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A NON-ERGODIC PHENOMENON FOR SOME RANDOM DYNAMICAL SYSTEM

BY

ANDRZEJ KOMISARSKI* (WARSZAWA)

Abstract. In [2] Jajte formulated the following question:
Let $h_0(x)$ and $h_1(x)$ be homeomorphisms of the interval $[0, 1]$ onto itself. Is it true that for any $x \in [0, 1]$ and almost any $t \in (0, 1)$ there exists a limit of a sequence

$$\frac{1}{n}\sum_{i=1}^{n} h_{t_i} \circ \ldots \circ h_{t_1}(x)$$

for $n \to \infty$, where $t = (0, t_1 t_2 \ldots)_2$ is a binary representation of t, i.e. $t = \sum_{i \geq 1} t_i 2^{-i}$ and $t_i \in \{0, 1\}$?
The answer is negative. We describe the set of condensation points of the sequence in some special cases.

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1. Introduction and main results. Let h_0 and h_1 be homeomorphisms of the interval $[0, 1]$ onto itself. Fix $x \in [0, 1]$ and $t \in (0, 1)$. We discuss the sequences

$$\frac{1}{n}\sum_{i=1}^{n} h_{t_i} \circ \ldots \circ h_{t_1}(x), \tag{1}$$

where $t = (0, t_1 t_2 \ldots)_2$ is a binary representation of t, i.e. $t = \sum_{i \geq 1} t_i 2^{-i}$ and $t_i \in \{0, 1\}$. For t chosen in a random way, one can consider (1) as ergodic means for an elementary example of a random dynamical system. In [2] Jajte asked if the sequence (1) converges with $n \to \infty$ for all $x \in [0, 1]$ and almost all (in the sense of Lebesgue measure) $t \in (0, 1)$. It emerges that the answer is negative. Moreover, for a large and easily describable class of pairs h_0, h_1 the limit does not exist for almost all $t \in (0, 1)$ and almost all $x \in [0, 1]$. More precisely, we have:

THEOREM 1. *There exists $T \subset [0, 1]$ with $\lambda(T) = 1$ such that, for any increasing homeomorphism h: $[0, 1] \to [0, 1]$ with 0 and 1 as the only fixed*

* Department of Mathematics, Warsaw University.

points and for any $x\in(0, 1)$ *and any* $t\in T$, *the set* $\{n^{-1}\sum_{i=1}^{n} h_{t_i}\circ \ldots \circ h_{t_1}(x)\colon n\in N\}$ *is dense in* $[0, 1]$, *where* $h_0 = h$, $h_1 = h^{-1}$, $t = \sum_{i\geqslant 1} t_i 2^{-i}$, $t_i\in\{0, 1\}$, *and* $\lambda(\,)$ *denotes Lebesgue measure.*

Roughly speaking, if one takes ergodic means of superpositions of homeomorphisms chosen in a random way, then instead of one limit a dense set of condensation points is obtained. In some sense, the result is opposite to that which would be expected by analogy to ergodic theorems and to behaviour of a simple dynamical system defined by a homeomorphism of the interval [0, 1] onto itself. An analogical result for an arbitrary increasing homeomorphism h is described in Section 3.

2. Proofs. Before proving the theorem we fix some notation. Let $R\subset [0, 1]$ be a set of numbers with more than one binary representation. Obviously, $\lambda(R) = 0$. On the probability space $(\Omega = [0, 1]\setminus R, \mathrm{Borel}(\Omega), \lambda)$ the Rademacher sequence $r_i = r_i(t) = 1-2t_i$ forms a family of independent random variables with distribution $\lambda(r_i = 1) = \lambda(r_i = -1) = 1/2$. For $t\in[0, 1]\setminus R$, $x\in[0, 1]$, $n\in N$, we put

$$a_{t,n}(x) = \frac{1}{n}\sum_{i=1}^{n} h_{t_i}\circ \ldots \circ h_{t_1}(x) = \frac{1}{n}\sum_{i=1}^{n} h^{r_i}\circ \ldots \circ h^{r_1}(x) = \frac{1}{n}\sum_{i=1}^{n} h^{r_i+\ldots+r_1}(x).$$

Proof of Theorem 1. The demanded set T can be defined by the following formula:

(2)

$$T = \Big\{t\in[0, 1]\setminus R\colon \forall_{\alpha\in(0,1)}\forall_{N\in N}\exists_{n\in N}\frac{1}{n}\#\{i = 1, \ldots, n\colon \sum_{k=1}^{i} r_k(t) > N\} > \alpha\Big\}$$
$$\cap\Big\{t\in[0, 1]\setminus R\colon \forall_{\alpha\in(0,1)}\forall_{N\in N}\exists_{n\in N}\frac{1}{n}\#\{i = 1, \ldots, n\colon \sum_{k=1}^{i} r_k(t) < -N\} > \alpha\Big\}.$$

The required properties of the set T are proved in Lemmas 1 and 2. ■

LEMMA 1. *For any increasing homeomorphism* $h\colon [0, 1]\to[0, 1]$ *with* 0 *and* 1 *being the only fixed points of* h *and any* $t\in T$ *the set* $\{a_{t,n}(x)\colon n\in N\}$ *is dense in* $[0, 1]$ *for any* $x\in(0, 1)$.

Proof. Fix a homeomorphism h and points $t\in T$, $x\in(0, 1)$. According to the definition of $a_{t,n}(x)$ we have

$$a_{t,n+1}(x) = \frac{1}{n+1}[h_{t_{n+1}}\circ \ldots \circ h_{t_1}(x)+na_{t,n}(x)],$$

$$\frac{na_{t,n}(x)}{n+1} \leqslant a_{t,n+1}(x) \leqslant \frac{na_{t,n}(x)+1}{n+1},$$

$$-\frac{1}{n+1} \leqslant -\frac{a_{t,n}(x)}{n+1} \leqslant a_{t,n+1}(x)-a_{t,n}(x) \leqslant \frac{1-a_{t,n}(x)}{n+1} \leqslant \frac{1}{n+1},$$

and

(3) $$|a_{t,n+1}(x)-a_{t,n}(x)| \leqslant \frac{1}{n+1} \to 0.$$

Now we prove that $\limsup_{n\to\infty} a_{t,n}(x)$ and $\liminf_{n\to\infty} a_{t,n}(x)$ are equal to 1 and 0, respectively. The number x is not a fixed point of h; hence $h(x) > x$ or $h(x) < x$. Both cases are analogical, so it is enough to consider the case $h(x) > x$. Numbers $h^n(x) > 0$ form an increasing bounded sequence of reals, so there exists $\lim_{n\to\infty} h^n(x) > 0$. Moreover,

$$h\big(\lim_{n\to\infty} h^n(x)\big) = \lim_{n\to\infty} h^n(x),$$

so $\lim_{n\to\infty} h^n(x)$ is a fixed point of h and must be equal to 1.

Consider $\limsup a_{t,n}(x)$ for $n\to\infty$. Let $0<\varepsilon<1$ be arbitrarily chosen. Fix $N\in N$ satisfying

$$\forall_{n>N}\ 1-h^n(x) < \varepsilon/2.$$

For $t\in T$

$$\exists_{n\in N}\ \frac{1}{n}\#\{i=1,\ldots,n:\ \sum_{k=1}^{i} r_k > N\} > \frac{1-\varepsilon}{1-\varepsilon/2}.$$

For such n we have

$$1 > a_{t,n}(x) = \frac{\sum_{i=1}^{n} h^{r_1+\ldots+r_i}(x)}{n} = \frac{\sum_{\substack{i=1\\ r_1+\ldots+r_i>N}}^{n} h^{r_1+\ldots+r_i}(x) + \sum_{\substack{i=1\\ r_1+\ldots+r_i\leqslant N}}^{n} h^{r_1+\ldots+r_i}(x)}{n}$$

$$\geqslant \frac{\sum_{\substack{i=1\\ r_1+\ldots+r_i>N}}^{n} (1-\varepsilon/2)}{n} \geqslant (1-\varepsilon/2)\frac{\#\{i=1,\ldots,n:\ r_1+\ldots+r_i>N\}}{n}$$

$$> (1-\varepsilon/2)\frac{1-\varepsilon}{1-\varepsilon/2} = 1-\varepsilon.$$

Hence

(4) $$\forall_{\varepsilon>0}\,\exists_{n\in N}\,|a_{t,n}(x)-1| < \varepsilon.$$

It is easy to prove in the same way that

(5) $$\forall_{\varepsilon>0}\,\exists_{n\in N}\,|a_{t,n}(x)-0| < \varepsilon.$$

Relations (3), (4) and (5) imply that $\{a_{t,n}(x):\ n\in N\}$ is dense in $[0, 1]$. ■

LEMMA 2. *The Lebesgue measure of the set T defined by* (2) *is equal to* 1.

To prove this lemma we need the following generalization of the classical arc sin law for a symmetric random walk. (For more details about arc sin law see [1].)

LEMMA 3. *For any* $N \in \mathbf{Z}$ *and any* $0 < \alpha < 1$ *we have*

$$\lambda\left(\left\{t \in [0, 1] \setminus R: \frac{1}{n} \# \{i = 1, \ldots, n: \sum_{k=1}^{i} r_k(t) > N\} > \alpha\right\}\right) \to f(\alpha) \qquad \text{for } n \to \infty,$$

where $f(\alpha) = 1 - 2\pi^{-1} \arcsin \sqrt{\alpha}$.

Proof of Lemma 2. For any $N \in \mathbf{N}$ and $0 < \alpha < 1$ let us put

$$T_{N,\alpha} = \left\{t \in [0, 1] \setminus R: \exists_{n \in \mathbf{N}} \frac{1}{n} \# \{i = 1, \ldots, n: \sum_{k=1}^{i} r_k(t) > N\} > \alpha\right\}.$$

According to the definition (2), the set T is an intersection of two sets. Denote them by T_1 and T_2, respectively. We have

$$T_1 = \bigcap_{N=1}^{\infty} \bigcap_{\alpha \in \mathbf{Q} \cap (0,1)} T_{N,\alpha}. \tag{6}$$

We will show that $\lambda(T_{N,\alpha}) = 1$. For a given $N \in \mathbf{N}$ and $0 < \alpha < 1$, fix $\alpha < \beta < 1$. Define by induction a sequence of sets $A_l \subset [0, 1] \setminus R$ and sequences of numbers n_l, $N_l \in \mathbf{N}$, as follows:

Assume that A_j, N_j, n_j have already been defined for all $j < l$. ($l = 1$ means that no A_j, N_j, n_j have been defined so far.) To define A_l, N_l, n_l observe that there exists N_l large enough to satisfy

$$\forall_{n \geqslant N_l} \frac{1}{n}\left(n + \sum_{j<l} n_j\right) < \frac{\beta}{\alpha},$$

and then

$$f\left(\frac{\alpha}{n}\left(n + \sum_{j<l} n_j\right)\right) > f(\beta).$$

By Lemma 3 the Lebesgue measure of the set

$$\left\{t \in [0, 1] \setminus R: \frac{1}{n} \# \{i = 1, \ldots, n: \sum_{k=1}^{i} r_{k+\Sigma_{j<l} n_j}(t) > N + \sum_{j<l} n_j\} > \frac{\alpha}{N_l}\left(N_l + \sum_{j<l} n_j\right)\right\}$$

tends to $f\left(\alpha N_l^{-1}\left(N_l + \sum_{j<l} n_j\right)\right) > f(\beta)$ when n tends to infinity, and hence there exists $n_l > N_l$ satisfying

$$\lambda\left(\left\{t \in [0, 1] \setminus R: \frac{1}{n_l} \# \{i = 1, \ldots, n_l: \sum_{k=1}^{i} r_{k+\Sigma_{j<l} n_j}(t) > N + \sum_{j<l} n_j\} \right.\right.$$
$$\left.\left. > \frac{\alpha}{N_l}\left(N_l + \sum_{j<l} n_j\right)\right\}\right) > f(\beta).$$

Let A_l be the latter set considered.

A_l are independent in $(\Omega = [0, 1]\backslash R, \text{Borel}(\Omega), \lambda)$ because r_k are independent and $\lambda(A_l) \geqslant f(\beta) > 0$. Consequently, by the Borel–Cantelli theorem, $\lambda(\limsup_{l\to\infty} A_l) = 1$. According to the definitions of $T_{N,\alpha}$ and A_l it is easy to verify that $A_l \subset T_{N,\alpha}$ for all $l \in N$. This implies $\lambda(T_{N,\alpha}) = 1$. By (6), T_1 is a countable intersection of sets $T_{N,\alpha}$ and $\lambda(T_1) = 1$. Similarly it can be proved that $\lambda(T_2) = 1$. A measure of the set $T = T_1 \cap T_2$ is also equal to 1. ∎

Proof of Lemma 3. Let $B_l = \{t \in [0, 1]\backslash R\colon \sum_{k=1}^{j} r_k(t) = N$ holds for $j = l$ and does not hold for $j < l\}$ and

$$A_{n,\alpha,N} = \left\{t \in [0, 1]\backslash R\colon \frac{1}{n} \# \left\{i = 1, \ldots, n\colon \sum_{k=1}^{i} r_k(t) > N\right\} > \alpha\right\}.$$

We have to prove that, for any fixed $N \in \boldsymbol{Z}$ and $\alpha \in (0, 1)$, $\lambda(A_{n,\alpha,N})$ tends to $f(\alpha)$ as n tends to ∞. It is easy to see that for any $\varepsilon > 0$

$$\limsup_{n\to\infty} \lambda(A_{n,\alpha,N} \mid B_l)$$
$$\leqslant \lim_{n\to\infty} \lambda\left(\left\{t \in [0, 1]\backslash R\colon \frac{1}{n} \# \left\{i = 1, \ldots, n\colon \sum_{k=l+1}^{l+i} r_k(t) > 0\right\} > \alpha - \varepsilon\right\}\right),$$

which is equal to $f(\alpha - \varepsilon)$ (due to the classical arcsin law).

The same argument gives us the inequality

$$\liminf_{n\to\infty} \lambda(A_{n,\alpha,N} \mid B_l) \geqslant f(\alpha + \varepsilon).$$

Since f is a continuous function and ε is arbitrary, $\lim_{n\to\infty} \lambda(A_{n,\alpha,N} \mid B_l)$ exists and is equal to $f(\alpha)$, which together with $\sum_{l=1}^{\infty} \lambda(B_l) = 1$ gives us the conclusion. ∎

3. Other generalizations. Now we formulate a simple generalization of Theorem 1.

THEOREM 2. *There exists $T \subset [0, 1]$ with $\lambda(T) = 1$ such that, for any increasing homeomorphism h: $[0, 1] \to [0, 1]$, for any $x \in [0, 1]$ and any $t \in T$, we have*

$$\text{cl}\left\{\frac{1}{n} \sum_{i=1}^{n} h_{t_i} \circ \ldots \circ h_{t_1}(x)\colon n \in N\right\} = [m_x, M_x],$$

where m_x is the maximal fixed point of h not greater than x, M_x is the minimal fixed point of h not less than x. As before $h_0 = h$, $h_1 = h^{-1}$ and $t = (0, t_1 t_2 \ldots)_2$ is a binary representation of t.

Proof. The set T is the same as in the proof of Theorem 1 and is defined by (2). To check that it satisfies the conclusion of the theorem we consider two cases:

If x is a fixed point of h, then

$$M_x = m_x = x \quad \text{and} \quad \mathrm{cl}\left\{\frac{1}{n}\sum_{i=1}^{n} h_{t_i}\circ \ldots \circ h_{t_1}(x)\colon n\in N\right\} = \mathrm{cl}\{x\} = [m_x, M_x].$$

If x is not a fixed point, then consider a restriction $h' = h|_{[m_x, M_x]}$ of the function h. The function h' is an increasing homeomorphism of the interval $[m_x, M_x]$ onto itself with m_x and M_x as the only two fixed points. It is easy to see that, as in Theorem 1, $\{n^{-1}\sum_{i=1}^{n} h'_{t_i}\circ \ldots \circ h'_{t_1}(x)\colon n\in N\}$ is dense in $[m_x, M_x]$, and this implies the conclusion. ∎

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Department of Mathematics, Warsaw University
ul. Banacha 2, 02-097 Warszawa, Poland
E-mail address: andkom@mimuw.edu.pl

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