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UPPER BOUNDS FOR THE EXPECTED JEFFERSON ROUNDING UNDER MEAN-VARIANCE-SKEWNESS CONDITIONS

BY

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Abstract. For the class of nonnegative random variables with given mean, variance, and skewness and support bound, we present a sharp upper bound for the expectation of rounding due to the Jefferson rule. The result gives an estimate for average extra gains due to rounding down payments. Arguments of four-dimensional geometric moment theory implemented in the proof provide tools for refined evaluations of rates of convergence of probability distributions and positive linear operators.

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1. INTRODUCTION AND NOTATION

The paper is organized as follows. In Section 1 we state the problem, discuss briefly related results and applications, and present a geometric optimal distance method implemented here and in the cited literature. We also introduce some notation and describe the range of reasonable assumptions for the problem. In Section 2 we provide geometric and analytic formulations of main results. Section 3 contains some auxiliary lemmas and proofs of the main results.

We consider the class of random variables which are nonnegative bounded above by a fixed possibly infinite number a, and satisfy three moment conditions $EX^i = m_i$, $i = 1, 2, 3$, where all m_i are given. The objective of this paper is to determine the sharp upper bound $U_a(M)$, $M = (m_1, m_2, m_3)$, for the expected integer part of the random variable $E\lfloor X \rfloor$ under the mentioned support and moment conditions. The choice of these moments is natural. They have meaningful statistical counterparts: the mean m_1, variance $\sigma^2 = m_2 - m_1^2$,

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and skewness $s = (m_3 - 3m_2 m_1 + 2m_1^3)/\sigma^3$. However, we prefer expressing the conditions in terms of ordinary moments m_i, $i = 1, 2, 3$, which is more convenient for the method of proof we apply and provides a simpler representation of final results. The function $\lfloor \cdot \rfloor$ is also referred to as the floor of the number and the Jefferson rule of rounding.

Considering the discrepancy between the expectations of the original and rounded variables we obtain an estimate of average extra gains per transaction of commercial establishments due to rounding down payments when the basic statistical parameters of transactions can be evaluated. Analogous results for more general c-rounding procedures (which round down and up when the fractional part is less or greater than $c \in [0, 1]$) under one and two moment conditions were solved in Anastassiou and Rachev [2] and Rychlik [13]. Our result allows to derive more subtle conclusions in the theory of apportionment which deals with problems of determining fair representations of regions and parties by assigning seats in parliaments fairly reflecting the vote distribution, allocating jobs and service facilities among administrative units proportionally to their population sizes and structures, etc. The Jefferson rounding is the most natural element of the class of divisor rules of rounding that were defined in the monograph of Balinski and Young [7] and proved to be the only efficient class not suffering from paradoxes and bias (for a recent account of the apportionment theory we refer to Balinski and Rachev [6]). Pukelsheim [10] pointed out that the method of Jefferson is the most suitable discretization for stratified sampling schemes.

The results were established by means of the geometric moment theory developed in Kemperman [9]. The crucial tool of the theory is based on the fact that for an arbitrary probabilistic measure μ on a measurable Hausdorff space $\mathscr{T}$ with given generalized moments $\int_{\mathscr{T}} f_i \, d\mu$ for some integrable functions f_i, $i = 1, \ldots, m$, there exists a probabilistic measure μ' supported on $m+1$ points at most with the same moments. The result was independently obtained by Richter [11] and Rogosinsky [12]. Accordingly, the moment point $(f_1(t), \ldots, f_m(t))$ corresponds with the degenerate measure concentrated at $t \in \mathscr{T}$, and the set of moment points for all probability measures coincides with the convex hull of the image of moment functions $\operatorname{conv}\{(f_1(t), \ldots, f_m(t)) : t \in \mathscr{T}\}$. In our case, the image points form the space curve

$$\widehat{\mathbf{0}A} = \{(t, t^2, t^3) : 0 \leqslant t \leqslant a\},$$

and, in consequence, the moment problem is well stated iff

$$M \in \mathscr{M}_a = \operatorname{conv} \widehat{\mathbf{0}A}. \tag{1.1}$$

Attaching the expected floor, we get

$$(M, \mathrm{E}\lfloor X \rfloor) \in \mathscr{H}_a = \operatorname{conv}\{(t, t^2, t^3, \lfloor t \rfloor) : 0 \leqslant t \leqslant a\} \tag{1.2}$$

for all random variables satisfying the support and moment conditions with

$M \in \mathcal{M}_a$. Hence $U_a(M)$ is the supremum of the last coordinate for all points $(M, x) \in \mathcal{H}_a$. It can be concluded from Kemperman ([9], Theorem 5, p. 99) that $U_a(M)$ is attained for all finite a and $M \in \mathcal{M}_a$. The result for the infinite support case will be deduced by analyzing solutions for $a \to +\infty$.

Summing up, our original problem can be replaced by a geometric one consisting in determining the upper envelope for the convex set (1.2) with various a. Note that (1.2) is a four-dimensional set and we cannot rely merely on intuitive geometric arguments applicable for planar and space objects as was possible in solving the rounding problems under one- or two-moment constraints. Our reasoning is based on combining geometric and analytic arguments. Tools introduced here can be used for solving analogous three-moment problems for other rounding rules. What is more important, they also enable us to improve evaluations of the rates of convergence of distributions with given moments in the Prokhorov metrics (see Anastassiou and Rychlik [4], [5]), and similar results for the Lévy and Kantorovich distances can be obtained as well. Further possible applications in the approximation theory include refinements of Korovkin type inequalities of Anastassiou [1] describing the rates of pointwise convergence of positive linear operators to the unit one.

Below we mainly concentrate on the most sophisticated case of noninteger $a \in (3, +\infty)$. For the remaining support bounds the number of possible types of solutions is reduced and they can be concluded from the proof of the former case. For fixed a, we shall try to represent various $M \in \mathcal{M}_a$ as convex combinations $M = \sum_{i=1}^{k} \alpha_i T_i$, where $T_i = (t_i, t_i^2, t_i^3)$ for some $0 \leqslant t_i \leqslant a$, $1 \leqslant i \leqslant k \leqslant 4$, so to maximize $\sum_{i=1}^{k} \alpha_i \lfloor t_i \rfloor$. Clearly, $k = 1$ iff $m_1 = m_2^{1/2} = m_3^{1/3} = t_1 \in [0, a]$. Below we represent M as combinations of $k = 2, 3, 4$ image points $T_i \in \widehat{0A}$, $i = 1, \ldots, k$, with coefficient adding up to 1. For the two-point representation which is possible iff

$$\begin{aligned} m_2 - m_1(t_1+t_2) + t_1 t_2 &= 0, \\ m_3 - m_2(t_1+t_2) + m_1 t_1 t_2 &= 0, \end{aligned} \tag{1.3}$$

the coefficients are

$$\alpha_i = \frac{m_1 - t_j}{t_i - t_j}, \qquad 1 \leqslant i \neq j \leqslant 2. \tag{1.4}$$

If M satisfies the equation

$$m_3 - m_2(t_1+t_2+t_3) + m_1(t_1 t_2 + t_1 t_3 + t_2 t_3) - t_1 t_2 t_3 = 0, \tag{1.5}$$

then M is a point of the plane $\mathrm{pl}(T_1 T_2 T_3)$ spanned by T_1, T_2, T_3, and the respective coefficients are

$$\alpha_i = \frac{m_2 - m_1 \sum_{j \neq i} t_j + \prod_{j \neq i} t_j}{\prod_{j \neq i} (t_i - t_j)}, \qquad 1 \leqslant i,\ j \leqslant 3. \tag{1.6}$$

Finally, with no restrictions, M can be represented as $M = \sum_{i=1}^{4} \alpha_i T_i$ with

$$\alpha_i = \frac{m_3 - m_2 \sum_{j \neq i} t_j + m_1 \sum_{j \neq i \neq k} t_j t_k - \prod_{j \neq i} t_j}{\prod_{j \neq i} (t_i - t_j)}, \quad 1 \leqslant i,\, j,\, k \leqslant 4. \tag{1.7}$$

Observe that (1.4), (1.6), (1.7) are not coefficients of convex combinations unless all are nonnegative. The facts that the left-hand side of (1.5) is less or greater than 0 may be geometrically interpreted by saying that M is located below or above $\mathrm{pl}(T_1\, T_2\, T_3)$. This is no justified to rewrite (1.5) as $\mathrm{pl}(T_1\, T_2\, T_3)(M) = 0$. Replacing M by $T \in \widehat{\mathbf{0}A}$ there, we get $(t-t_1)\,(t-t_2)(t-t_3) = 0$, which immediately implies the following lemma that will be repeatedly referred to:

LEMMA 1. *If T, $T_i \in \widehat{\mathbf{0}A}$, $i = 1, 2, 3$, then T is located below* $\mathrm{pl}(T_1\, T_2\, T_3)$ *iff t is either the smallest or the second greatest among t, t_1, t_2, t_3. Also, t lies above* $\mathrm{pl}(T_1\, T_2\, T_3)$ *iff t is either the second smallest or the greatest.*

Note that the numerator in (1.7) describes the location of M with respect to the plane determined by T_j for $j \neq i$, and therefore can be written as $\mathrm{pl}(T_1\, T_2\, T_3\, T_4 \backslash T_i)(M)$. A straightforward computation shows that

$$\frac{\partial \alpha_j}{\partial t_i} = \frac{\mathrm{pl}(T_1\, T_2\, T_3\, T_4 \backslash T_i)(M)}{(t_j - t_i)^2 \prod_{i \neq k \neq j} (t_j - t_k)}, \quad 1 \leqslant j \neq i \leqslant 4. \tag{1.8}$$

In the case we let t_i vary and keep the other t_j fixed, this formula enables us to characterize qualitative changes of coefficients α_j provided that we know the ordering of t_1, t_2, t_3, t_4, and the location of M with respect to the plane spanned by the immobile T_j. The direction of change for the contribution of the mobile point T_i can be deduced directly: α_i increases iff T_i approaches M.

In the sequel we distinguish some specific $T = (t, t^2, t^3)$ of the graph of the moment function. We shall write $\mathbf{0}$, $\mathbf{1}$, K, N and A for the points generated by 0, 1, an integer k, $n = \lfloor a \rfloor$ and a, respectively. For convenience, we put $l = k+1$ and, consequently, write $L = (l, l^2, l^3)$.

We now describe the class of possible moment conditions. We cite here a classic result of the moment theory (see Karlin and Studden [8], Chapter 4) asserting that (1.1) holds for $a = 1$ iff the following matrices:

$$\begin{bmatrix} 1 & m_1 \\ m_1 & m_2 \end{bmatrix}, \quad \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}, \quad \begin{bmatrix} 1-m_1 & m_1-m_2 \\ m_1-m_2 & m_2-m_3 \end{bmatrix}, \quad [m_1 - m_2] \tag{1.9}$$

are positive semidefinite. A standard rescaling yields the conclusion that $M \in \mathcal{M}_a$ for some positive a iff

$$0 \leqslant m_1 \leqslant a, \tag{1.10}$$

$$m_1^2 \leqslant m_2 \leqslant a m_1, \tag{1.11}$$

$$\frac{m_2^2}{m_1} \leqslant m_3 \leqslant a m_2 - \frac{(a m_1 - m_2)^2}{a - m_1}, \tag{1.12}$$

excluding $m_1 = 0$ and a in (1.12) for which $M = \mathbf{0}$ and A, respectively. The lower bound in (1.12) is attained iff X has the two-point distribution supported on 0 and m_2/m_1 with probabilities σ^2/m_2 and m_1^2/m_2, respectively. The upper inequality becomes an equality iff X has the two-point distribution supported on $(m_1 a - m_2)/(a - m_1)$ and a with the following probabilities: $\sigma^2/[\sigma^2 + (a-m_1)^2]$ and $(a-m_1)^2/[\sigma^2 + (a-m_1)^2]$, respectively. This implies that the lower and upper envelopes $\underline{\mathcal{M}_a}$, $\overline{\mathcal{M}_a}$ of $\mathcal{M}_a$ consist of the line segments $\overline{\mathbf{0}T}$ and $\overline{TA}$, respectively, where T runs along the curve $\widehat{\mathbf{0}A}$. The envelopes can be visualized as the membranes $\operatorname{mem}(\mathbf{0}, \widehat{\mathbf{0}A})$ and $\operatorname{mem}(A, \widehat{\mathbf{0}A})$ spanned by $\widehat{\mathbf{0}A}$ and either $\mathbf{0}$ or A, respectively. Formulae (1.3) provide analytic expressions for the points of the membranes with t_1 equal to 0 and a, respectively, and $t_2 \in [0, a]$.

We now decompose the moment space (1.1) into smaller pieces. The reason of partitioning is that extreme expected roundings and the distributions that achieve them are expressed differently in various regions of $\mathcal{M}_a$. Since U_a is a continuous function of moment conditions, for simplicity we consider closed subsets of (1.1) with nonoverlapping interiors. As we shall see, the solutions for border points of adjacent parts coincide although they are determined by different formulae. We first define

$$\mathcal{P} = \operatorname{conv}\{K:\ k = 0, \ldots, n\}. \tag{1.13}$$

It can be easily verified that $\mathcal{P}$ is a polygon with $n+1$ vertices K, $0 \leqslant k \leqslant n$, and $3n-3$ edges $\overline{\mathbf{0}K}$, $1 \leqslant k \leqslant n$, $\overline{KN}$, $1 \leqslant k \leqslant n-1$, and $\overline{KL}$, $1 \leqslant k \leqslant n-2$. If $k+1 < j$ and $n(n-j) > 0$, then $\overline{KJ}$ is a diagonal of $\mathcal{P}$. Moreover, the lower and upper envelopes of $\mathcal{P}$ consist of triangles:

$$\underline{\mathcal{P}} = \bigcup_{k=1}^{n-1} \Delta(\mathbf{0}KL), \tag{1.14}$$

$$\overline{\mathcal{P}} = \bigcup_{k=0}^{n-2} \Delta(KLN). \tag{1.15}$$

We can verify it immediately embedding $\mathcal{P}$ into $\mathcal{M}_n$ and noticing that $\overline{\mathbf{0}K} \subset \underline{\mathcal{M}_n}$, $k = 1, \ldots, n$, and $\overline{KN} \subset \overline{\mathcal{M}_n}$, $k = 0, \ldots, n-1$. Consider also

$$\mathcal{T}_k = \operatorname{conv}\{K, L, N, A\}, \qquad k = 0, \ldots, n-2. \tag{1.16}$$

If a is noninteger, then for each k the point A lies above $\Delta(KLN)$, and so $\mathcal{T}_k$ is a tetragon. Each $\mathcal{T}_k$ has the common sides $\Delta(KNA)$ and $\Delta(LNA)$ with $\mathcal{T}_{k-1}$ and $\mathcal{T}_l$, respectively (except of the extreme ones) and their bottoms go into the making of $\overline{\mathcal{P}}$. Applying (1.9) and a standard linear change of variables we can assert that $\operatorname{mem}(N, \widehat{NA})$ and $\operatorname{mem}(A, \widehat{NA})$ are the upper and lower borders of

$$\mathcal{N} = \operatorname{conv} \widehat{NA}. \tag{1.17}$$

Note that the latter membrane is a part of $\overline{\mathcal{M}_a}$.

We also distinguish n surfaces $\operatorname{mem}(L, \widehat{KL})$, $k = 0, \ldots, n-1$, in the moment space. The surfaces divide $\mathcal{M}_a \backslash (\mathcal{P} \cup \bigcup_{k=0}^{n-2} \mathcal{T}_k \cup \mathcal{N})$ into $2n$ parts with disjoint interiors. $n+1$ of them, denoted by $\mathcal{L}_0, \ldots, \mathcal{L}_n$, adhere to the lower envelope of $\mathcal{M}_a$, and so their bottom sides coincide with parts of $\underline{\mathcal{M}_a}$. The top side of $\mathcal{L}_0$ is $\operatorname{mem}(\mathbf{1}, \widehat{\mathbf{01}})$. Each of $\mathcal{L}_k$, $k = 1, \ldots, n-1$, is bounded above by a lower side $\triangle(\mathbf{0}KL)$ of the polygon and $\operatorname{mem}(L, \widehat{KL})$. The upper cover of the last one consists of $\triangle(\mathbf{0}NA)$ which is a side of $\mathcal{T}_0$, and $\underline{\mathcal{N}} = \operatorname{mem}(N, \widehat{NA})$. Observe that we can represent $\mathcal{L}_k$ as

$$\mathcal{L}_0 = \operatorname{conv} \widehat{\mathbf{01}}, \tag{1.18}$$

$$\mathcal{L}_k = \operatorname{conv} \{\mathbf{0}\} \cup \widehat{KL}, \quad k = 1, \ldots, n-1, \tag{1.19}$$

$$\mathcal{L}_n = \overline{(\operatorname{conv} \{\mathbf{0}\} \cup \widehat{NA})} \backslash \mathcal{N}. \tag{1.20}$$

The remaining sets $\mathcal{U}_k$, $k = 0, \ldots, n-1$, adhere to $\overline{\mathcal{M}_a}$. The respective bottoms $\underline{\mathcal{U}_k}$ consist of $\triangle(KLA)$, the top sides of $\mathcal{T}_k$, and $\operatorname{mem}(L, \widehat{KL})$. We can also write

$$\mathcal{U}_k = \operatorname{conv} \{A\} \cup \widehat{KL}, \quad k = 1, \ldots, n-1. \tag{1.21}$$

Summing up, we propose the following partition:

$$\mathcal{M}_a = \mathcal{P} \cup \bigcup_{k=0}^{n-2} \mathcal{T}_k \cup \mathcal{N} \cup \bigcup_{k=0}^{n} \mathcal{L}_k \cup \bigcup_{k=0}^{n-1} \mathcal{U}_k. \tag{1.22}$$

2. MAIN RESULTS

In Theorem 1 we describe the upper bounds $U_a(M)$ for general finite a and all $M \in \mathcal{M}_a$. Decomposition (1.22) will be used to characterize specific solutions for moment points belonging to respective regions. For the sake of brevity, these will be formulated by means of geometric notions. Theorem 1 is followed by a comprehensive analytic presentation of the specific assumptions and corresponding solutions with explanatory comments. All elements of partition (1.22) essentially contribute to $\mathcal{M}_a$ iff a is noninteger and greater than 3. Otherwise, some of them either vanish or have empty interiors, and so can be absorbed by adjacent regions. For the specific a, we present explicitly maximally reduced representations of the moment space and indicate the solutions for the respective subregions in Corollary 1. The case $a = +\infty$ is treated in Theorem 2.

THEOREM 1. *Let* $0 < a < +\infty$.

(i) *If* $M \in \mathscr{P}$ (see (1.13)), *then* $U_a(M) = m_1$.

(ii) *If* $M \in \mathscr{N}$ (see (1.17)), *then* $U_a(M) = n$.

(iii) *If* $M \in \mathscr{T}_k$, $k = 0, \ldots, n-2$ (see (1.16)), *then*

$$U_a(M) = \alpha_k k + \alpha_l l + (\alpha_n + \alpha_a) n,$$

where

$$M = \alpha_k K + \alpha_l L + \alpha_n N + \alpha_a A$$

is a unique convex combination of K, L, N *and* A.

(iv) *If* $M \in \mathscr{L}_k$, $k = 0, \ldots, n-1$ (see (1.18), (1.19)), *then there exists a unique* $t \in [k, l]$ *such that*

$$M = \alpha_0 \mathbf{0} + \alpha_t T + \alpha_l L \tag{2.1}$$

is a convex combination and

$$U_a(M) = \alpha_t k + \alpha_l l.$$

(v) *If* $M \in \mathscr{L}_n$ ((see (1.20)), *then there exists a unique* $t \in [n, a]$ *such that*

$$M = \alpha_0 \mathbf{0} + \alpha_n N + \alpha_t T$$

is a convex combination and

$$U_a(M) = (\alpha_n + \alpha_t) n. \tag{2.2}$$

(vi) *If* $M \in \mathscr{U}_k$, $k = 0, \ldots, n-1$ (see (1.21)), *then there exists a unique* $t \in [k, l]$ *such that*

$$M = \alpha_t T + \alpha_l L + \alpha_a A$$

is a convex combination and

$$U_a(M) = \alpha_t k + \alpha_l l + \alpha_a n. \tag{2.3}$$

Analytic representations and comments

(i) Combining (1.6) with (1.14) and (1.15), we verify that $M \in \mathscr{P}$ iff

$$\begin{aligned} \max_{k=1,\ldots,n-1} \{m_2(k+l) - m_1 kl\} &\leqslant m_3 \\ &\leqslant \min_{k=0,\ldots,n-2} \{m_2(k+l+n) - m_1(kl+kn+ln) + kln\}. \end{aligned} \tag{2.4}$$

It is evident that $\mathrm{E}\lfloor X \rfloor = \mathrm{E}X$ iff X is supported on integer points. Every $M \in \operatorname{int} \mathscr{P}$ admits nonunique solutions.

(ii) The condition $M \in \mathscr{N}$ is equivalent to

$$\begin{gathered} n \leqslant m_1 \leqslant a, \quad m_1^2 \leqslant m_2 \leqslant (n+a) m_1 - na, \\ nm_2 + \frac{(m_2 - nm_1)^2}{m_1 - n} \leqslant m_3 \leqslant am_2 - \frac{(am_1 - m_2)^2}{a - m_1}. \end{gathered} \tag{2.5}$$

Clearly, $\mathrm{E}\lfloor X\rfloor = n$ iff X is supported on $[n, a]$. Interior points of $\mathcal{N}$ admit nonunique solutions.

(iii) If $M \in \operatorname{int} \mathcal{T}_k$ for some k, then this is a nontrivial combination of A and $P \in \triangle(KLN)$. Since A lies above $\mathrm{pl}(KLN)$, so does M. Likewise we check that M is located above $\mathrm{pl}(KNA)$ and beneath $\mathrm{pl}(KLA)$ and $\mathrm{pl}(LNA)$. These, together with (1.6), yield that the conditions

$$\begin{aligned}(2.6)\quad & \max\{m_2(k+l+n)-m_1(kl+ka+ln)+kln,\\ & \quad m_2(k+n+a)-m_1(kn+ka+na)+kna\}\\ & \qquad \leqslant m_3 \leqslant \min\{m_2(k+l+a)-m_1(kl+ka+la)+kla,\\ & \qquad\qquad m_2(l+n+a)-m_1(ln+la+na)+lna\}\end{aligned}$$

are equivalent to $M \in \mathcal{T}_k$, $k = 0, \ldots, n-2$. The representation of M by means of vertices of $\mathcal{T}_k$ is unique and (1.7) is used for calculating the respective coefficients. The ultimate formula is huge.

The existence of points appearing in statements (iv)–(vi) of Theorem 1 will be proved in Lemma 4 of Section 3. Below the following analytic expression:

$$(2.7)\quad k \leqslant m_1 \leqslant l, \quad m_1^2 \leqslant m_2 \leqslant (k+l)m_1-kl, \quad m_3 = lm_2-\frac{(lm_1-m_2)^2}{l-m_1}$$

for $M \in \mathrm{mem}(L, \widehat{KL})$ will be implemented.

(iv) Cases $k = 0$ and $k \geqslant 1$ will here be treated separately. The former has simpler representations of assumptions and solution, and by far easier proof.

Case $k = 0$. Notice that $\mathcal{L}_0 = \mathcal{M}_1$, and so $M \in \mathcal{L}_0$ iff (1.9) or, equivalently, (1.10)–(1.12) with $a = 1$ hold. Here

$$(2.8)\quad t = \frac{m_2-m_3}{m_1-m_2},$$

$$(2.9)\quad U_1(M) = \alpha_t = \frac{m_1 m_3-m_2^2}{m_1-2m_2+m_3}.$$

Case $k \geqslant 1$. Then $M \in \mathcal{L}_k$ iff

$$(2.10)\quad 0 \leqslant m_1 \leqslant l,$$

$$(2.11)\quad \max\{m_1 k, m_1^2\} \leqslant m_2 \leqslant m_1 l,$$

$$(2.12)\quad \frac{m_1^2}{m_2} \leqslant m_3 \leqslant \begin{cases} m_2(k+l)-m_1 kl & \text{if } m_2 \geqslant m_1(k+l)-kl,\\ lm_2-(lm_1-m_2)^2/(l-m_1) & \text{otherwise,}\end{cases}$$

$k = 1, \ldots, n-1$ (cf. (1.12), (1.18), (1.19), (2.4) and (2.7)). The solution is supported on 0, l and

$$(2.13)\quad t = \frac{m_2 l-m_3}{m_1 l-m_2},$$

with weights described by (1.6) and

$$U_a(M) = \frac{m_2(t-k)+m_1(kl-t^2)}{t(l-t)}. \tag{2.14}$$

(v) We have $m \in \mathscr{L}_n$ iff (2.10)–(2.11) with k and l replaced by n and a, respectively, and

$$\frac{m_1^2}{m_2} \leqslant m_3 \leqslant \begin{cases} m_2(n+a)-m_1 na & \text{if } m_2 \geqslant m_1(n+a)-na, \\ nm_2+(m_2-nm_1)^2/(m_1-n) & \text{otherwise,} \end{cases}$$

hold (cf. (1.6) for $(t_1, t_2, t_3) = (0, n, a)$, (1.12), (1.20), and (2.5)). Then for

$$t = \frac{m_3-m_2 n}{m_2-m_1 n}$$

(see (1.6)) we have

$$U_a(M) = [m_1(t+n)+m_2]/t,$$

which is attained for the distribution supported on 0, n and t with the weights determined by (1.6) (see (2.2)).

(vi) For $k = 0, \ldots, n-1$ it follows that $M \in \mathscr{U}_k$ iff

$$k \leqslant m_1 \leqslant a,$$
$$\max\{m_1^2, m_1(l+a)-la\} \leqslant m_2 \leqslant m_1(k+a)-ka,$$
$$\left.\begin{matrix} lm_2-(lm_1-m_2)^2/(l-m_1) & \text{if } m_2 \leqslant m_1(k+l)-kl, \\ m_2(k+l+a)-m_1(kl+ka+la)+kla & \text{otherwise} \end{matrix}\right\} \leqslant m_3 \leqslant am_2+\frac{(am_1-m_2)^2}{a-m_1}$$

(cf. (1.12), (1.21), (2.6), and (2.7)). The support of the solution consists of points

$$t = \frac{m_3-m_2(l+a)+m_1 la}{m_2-m_1(l+a)+la},$$

l and a (cf. (1.5)), and the respective probabilities α_t, α_l and α_a, and $U_a(M)$ can be calculated by means of (1.6) and (2.3).

COROLLARY 1. (i) *If* $0 < a < 1$, *then* $\mathscr{M}_a = \mathscr{N}$ (cf. Theorem 1 (ii)) *and* $U_a(M) = 0$ *for all* M.

(ii) *If* $a = 1$, *then* $\mathscr{M}_a = \mathscr{L}_0$ *and Theorem* 1 (iv), *case* $k = 0$, *can be applied.*

(iii) *If* $1 < a < 2$, *then*

$$\mathscr{M}_a = \mathscr{N} \cup \mathscr{L}_0 \cup \mathscr{L}_n \cup \mathscr{U}_0,$$

and for M *for the consecutive subsets of the partition, the statements of Theorem* 1 (ii) *and* (iv)–(vi) *hold true.*

(iv) *If* $2 < a < 3$, *then*

$$\mathcal{M}_a = \mathcal{N} \cup \mathcal{T}_0 \cup \bigcup_{k=0}^{1} \mathcal{L}_k \cup \mathcal{L}_n \cup \bigcup_{k=0}^{1} \mathcal{U}_k,$$

and the respective solutions are presented in parts (ii)–(vi) *of Theorem* 1.

(v) *If* $a \geqslant 2$ *is integer, then* (1.1) *can be written as*

$$\mathcal{M}_a = \mathcal{P} \cup \bigcup_{k=0}^{a-1} \mathcal{L}_k \cup \bigcup_{k=0}^{a-2} \mathcal{U}_k$$

(*when* $a = 2$, $\mathcal{P} = \triangle(\mathbf{01}A)$ *can be dropped*), *and the assertions of Theorem* 1 (i), (iv), *and* (vi) *hold for the respective three summands.*

Analytic counterparts of the statements of Corollary 1 can be found in the comments on the parts of Theorem 1 they refer to.

For $a = +\infty$ we have $M \in \mathcal{M}_\infty$ iff only the lower bounds in (1.10)–(1.12) are satisfied. We see that $\mathcal{M}_\infty$ has a lower envelope $\operatorname{mem}(\mathbf{0}, \widehat{\mathbf{0}\infty}) = \bigcup_{0<t<\infty} \overline{\mathbf{0}T}$, and two side borders, and is unbounded above. We split it as follows:

$$\mathcal{M}_a = \mathcal{R} \cup \bigcup_{k=0}^{\infty} \mathcal{L}_k \cup \bigcup_{k=0}^{\infty} \mathcal{V}_k.$$

Here $\mathcal{R} = \operatorname{conv}\{K\colon k = 0, 1, \ldots\}$ represents the limit set of the increasing family of polygons for $a \to +\infty$, whose points satisfy

$$m_1 \geqslant 0, \tag{2.15}$$

$$m_2 > \max_{k=0,1,\ldots} m_1(k+l) - kl, \tag{2.16}$$

$$m_3 \geqslant \max_{k=0,1,\ldots} m_2(k+l) - m_1 kl, \tag{2.17}$$

or (2.15) with equalities in (2.16) and (2.17). Formulae (1.18)–(1.19) and (2.10)–(2.12) describe $\mathcal{L}_k$, $k = 0, 1, \ldots$ Finally, $\mathcal{V}_k$, $k = 0, 1, \ldots$, arise from respective $\mathcal{U}_k$ as $a \to +\infty$. The relation $M \in \mathcal{V}_k$ is characterized by

$$k \leqslant m_1 \leqslant l, \qquad m_1^2 \leqslant m_2 \leqslant (k+l)m_1 - kl, \qquad m_3 \geqslant lm_2 - \frac{(lm_1 - m_2)^2}{l - m_1}.$$

Geometrically, each $\mathcal{V}_k$ is an unbounded above set with bottom $\operatorname{mem}(L, \widehat{KL})$ adhering to $\mathcal{L}_k$, one vertical side touching $\mathcal{R}$, and the other belonging to the border of $\mathcal{M}_\infty$.

THEOREM 2. *Let* $a = +\infty$.

(i) *If* $M \in \mathcal{R}$ (see (2.15)–(2.17)), *then* $U_\infty(M) = m_1$, *which is attained by any distribution supported on integers and satisfying the moment conditions.*

(ii) *If* $M \in \mathcal{L}_k$ *for some* $k = 0, 1, \ldots$, *then the conclusion of Theorem* 1 (iv) *holds* (see also (2.8)–(2.14)).

(iii) *If* $M \in \mathscr{V}_k$ *for some* $k = 0, 1, \ldots,$ *then*

$$U_\infty(M) = k + \frac{\sigma^2}{l^2 - 2m_1 l + m_2},$$

which is attained asymptotically for the distributions supported on l, $a \to +\infty$, *and* $t(a) \to t = (m_1 l - m_2)/(l - m_1)$.

Each limit distribution of case (iii) has two support points l and t, provides the upper bound for the mean-variance conditions (see Rychlik [13]) and does not satisfy the third one.

3. PROOFS

Lemmas 2 and 3 can be expressed more generally in forms suitable for the abstract moment theory. Lemma 2 was proved by Anastassiou and Rychlik ([3], Lemma 1). The statement of Lemma 3 was a basic argument in the proof. This was formulated and verified in the proof of the former, but we display it here, because this will be referred explicitly in the sequel.

LEMMA 2. *A distribution attaining the maximal expected rounding* $U_a(M)$ *for a moment point* M *of an open* $\mathscr{M}' \subset \mathscr{M}_a$, $\mathscr{M}' \cap \widehat{\mathbf{0}A} = \emptyset$, *is a mixture of distributions attaining* $U_a(M_i)$ *for some moment points* M_i *of the border of* $\mathscr{M}'$. *Accordingly, the subset of support points of the distributions with maximal expected roundings for border points contains that for the inner ones.*

LEMMA 3. *Consider distributions* μ_P, μ_Q *supported on* $[0, a]$ *and generating moment points* P, Q, *respectively. Suppose that* $M = \alpha P + (1-\alpha) Q$ *for* $0 < \alpha < 1$. *Take a point* $R \in \overline{PM}$ *and a respective distribution* $\bar{\mu}_R$ *attaining* $U_a(R)$. *Then replacing* $\alpha\mu_P + (1-\alpha)\mu_Q$ *by* $\beta\bar{\mu}_R + (1-\beta)\mu_Q$ *with* β *satisfying* $\beta R + (1-\beta) Q = M$ *does not result in decreasing the expected rounding.*

LEMMA 4. *Let* $0 \leqslant x \leqslant y \leqslant a$.

(i) *If* $M \in \operatorname{conv} \widehat{XY}$, *then there is* $T \in \widehat{XY}$ *such that* $M \in \triangle(XTY)$.

(ii) *If* $M \in \operatorname{conv}\{\mathbf{0}\} \cup \widehat{XY}$, *then there is* $T \in \widehat{XY}$ *such that* $M \in \triangle(\mathbf{0}TY)$.

(iii) *If* $M \in (\operatorname{conv}\{\mathbf{0}\} \cup \widehat{XY}) \setminus \operatorname{conv}\widehat{XY}$, *then there is* $T \in \widehat{XY}$ *such that* $M \in \triangle(\mathbf{0}XT)$.

(iv) *Replacing* $\mathbf{0}$ *by* A *in the assumptions of* (ii) *and* (iii) *yields* $M \in \triangle(XTA)$ *and* $M \in \triangle(TYA)$, *respectively, for some* $T \in \widehat{XY}$.

Proof. (i) Consider the vertical line segment $\overline{PQ}$ containing M whose end-points P and Q belong to the lower and upper envelopes $\operatorname{mem}(X, \widehat{XY})$ and $\operatorname{mem}(Y, \widehat{XY})$ of $\operatorname{conv}\widehat{XY}$, respectively. Since $P \in \overline{XR}$, $Q \in \overline{SY}$ for some $R, S \in \widehat{XY}$, $M \in \operatorname{conv}\{X, R, S, Y\}$. Consider the ray $YM^{\rightarrow}$ running down from Y through M, $\triangle(XRS)$ and the lower envelope of $\operatorname{conv}\widehat{XY}$, and denote the piercing point of the envelope by V. Since $M \in \overline{VY}$ and $V \in \overline{XT}$ for some $T \in \widehat{XY}$, we can write $M \in \triangle(XTY)$.

(ii) The sides of the hull we analyze are $\triangle(\mathbf{0}XY)$, and $\mathrm{mem}(\mathbf{0}, \widehat{XY})$, and $\mathrm{mem}(Y, \widehat{XY})$. For any M located therein, the half-line $\mathbf{0}M^{\mapsto}$ leaves the hull at some $P \in \overline{TY} \subset \mathrm{mem}(Y, \widehat{XY})$ for some $T \in \widehat{XY}$ so that $M \in \triangle(\mathbf{0}TY)$.

(iii) We can repeat the above arguments, substituting $\mathrm{mem}(X, \widehat{XY})$ for $\mathrm{mem}(Y, \widehat{XY})$.

(iv) can be proved similarly. ∎

LEMMA 5. *Let $M \in \mathcal{M}_a$, let $z \in [0, a]$ be fixed, and $x, y \in [0, a]$ vary so that*

$$M = \alpha_x X + \alpha_y Y + \alpha_z Z \in \triangle(XYZ).$$

(i) *If $x, y < z$ and x increases (decreases), then y increases (decreases) and α_z decreases (increases).*

(ii) *If $z < x, y$ and x increases (decreases), then both y and α_z increase (decrease).*

Proof. Applying (1.6) we establish the dependence of y on x:

$$y(x) = \frac{(m_1 z - m_2)x - (m_2 z - m_3)}{(z - m_1)x - (m_1 z - m_2)}.$$

It follows that

$$y'(x) = \frac{(z - m_1)(m_2 z - m_3) - (m_1 z - m_2)^2}{[(z - m_1)x - (m_1 z - m_2)]^2}. \tag{3.1}$$

We claim that the numerator of (3.1) is positive once z is extreme. To show this we refer to the probabilistic interpretation of moments:

$$[\mathrm{E}X(z-X)]^2 \leqslant \mathrm{E}X^2(z-X)\,\mathrm{E}(z-X)$$

is the Schwarz inequality for a random variable $X \leqslant z$ with respect to the probability law generated by $(z-X)/\mathrm{E}(z-X)$ if z is maximal, and for $X \geqslant z$ with $(X-z)/\mathrm{E}(X-z)$ otherwise. Positivity of (3.1) implies that the directions of changes of x and $y(x)$ are the same.

In the proof of the latter statements of (i) and (ii) we use geometric arguments. Actually, we only examine the case (i), the other can be handled in much the same way. Suppose that $x_i < y_i = y(x_i) < z$, $i = 1, 2$, and $x_1 < x_2$. Let T_i stand for the points of $\widehat{X_i Y_i}$ such that $M \in \overline{T_i Z}$, $i = 1, 2$. Note that T_1 and T_2 belong to the lower and upper envelopes of $\mathrm{conv}\,\widehat{X_1 Y_2}$. The half-line $ZM^{\mapsto}$ runs downwards and intersects $\mathrm{conv}\,\widehat{X_1 Y_2}$ crossing first the upper envelope at T_2, and then the lower one at T_1. The contributions of Z in the convex representations of M by X_i, Y_i and Z are $|MT_i|/(|ZM| + |MT_i|)$, $i = 1, 2$. Clearly, the latter is smaller as required. ∎

Proof of Theorem 1. (i) Observe that a natural bound $\mathrm{E}\lfloor X \rfloor \leqslant m_1$ becomes equality iff X is supported on integers only. This is equivalent to saying that the respective moment point M can be represented as a convex combination of K for all integer k contained in $[0, a]$, which simply means that $M \in \mathscr{P}$.

(ii) Another trivial bound is $\mathrm{E}\lfloor X \rfloor \leqslant \operatorname{ess\,sup}\lfloor X \rfloor \leqslant n$. This is attained iff X is supported on an arbitrary subset of $[n, a]$. This is possible iff $M \in \mathscr{N}$.

Analogous solutions appeared obviously in the respective problems subject to two-moment conditions. We shall further use the solutions of the remaining cases of the two-moment problem. Anastassiou and Rachev [2] proved that all combinations of 0, n and a attain the maximal expected floor rounding for the pairs of moment conditions (m_1, m_2) and (m_1, m_3) they satisfy. They remain optimal when we add the third condition which they obey. Accordingly, for every $M \in \triangle(\mathbf{0}NA)$ a unique combination of 0, n and a attains $U_a(M)$. By similar arguments we conclude from Rychlik [13] that, for an arbitrary $M \in \operatorname{mem}(L, \widehat{KL}) = \bigcup_{k \leqslant t \leqslant l} \overline{TL}$ and some $k = 0, \ldots, n-1$, $U_a(M)$ is attained by a properly chosen combination of respective t and l.

(iv) Case $k = 0$. The border of $\mathscr{L}_0$ consists of the upper side $\operatorname{mem}(\mathbf{1}, \widehat{\mathbf{01}})$ on which the optimal expected rounding is achieved by combinations of a single $t \in [0, 1)$ and 1, and the lower one $\operatorname{mem}(\mathbf{0}, \widehat{\mathbf{01}})$ which has the unique (and so optimal) representation by two-point mixtures of 0 with various $t \in (0, 1]$. Suppose that $M \in \operatorname{int} \mathscr{L}_0$. By Lemma 2, $U_a(M)$ is attained by a distribution on some $t_i \in [0, 1)$ and 1. Since 1 has only a non-zero rounding, we should look for a convex combination

$$M = \sum \alpha_{s_i} S_i + \alpha_1 \mathbf{1} = (1-\alpha_1) S + \alpha_1 \mathbf{1},$$

with maximal α_1, where S is a convex combination of $S_i \in \widehat{\mathbf{01}} \backslash \{\mathbf{1}\}$ such that $M \in \overline{S\mathbf{1}}$. Arguing as in the proof of Lemma 5 we conclude that α_1 is maximized if $S \in \overline{\mathbf{0}T} \subset \mathscr{L}_0$ for a unique $t \in (0, 1)$.

(iv) Case $k \geqslant 1$. The upper envelope of $\mathscr{L}_k$ has two pieces: $\triangle(\mathbf{0}KL)$, where the combinations of the points generating the vertices provide the maximal expectation of rounding, and $\operatorname{mem}(L, \widehat{KL})$ on which mixtures of l with single $t \in [k, l)$ are optimal. The lower one is uniquely represented by pairs $\mathbf{0}$ with $T \in \widehat{KL}$. Lemma 2 implies that the distribution achieving $U_a(M)$ for $M \in \operatorname{int} \mathscr{L}_k$ is supported on at most four points among 0, $t_i \in [k, l)$ and l. Below we consider all possible combinations.

(a) We start with treating a specific representation

$$M = \alpha_0 \mathbf{0} + \alpha_k K + \alpha_t T + \alpha_l L. \tag{3.2}$$

By Lemma 4 (ii), (3.2) is possible with $\alpha_k = 0$ and the other coefficients being positive. However, α_k may be positive as well: increasing slightly t, we raise $\triangle(\mathbf{0}TL)$ so that M is located between K and $\triangle(\mathbf{0}TL)$. Therefore, we assume that M is an inner point of the tetragon $\operatorname{conv}\{\mathbf{0}, K, T, L\}$ and letting t vary we observe the respective changes of expected Jefferson rounding:

$$E(t) = (\alpha_k+\alpha_t)k+\alpha_l l = m_1-\alpha_t(t-k) = m_1+\frac{\mathrm{pl}(\mathbf{0}KL)(M)}{t(l-t)}, \tag{3.3}$$

by (1.5) and (1.7). Due to Lemma 1, T and M are situated beneath $\mathrm{pl}(\mathbf{0}KL)$, and hence the numerator in (3.3) is negative and fixed. The denominator is positive and decreasing for $l/2 \leqslant k < t < l$. It follows that (3.3) is a decreasing function of t.

We increase $E(t)$ slipping T down along $\widehat{KL}$ as far as possible, i.e. until $\triangle(\mathbf{0}TL)$ absorbs M. Moving T upwards would decrease $E(t)$ and lead to two possible ultimate cases: either $M\in\triangle(KTL)$ if $M\in\operatorname{conv}\widehat{KL}$ or $M\in\triangle(\mathbf{0}KT)$ otherwise (cf. Lemma 4 (i), (iii)). These conclusions become more apparent as we observe the movement of the projection point P of $LM^{\mapsto}$ onto $\triangle(\mathbf{0}KT)$, as t varies. For t increasing, P approaches $\overline{\mathbf{0}T}$, and $M\in\triangle(\mathbf{0}TL)$ when finally $P\in\overline{\mathbf{0}T}$. In the remaining cases P moves towards $\overline{TL}$ and $\overline{\mathbf{0}K}$, respectively.

Summing up, we proved that the combinations of 0, k, t, l, and 0, k, t, and k, t, l provide smaller expected rounding than that of 0, t, and l. Below we show that each other possibility is inferior with respect to one of the above mentioned.

(b) Consider the combinations of $t_i \geqslant k$, possibly including l and not including 0. They generate $M\in\operatorname{conv}\widehat{KL}$. Analysis similar to that in the proof of the case $k=0$ shows that the best combination (excluding 0) is that of k, l and some s in (k, l) which can be further improved by taking 0, $t\in[k, l)$ and l, as we concluded above.

(c) Now we start from a distribution supported on 0 and $t_1, t_2, t_3\in[k, l)$, $t_1<t_2<t_3$, say. Then $M\in\overline{\mathbf{0}P}$ for some $P\in\triangle(T_1T_2T_3)$. We increase the expected rounding by bringing P nearer to M. For this purpose we lower $\triangle(T_1T_2T_3)$. By Lemma 1, this can be achieved by increasing t_2 or decreasing t_3. In any case we proceed until P reaches an edge, either $\overline{T_1T_2}$ or $\overline{T_1T_3}$, respectively. This shows the advantage of reducing one of t_i, say t_3. By Lemma 5 (ii), we can further increase the contribution of P by decreasing both t_1, t_2 so that $M\in\triangle(\mathbf{0}T_1T_2)$. If $M\in\operatorname{conv}\widehat{KL}$, the edge $\overline{T_1T_2}$ will ultimately absorb it, and we arrive at a combination analyzed in subcase (b). If $M\notin\operatorname{conv}\widehat{KL}$, we stop when t_1 reaches k, i.e. $\overline{T_1T_2}\in\operatorname{mem}(K, \widehat{KL})$. This is a combination treated in subcase (a). Anyway, we get more using (2.1).

(d) It remains to analyze distributions on 0, l and some $k<t<s<l$. Then for some $P\in\triangle(\mathbf{0}TS)$ we have

$$M = (1-\alpha_l)P+\alpha_l L = (1-\alpha_l)(\beta_0\mathbf{0}+\beta_t T+\beta_s S)+\alpha_l L,$$

with $0<\alpha_l, \beta_0, \beta_t, \beta_s<1$, $\beta_0+\beta_t+\beta_s=1$. We aim in increasing

$$E(t, s) = (1-\alpha_l)(\beta_t+\beta_s)k+\alpha_l l. \tag{3.4}$$

To this end we keep P fixed and decrease t and s simultaneously so the P still belongs to $\triangle(\mathbf{0}TS)$. By Lemma 5 (ii), β_0 decreases and $\beta_t+\beta_s$ increases. Since, moreover, α_l does not change, (3.4) actually becomes greater. We now verify the limitations of this approach. If $M\in\operatorname{conv}\widehat{KL}$, then for some $t>k$ we get $P\in\overline{TS}$ and $M\in\triangle(TSL)$, and we can go to the desired claim through (b) and (a). If $M\in\mathscr{L}_k\backslash\operatorname{conv}\widehat{KL}$, we eventually reach a combination of 0, $t=k$, $s\in(k,l)$ and l, which was analyzed in (a).

(v) $\mathscr{L}_n$ is located among $\operatorname{mem}(\mathbf{0},\widehat{NA})$, $\triangle(\mathbf{0}NA)$ and $\operatorname{mem}(N,\widehat{NA})$. Optimal distributions for the border moment points are supported on 0, $t\in[n,a]$, and 0, n, a, and n with $t\in[n,a]$, respectively. Lemma 2 implies that the optimal distribution for $M\in\operatorname{int}\mathscr{L}_n$ is supported on 0 and some $t_i\in[n,a]$. We have

$$M=\alpha_0\mathbf{0}+(1-\alpha_0)\sum\alpha_i S_i=\alpha_0\,\mathbf{0}+(1-\alpha_0)\,S$$

for some $S\in\mathscr{N}$, and we aim at maximizing $(1-\alpha_0)n$. We obtain minimal α_0 choosing S closest to M. Such an S should be an element of $\underline{\mathscr{N}}$ and have a representation $S=\alpha_n N+\alpha_t T$ for some $t\in(n,a)$.

(vi) Case: a is integer. Every $\mathscr{U}_k$ is bounded by $\triangle(KLA)$, $\operatorname{mem}(L,\widehat{KL})$ and $\operatorname{mem}(A,\widehat{KL})$, with the respective supports $\{k,l,a\}$, $\{t,l\}$ and $\{s,a\}$ of optimal distributions for some t, $s\in(k,l)$. Accordingly, the optimal distribution for a moment point lying inside $\mathscr{U}_k$ is supported on a and some $s_i\in[k,l]$. Here a is necessary since $M\in\operatorname{conv}\widehat{KL}$ otherwise. If we take a and two or more s_i from $[k,l)$ (including l or not), we obtain $M\in\overline{AS}$ for some $S\in\operatorname{int}\operatorname{conv}\widehat{KL}$. This segment crosses $\operatorname{mem}(L,\widehat{KL})$. By Lemma 3, we increase the expectation of rounding once we replace the combination of s_i by a pair t, l such that $\overline{TL}\in\operatorname{mem}(L,\widehat{KL})$ and $\overline{SA}$ cross each other, because the pair is optimal for $M\in\overline{TL}$.

(iii) and (vi). Case: a is noninteger. We consider M from the region situated above $\triangle(\mathbf{0}NA)$, $\triangle(KLN)$, $k=0,\ldots,n-2$, $\operatorname{mem}(L,\widehat{KL})$, $k=0,\ldots,n-1$, and beneath $\mathscr{M}_a=\operatorname{mem}(A,\widehat{\mathbf{0}A})$. We can write $M\in\overline{TS}$, where $T\in\operatorname{conv}\widehat{\mathbf{0}N}$ and $S\in\operatorname{conv}\widehat{NA}\backslash N$. These moment points are generated by distributions supported on $[0,n]$ and $(n,a]$, respectively. However, the half-line running from S towards M crosses $\triangle(\mathbf{0}NA)$ first. Due to Lemma 3, the distribution generating S can be replaced by one supported on 0, n, and a. Hence the only distribution on $(n,a]$ which is of interest is that concentrated at a.

(iii) (cont.) Suppose that $M\in\mathscr{T}_k$, $k=0,\ldots,n-2$. The optimal distribution for M is a mixture of ones concentrated on $[0,n]$ and $\{a\}$, where the former has a moment point $S\in AM^{\mapsto}$ that lies between M and $\underline{\mathscr{M}}_a$, and is optimal for S among the distributions on $[0,n]$ (see Lemma 3). Observe that $AM^{\mapsto}$ runs first above $\operatorname{conv}\widehat{\mathbf{0}N}$, then through $\operatorname{conv}\widehat{KL}\cup\{N\}\backslash\operatorname{conv}\widehat{KL}$, where the support points n, l with some $t\in[k,l)$ are optimal among distributions on $[0,n]$ (see (vi), case: a is integer), and pierces $\triangle(KLN)$. There is no need to follow the path of $AM^{\mapsto}$ further, because we know the optimal distribution supported on $[0, a]$ for moment points in $\triangle(KLN)$ and we can replace it for the distributions optimal for S located below.

In conclusion, we can confine ourselves on the distributions supported on a single $t \in [k, l)$, l, n, and a. For various choices of t, the expected rounding takes the form

$$E(t) = k\alpha_t(t) + l\alpha_l(t) + n[\alpha_n(t) + \alpha_a(t)] = k + \alpha_l(t) + (n-k)[\alpha_n(t) + \alpha_a(t)].$$

Our purpose is to show that $E(t)$ is nonincreasing, which would imply that the combination of k, l, n and a is optimal for $M \in \mathcal{T}_k$. It suffices to show that

$$E'(t) = \alpha_l'(t) + (n-k)[\alpha_n'(t) + \alpha_a'(t)] \leqslant 0 \tag{3.5}$$

for $k \leqslant t < l$, where

$$\alpha_l'(t) = \frac{\mathrm{pl}(LNA)(M)}{(l-t)^2(n-l)(a-l)}, \tag{3.6}$$

$$\alpha_n'(t) = -\frac{\mathrm{pl}(LNA)(M)}{(n-t)^2(n-l)(a-n)}, \tag{3.7}$$

$$\alpha_a'(t) = \frac{\mathrm{pl}(LNA)(M)}{(a-t)^2(a-l)(a-n)} \tag{3.8}$$

(cf. (1.8)). Observe that T, and so M are located beneath $\mathrm{pl}(LNA)$ (see Lemma 1), and hence

$$\mathrm{pl}(LNA)(M) < 0. \tag{3.9}$$

Using (3.6)–(3.9) and simple calculations, we represent (3.5) equivalently as

$$\frac{1}{n-k}\left(\frac{1}{n-l} - \frac{1}{a-l}\right) \geqslant \left(\frac{l-t}{n-t}\right)^2 \frac{1}{n-l} - \left(\frac{l-t}{a-t}\right)^2 \frac{1}{a-l}. \tag{3.10}$$

Note that we only need to check (3.10) for $k = 0$, because this is expressed in terms of differences and we can simply subtract k from t and eaeh parameter of (3.10).

Our reasoning is therefore reduced to proving

$$\frac{1}{n}\left(\frac{1}{n-1} - \frac{1}{a-1}\right) \geqslant \left(\frac{1-t}{n-t}\right)^2 \frac{1}{n-1} - \left(\frac{1-t}{a-t}\right)^2 \frac{1}{a-1} \tag{3.11}$$

for given $2 \leqslant n < a < n+1$ and all $0 \leqslant t < 1$. We check (3.11) for $t = 0$ and show that the derivative of the right-hand side is negative. Substituting $t = 0$ yields

$$\frac{a-n}{n(n-1)(a-1)} \geqslant \frac{(a-n)(a^2+an+n^2-a-n)}{n^2(n-1)a^2(a-1)},$$

which can be reduced to $a^2 \geqslant a+n$. This is satisfied for all $a > n$ when $n \geqslant 2$. Differentiating the right-hand side of (3.11) we obtain

$$-2(1-t)[(n-t)^{-3}-(a-t)^{-3}],$$

which is negative when $t < 1$ and $t < n < a$.

(vi) Case: a is noninteger (cont.). We are left with the task of determining the solutions for the moment points contained among $\triangle(KLA)$, $\mathrm{mem}(L, \widehat{KL})$ and $\mathrm{mem}(A, \widehat{KL})$ for some $k = 0, \ldots, n-1$. The solutions for the borders are known: their supports sum up to some $t_i \in [k, l)$, l and a. Standard arguments lead to the conclusion that any combination of t_i and l can be replaced by a unique pair t, l such that $\overline{TL} \in \mathrm{mem}(L, \widehat{KL})$ and $M \in \overline{TA}$. This completes the proof of Theorem 1. ∎

Proof of Theorem 2. (i) The universal bound m_1 for the expected roundings is attained by the distributions supported on integers. It suffices to consider four-point distributions. Three first moments of all four-point distributions on nonnegative integers coincide with $\mathscr{R} = \lim_{a\to\infty} \mathscr{P}_a$, where $\mathscr{P}_a$ denotes the polygon constructed for $\mathscr{M}_a$. Consequently, for given $M \in \mathscr{R}$ there is a possibly large but finite a such that M is a convex combination of four K_i with all $k_i \leqslant a$.

(ii) Observe that once $k \leqslant n-1$ for some a, the solution for $M \in \mathscr{L}_k$ does not change with the increase of a. None four-point distribution with the moment point $M \in \mathscr{L}_k$ (which has a bounded support by some a) provides greater expected rounding than one defined in Theorem 1 (iv).

(iii) If $M \in \mathscr{V}_k$, then there is a sufficiently large a_0 such that $M \in \mathscr{U}_k$ for all $a > a_0$. As a increases, $U_a(M)$ does as well. Any sequence μ_m, $m \geqslant 1$, of four-point distributions satisfying the moment conditions can be replaced by one whose elements, defined in Theorem 1 (vi) provide $U_{a_m}(M)$ for $a_m \to +\infty$ such that

$$\int \lfloor x \rfloor \, \mu_m(dx) \leqslant U_{a_m}(M) \quad \text{for all } m.$$

It is a simple matter to determine the limiting values of the support point $t \in (k, l)$, coefficients α_t, α_l, α_a, and the extreme expected Jefferson rounding. ∎

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