0^*=E^{(u^*,v^*)}\Big[\int_0^{\hat\tau\wedge\hat\sigma} C\big(s,X,u^*(s,X,Z_s^*),v^*(s,X,Z_s^*)\big)\,ds+Y_{\hat\tau\wedge\hat\sigma}^*\Big].$$

From the equality

$$Y_{\hat\tau\wedge\hat\sigma}^*=S_{\hat\tau}1_{\{\hat\tau\leq\hat\sigma,\hat\sigma<\infty\}}+U_{\hat\sigma}1_{\{\hat\sigma<\hat\tau\}}+\xi 1_{\{\hat\tau=\hat\sigma=\infty\}}\quad P^{(u^*,v^*)}\text{-a.s.}$$

we obtain $Y_0^*=J(u^*,\hat\sigma;v^*,\hat\tau)$.

Now, let u be an element of $\mathscr{U}$, and σ be a stopping time. Since P and $P^{(u,v^*)}$ are equivalent probabilities on $(\Omega,\mathscr{F})$, we get

$$Y_0^*=E^{(u,v^*)}[Y_0^*]$$

$$=E^{(u,v^*)}\Big[\xi+\int_0^\infty H\big(s,X,Z_s^*,u^*(s,X,Z_s^*),v^*(s,X,Z_s^*)\big)\,ds$$

$$+K_\infty^{*+}-K_\infty^{*-}-\int_0^\infty Z_s^*\,dB_s\Big]$$

$$=E^{(u,v^*)}\Big[\int_0^{\sigma\wedge\hat\tau} H\big(s,X,Z_s^*,u^*(s,X,Z_s^*),v^*(s,X,Z_s^*)\big)\,ds$$

$$+K_{\sigma\wedge\hat\tau}^{*+}-K_{\sigma\wedge\hat\tau}^{*-}-\int_0^{\sigma\wedge\hat\tau} Z_s^*\,dB_s+Y_{\sigma\wedge\hat\tau}^*\Big]$$

$$=E^{(u,v^*)}\Big[\int_0^{\sigma\wedge\hat\tau} C\big(s,X,u_s,v^*(s,X,Z_s^*)\big)\,ds+K_{\sigma\wedge\hat\tau}^{*+}-K_{\sigma\wedge\hat\tau}^{*-}-\int_0^{\sigma\wedge\hat\tau} Z_s^*\,dB_s^{(u,v^*)}+Y_{\sigma\wedge\hat\tau}^*$$

$$+\int_0^{\sigma\wedge\hat\tau}\Big(H\big(s,X,Z_s^*,u^*(s,X,Z_s^*),v^*(s,X,Z_s^*)\big)$$

$$-H\big(s,X,Z_s^*,u_s,v^*(s,X,Z_s^*)\big)\Big)\,ds\Big].$$

But $K^{*+}_{\sigma\wedge\hat{\tau}} = 0$ $P^{(u,v^*)}$-a.s, and by (6.3) we have

$$H(s, X, Z_s^*, u^*(s, X, Z_s^*), v^*(s, X, Z_s^*)) - H(s, X, Z_s^*, u_s, v^*(s, X, Z_s^*)) \leqslant 0 \quad \text{for all } s \in [0, \infty).$$

On the other hand, $\left(\int_0^t Z_s^* \, dB_s^{(u,v^*)}\right)_{t \geqslant 0}$ is a $P^{(u,v^*)}$-martingale,

$$\begin{aligned} Y^*_{\sigma\wedge\hat{\tau}} &= Y^*_{\hat{\tau}} 1_{\{\hat{\tau} \leqslant \sigma, \sigma < \infty\}} + Y^*_\sigma 1_{\{\sigma < \hat{\tau}\}} + \xi 1_{\{\hat{\tau} = \sigma = \infty\}} \\ &\leqslant S_{\hat{\tau}} 1_{\{\hat{\tau} \leqslant \sigma, \sigma < \infty\}} + U_\sigma 1_{\{\sigma < \hat{\tau}\}} + \xi 1_{\{\hat{\tau} = \sigma = \infty\}}. \end{aligned}$$

It follows that

$$\begin{aligned} J(u^*, \hat{\sigma}; v^*, \hat{\tau}) &= Y_0^* \\ &\leqslant E^{(u,v^*)}\Big[\int_0^{\sigma\wedge\hat{\tau}} C(s, X, u_s, v^*(s, X, Z_s^*))\, ds + S_{\hat{\tau}} 1_{\{\hat{\tau} \leqslant \sigma, \sigma < \infty\}} + U_\sigma 1_{\{\sigma < \hat{\tau}\}} + \xi 1_{\{\hat{\tau} = \sigma = \infty\}}\Big] \\ &= J(u, \sigma; v^*, \hat{\tau}). \end{aligned}$$

In the same way we can show that, for any $(v, \tau) \in \mathscr{V} \times \mathscr{T}$ we have

$$\begin{aligned} J(u^*, \hat{\sigma}; v^*, \hat{\tau}) &= Y_0^* \\ &\geqslant E^{(u^*,v)}\Big[\int_0^{\hat{\sigma}\wedge\tau} C(s, X, u^*(s, X, Z_s^*), v_s)\, ds + S_\tau 1_{\{\tau \leqslant \hat{\sigma}, \hat{\sigma} < \infty\}} + U_{\hat{\sigma}} 1_{\{\hat{\sigma} < \tau\}} + \xi 1_{\{\hat{\sigma} = \tau = \infty\}}\Big] \\ &= J(u^*, \hat{\sigma}; v, \tau). \end{aligned}$$

Henceforth the strategy $(u^*, \hat{\sigma}; v^*, \hat{\tau})$ is a saddle-point for the mixed stochastic game problem with infinite horizon.

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