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A GENERAL CONTRACTION PRINCIPLE FOR VECTOR-VALUED MARTINGALES

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Abstract. We prove a contraction principle for vector-valued martingales of type

$$\Big\|\sum_{i=1}^{n} \Delta_i x_i\Big\|_{L_p^X} \leqslant c_p \Big\|\sup_{1\leqslant i\leqslant n} A_i(\Delta_i)\Big\|_{L_p} \Big\|\sum_{i=1}^{n} H_i x_i\Big\|_{L_1^X} \quad (1 \leqslant p < \infty),$$

where X is a Banach space with elements $x_1, \ldots, x_n$, $(\Delta_i)_{i=1}^n \subset L_1(\Omega, \boldsymbol{P})$ a martingale difference sequence belonging to a certain class, $(H_i)_{i=1}^n \subset L_1(M, \nu)$ a sequence of independent and symmetric random variables exponential in a certain sense, and A_i operators mapping each Δ_i into a non-negative random variable. Moreover, special operators A_i are discussed and an application to Banach spaces of Rademacher type α $(1 < \alpha \leqslant 2)$ is given.

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INTRODUCTION

For vector-valued random variables we relate the property *martingale difference sequence* to the property *independent and symmetric*. This is done by the consideration of inequalities of type

$$(1) \qquad \Big\|\sum_{i=1}^{n} \Delta_i x_i\Big\|_{L_p^X} \leqslant c_p \Big\|\sup_{1\leqslant i\leqslant n} A_i(\Delta_i)\Big\|_{L_p} \Big\|\sum_{i=1}^{n} H_i x_i\Big\|_{L_1^X} \quad (1 \leqslant p < \infty),$$

where X is a Banach space with elements $x_1, \ldots, x_n$, $(\Delta_i)_{i=1}^n \subset L_1(\Omega, \boldsymbol{P})$ a martingale difference sequence belonging to a certain class, $(H_i)_{i=1}^n \subset L_1(M, \nu)$ a sequence of independent and symmetric random variables, and A_i operators mapping each Δ_i into a non-negative random variable. Our interest in inequalities of form (1) comes from the following two aspects. First, they extend the classical contraction principle to the martingale setting. The classical contraction principle corresponds to the case where $\Delta_1, \ldots, \Delta_n$ are independent and of mean zero, $H_1, \ldots, H_n$ are the Rademacher variables, and $A_i(\Delta_i) = |\Delta_i|$. Secondly, in Corollary 7.2 these inequalities lead us to martingale inequalities in Banach spaces having type α, which extend the defining inequality from Definition 7.1, in which the Rademacher variables are involved. Herewith we want to indicate a way for further applications for inequalities of type (1).

Let us recall known results with respect to (1). Assume that $I := \{0, \ldots, N\}$ with $N \geqslant 1$ or $I := \{0, 1, 2, \ldots\}$ and that $(\mathcal{G}_k)_{k\in I}$ is a filtration on a probability space $[\Omega, \mathcal{G}, \boldsymbol{P}]$ such that $\mathcal{G}_\infty = \mathcal{G} = \bigvee_{k\in I} \mathcal{G}_k$. We let

$$\begin{aligned}\mathcal{M}\left((\mathcal{G}_k)_{k\in I}\right) := \{f = (f_k)_{k\in I} \subset L_1(\Omega, \mathcal{G}, \boldsymbol{P}) \text{ adapted} \mid f_0 = 0,\\ f_k = \boldsymbol{E}(f_\infty \mid \mathcal{G}_k) \text{ a.s. for } k \in I \text{ and some } f_\infty \in L_1(\Omega, \mathcal{G}, \boldsymbol{P})\},\end{aligned}$$

$$\mathcal{P}\left((\mathcal{G}_k)_{k\in I}\right) := \{f \in \mathcal{M}\left((\mathcal{G}_k)_{k\in I}\right) \mid (|df_k|)_{k\geqslant 1, k\in I} \text{ is predictable}\}$$

with $df_k := f_k - f_{k-1}$ for $k \geqslant 1$ and $df_0 := f_0$. The sequence $(h_k)_{k=0}^\infty \subset L_\infty[0, 1)$ stands for the Haar functions

$$h_0 := \chi_{[0,1)}, \quad h_1 := \chi_{[0,1/2)} - \chi_{[1/2,1)}, \quad h_2 := \chi_{[0,1/4)} - \chi_{[1/4,1/2)}, \quad \ldots$$

normalized in $L_\infty[0, 1)$, $r_1, r_2, \ldots \in L_\infty[0, 1)$ for the Rademacher variables

$$r_i := h_{2^{i-1}} + \ldots + h_{2^i - 1},$$

the sequence $g_1, g_2, \ldots$ for independent standard Gaussian variables, and $g_{\alpha,1}, g_{\alpha,2}, \ldots$ from $L_1(M, \nu)$ with $2 < \alpha < \infty$ for independent random variables distributed like

$$\nu(g_{\alpha,i} > \lambda) = \kappa_\alpha \int_\lambda^\infty \exp(-|\xi|^\alpha)\, d\xi \text{ for } \lambda \in \boldsymbol{R}, \quad \text{where } \kappa_\alpha := \Big(\int_{\boldsymbol{R}} \exp(-|\xi|^\alpha)\, d\xi\Big)^{-1}.$$

The known cases in which (1) is satisfied can be listed as follows:

	$(H_1, \ldots, H_n)$	$\Delta_i = \sum_{I_i} df_k$ with	$A_i(\Delta_i)(\omega)$
(a)	$(r_1, \ldots, r_n)$	$f \in \mathcal{M}\left((\mathcal{G}_k)_{k=0}^N\right)$	$\lvert\Delta_i(\omega)\rvert$
(b)	$(g_{\alpha,1}, \ldots, g_{\alpha,n})$	$f \in \mathcal{P}\left((\mathcal{G}_k)_{k=0}^N\right)$	$\sup_k \sqrt[\beta]{k}\, \xi_k^{(i)}(\omega)$
(c)	$(g_1, \ldots, g_n)$	$f \in \mathcal{P}\left((\mathcal{G}_k)_{k=0}^N\right)$	$\sqrt{\sum_{I_i} \lvert df_k(\omega)\rvert^2}$

where $1 = 1/\alpha + 1/\beta$, $0 = \tau_0 \leqslant \ldots \leqslant \tau_n = N$ is any sequence of stopping times,

$$I_i := \{1 \leqslant k \leqslant N \mid \tau_{i-1} < k \leqslant \tau_i\},$$

and $(\xi_k^{(i)}(\omega))_{k=1}^N$ is a non-increasing rearrangement of $\left(|\chi_{\{\tau_{i-1} < k \leqslant \tau_i\}}\, df_k(\omega)|\right)_{k=1}^N$. Statements (a) and (c) are proved in [9], statement (b) can be found in [8]. There is a basic difference in the proofs of (a) on the one hand, and (b) and (c) on the other hand. In (a) an induction argument due to Kwapień and Woyczyński is implicitly used, whereas (b) and (c) are based on majorizing measure type theorems due to Talagrand. The aim of this paper is the further development of the method used in (a). This is done as follows:

(i) Our basic result is Theorem 3.4. It is based on extrapolation and on Lemma 3.5, which contains the arguments of Kwapień and Woyczyński. Theorem 3.4 allows variables $H_1, \ldots, H_n$ on the right-hand side of (1) not necessarily identically distributed (the results, mentioned above, use identically distributed variables). Moreover, Theorem 3.4 provides an alternative approach to assertions (b) and (c) which does not use deep majorizing measure type theorems (see Corollary 6.6 and the remark below).

(ii) The assumptions of Theorem 3.4 involve operators A_i defined on martingales satisfying $BMO^*_{\psi_i}$-L_∞ estimates. In Theorems 4.7 and 5.3 (and implicitly in Example 4.5) we extend the known examples of such operators. The corresponding applications to Theorem 3.4 are given in Section 6.

(iii) In Section 7 we deduce a martingale inequality in Banach spaces having the Rademacher type α $(1 < \alpha \leqslant 2)$ and relate this inequality to a corresponding inequality in Banach spaces having a modulus of smoothness of power type α.

1. SOME GENERAL NOTATION

Throughout this paper all Banach spaces and random variables are assumed to be real. For a probability space $[\Omega, \mathscr{G}, P]$ and a Banach space X we let $L_0^X(\Omega, \mathscr{G}, P)$ be the space of all Borel-measurable $f\colon \Omega \to X$ such that there is a closed separable linear subspace $X_0 \subseteq X$ with $P(f \in X_0) = 1$, where

$$L_0(\Omega, \mathscr{G}, P) := L_0^{\mathbb{R}}(\Omega, \mathscr{G}, P)$$

$$\text{and} \quad L_0^+(\Omega, \mathscr{G}, P) := \{f \in L_0(\Omega, \mathscr{G}, P) \mid f \geqslant 0 \text{ a.s.}\}.$$

Given a compatible couple of Banach spaces (X_0, X_1) and $0 < \eta < 1$ we use

$$\|x\|_{(X_0,X_1)_{\eta,\infty}} := \sup_{0<t<\infty} t^{-\eta} K(x, t; X_0, X_1) \quad \text{for } x \in X_0 + X_1,$$

where, for $t \geqslant 0$,

$$K(x, t; X_0, X_1) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} \mid x = x_0 + x_1,\ x_i \in X_i\}$$

is the usual K-functional (see [1] for more information concerning the K-functional and interpolation spaces). Moreover, we make the conventions that $\inf \emptyset := \infty$ and that $A \sim_c B$ stands for $c^{-1}A \leqslant B \leqslant cA$ if $c > 0$ and $A, B \geqslant 0$. Finally, we shall use the Khintchine–Kahane inequality for the Rademacher variables stating that

$$\Big\| \sum_{i=1}^{n} r_i x_i \Big\|_{L_p^X} \sim_{c_p} \Big\| \sum_{i=1}^{n} r_i x_i \Big\|_{L_2^X} \tag{2}$$

for all Banach spaces X, $n = 1, 2, \ldots$, elements $x_1, \ldots, x_n \in X$, and $0 < p < \infty$, where $c_p > 0$ depends on p only.

2. *BMO*-SPACES OF CÀDLÀG PROCESSES

Here we introduce the BMO-spaces we are going to exploit. For a complete probability space $[\Omega, \mathscr{F}, P]$ we use $T := [0, \infty)$ and a filtration $(\mathscr{F}_t)_{0 \leqslant t \leqslant \infty}$ such that, see [15], p. 3,

(C1) $\mathscr{F} = \mathscr{F}_\infty = \bigvee_{t \in T} \mathscr{F}_t$,

(C2) $\mathscr{F}_0$ contains all P-null sets of $\mathscr{F}$,

(C3) $\mathscr{F}_t = \bigcap_{u > t} \mathscr{F}_u$ for $t \in T$.

DEFINITION 2.1. (i) We let $\mathscr{CL}\big((\mathscr{F}_t)_{t \in T}\big)$ be the set of all processes $f = (f_t)_{t \in T} \subseteq L_0(\Omega, \mathscr{F}, P)$ adapted to $(\mathscr{F}_t)_{t \in T}$ such that $\big(f_t(\omega)\big)_{t \in T}$ is right continuous and has finite left limits a.s. (i.e. f is *càdlàg*) such that $f_0 = 0$ and such that there is some $f_\infty \in L_0(\Omega, \mathscr{F}, P)$ with $\lim_{t \to \infty} f_t = f_\infty$ a.s.

(ii) For $f \in \mathscr{CL}\big((\mathscr{F}_t)_{t \in T}\big)$ and stopping times $0 \leqslant \sigma \leqslant \tau \leqslant \infty$ we let

$${}^{\sigma}f_t^{\tau}(\omega) := f_{\tau(\omega) \wedge t}(\omega) - f_{\sigma(\omega) \wedge t}(\omega), \qquad {}^{\sigma}f^{\tau} := ({}^{\sigma}f_t^{\tau})_{t \in T}, \qquad \text{and} \qquad f^{\tau} := {}^{0}f^{\tau}.$$

(iii) A subset $E \subseteq \mathscr{CL}\big((\mathscr{F}_t)_{t \in T}\big)$ is closed under starting and stopping provided that ${}^{\sigma}f^{\tau} \in E$ for all $f \in E$ and all stopping times $0 \leqslant \sigma \leqslant \tau \leqslant \infty$.

(iv) We let $\mathscr{M}\big((\mathscr{F}_t)_{t \in T}\big)$ be the set of uniformly integrable martingales $f = (f_t)_{t \in T}$ from $\mathscr{CL}\big((\mathscr{F}_t)_{t \in T}\big)$.

So, given $f \in \mathscr{CL}\big((\mathscr{F}_t)_{t \in T}\big)$ and a stopping time $\tau\colon \Omega \to [0, \infty]$, we also have f_∞ and f_τ, which are unique a.s.

DEFINITION 2.2. (i) Let $\mathscr{D}$ be the set of all increasing bijections $\psi\colon [1, \infty) \to [1, \infty)$ and $\bar{\mathscr{D}} \subset \mathscr{D}$ the subset of all $\psi \in \mathscr{D}$ such that

$$\psi(\lambda + \mu) + 1 \geqslant \psi(\lambda) + \psi(\mu) \quad \text{ for } \lambda, \mu \geqslant 1.$$

(ii) Given $\psi \in \mathscr{D}$ we let

$$\bar{\psi}(\lambda) - 1 := \sup\Big\{ \sum_{i=1}^{M} [\psi(\lambda_i) - 1] \mid \lambda = \sum_{i=1}^{M} \lambda_i, \lambda_i \geqslant 1, M = 1, 2, \ldots \Big\}.$$

Obviously, $\bar{\psi} \in \bar{\mathscr{D}}$. For a stopping time $\tau\colon \Omega \to [0, \infty]$ and $f = (f_t)_{t \in T} \in \mathscr{CL}\big((\mathscr{F}_t)_{t \in T}\big)$ we use

$$f_{\tau-} := \chi_{\{\tau = \infty\}} f_\infty + \lim_{n \to \infty} \big[\chi_{\{\tau < \infty\}} \chi_{\Omega_0} f_{(\tau - 1/n) \vee 0}\big],$$

where Ω_0 is a set of measure one on which $(f_t)_{t\in T}$ is right continuous and has finite left limits. The random variable $f_{\tau-}$ is unique a.s. Moreover, given $B\in\mathscr{F}$ with $P(B)>0$, we let P_B be the normalized restriction of P to B, otherwise we set $P_B:=0$.

DEFINITION 2.3. Let $f\in\mathscr{CL}\big((\mathscr{F}_t)_{t\in T}\big)$ and $\psi\in\mathscr{D}$.

(i) $\|f\|_{BMO_\psi}:=\inf c$, where the infimum is taken over all $c>0$ such that for all stopping times $\tau:\Omega\to[0,\infty]$ and $B\in\mathscr{F}_\tau$ one has

$$P_B(|f_\infty-f_{\tau-}|>\lambda)\leqslant \exp\big(1-\psi(\lambda/c)\big) \quad \text{for } \lambda\geqslant c.$$

(ii) $\|f\|_{BMO^*_\psi}:=\inf c$, where the infimum is taken over all $c>0$ such that for all stopping times $\tau:\Omega\to[0,\infty]$ and $B\in\mathscr{F}_\tau$ one has

$$P_B\big(\sup_{\tau\leqslant t\leqslant\infty}|f_t-f_{\tau-}|>\lambda\big)\leqslant \exp\big(1-\psi(\lambda/c)\big) \quad \text{for } \lambda\geqslant c.$$

For the classical notion of bounded mean oscillation for adapted càdlàg processes the reader is referred to [5], Chapters VI and VII. In [7] it is shown that $\bar{\psi}$ is the right tool to classify BMO_ψ-spaces of adapted sequences. The following assertion is proved in [7], Theorem 4.6, for the discrete time setting. For the convenience of the reader we recall its proof for the continuous time setting in the appendix.

THEOREM 2.4. *For $\psi\in\mathscr{D}$ one has*

(i) $\|\cdot\|_{BMO^*_{\bar\psi}}=\|\cdot\|_{BMO^*_\psi}\leqslant 4\psi^{-1}(3)\|\cdot\|_{BMO_\psi}$.

(ii) *If* $\|f\|_{BMO^*_\psi}=1$, $\lambda>0$, $\mu\geqslant 1$, *and* $f^*:=\sup_{t\in T}|f_t|$, *then*

$$P(f^*>\lambda+\mu)\leqslant e^{1-\bar\psi(\mu)}P(f^*>\lambda).$$

Besides the above theorem we shall use the relation

$$\|f_\tau-f_{\tau-}\|_{L_\infty(\Omega,P)}\leqslant\|f\|_{BMO^*_\psi},$$

where $\tau:\Omega\to[0,\infty]$ is a stopping time and $\psi\in\mathscr{D}$.

3. A GENERAL CONTRACTION PRINCIPLE

Throughout this section we assume that conditions (C1), (C2), and (C3) are satisfied. Let us first summarize some assumptions needed in the formulation of the main result, i.e. Theorem 3.4.

DEFINITION 3.1. An operator $A: E\to L_0^+(\Omega,\mathscr{F},P)$ satisfies *property* (S) [1] with constant $d>0$ provided that the following conditions are satisfied:

(S1) $E\subseteq\mathscr{CL}\big((\mathscr{F}_t)_{t\in T}\big)$ is closed under starting and stopping.

[1] The symbol (S) should indicate an assumption related to stopping of càdlàg processes.

(S2) $\|A(g^{\varrho})\|_{L_\infty(\Omega,P)} \leqslant d\,\|Ag\|_{L_\infty(\Omega,P)}$ for all $g \in E$ and stopping times ϱ.

(S3) For all $0 < \lambda < \infty$ and $f \in E$ there is a stopping time ϱ such that (i) $\varrho = \infty$ a.s. on $\{Af \leqslant \lambda\}$, (ii) $\|A(f^{\varrho})\|_{L_\infty(\Omega,P)} \leqslant d\lambda$.

Although the above condition looks quite technical and somewhat artificial, it seems that this condition is a right one to guarantee the extrapolation of a BMO-L_∞ estimate to an L_p-L_p estimate, needed in the proof of Theorem 3.4. Moreover, this condition is satisfied in the situations relevant for our purpose (see Lemma 6.3). The next definition we need is

DEFINITION 3.2. For $F \in L_0^X(M, \nu)$ and $\psi \in \mathscr{D}$ let

$$\|F\|_\psi := \sup_{1 \leqslant r < \infty} \frac{\|F\|_{L_r^X}}{\psi^{-1}(r)}.$$

Remark 3.3. First, note that $\|F\|_\psi < \infty$ implies $F \in L_r^X(M, \nu)$ for $1 \leqslant r < \infty$. The quantity $\|\cdot\|_\psi$ is often used because of the following:

(i) One has

$$\inf\{c > 0 \mid \nu(\|F\| > \lambda) \leqslant \exp(1 - \psi(\lambda/c)) \text{ for } \lambda \geqslant c\} \leqslant e\,\|F\|_\psi$$

(a converse inequality fails to be true in general).

(ii) If there are $\alpha, \beta > 1$ with $\alpha\psi(\lambda) \leqslant \psi(\beta\lambda)$ for all $\lambda \geqslant 1$, then there is a converse inequality: For example, by Lemma 3.7 one can see that $\nu(\|F\| > \lambda) \leqslant e^{1-\psi(\lambda)}$ for $\lambda \geqslant 1$ implies that $\|F\|_\psi \leqslant c(\psi, \alpha, \beta) < \infty$.

THEOREM 3.4. *Let $\psi_1, \ldots, \psi_n \in \bar{\mathscr{D}}$, $H_1, \ldots, H_n \in L_0(M, \nu)$ be independent and symmetric with*

$$\nu(|H_i| > \lambda) = \exp(1 - \psi_i(\lambda \vee 1)) \text{ for } \lambda \geqslant 0, \quad \text{and} \quad \tilde{H}_i := 4\psi_i^{-1}(3)\,H_i.$$

Assume that $A_1, \ldots, A_n \colon \mathscr{M}((\mathscr{F}_t)_{t\in T}) \supseteq E \to L_0^+(\Omega, \mathscr{F}, P)$ satisfy property (S) with constant $d > 0$, that $0 = \tau_0 \leqslant \tau_1 \leqslant \ldots \leqslant \tau_n \leqslant \infty$ are stopping times, and that

$$\|{}^{\tau_{i-1}}f^{\tau_i}\|_{BMO^*_{\psi_i}} \leqslant \|A_i({}^{\tau_{i-1}}f^{\tau_i})\|_{L_\infty(\Omega,P)} \quad \text{for } 1 \leqslant i \leqslant n \text{ and } f \in E.$$

Then the following holds:

(i) *For all $\psi \in \mathscr{D}$ there is a $c > 0$ depending on d and ψ only, such that for $f \in E$, elements $x_1, \ldots, x_n$ of a Banach space X, and $1 \leqslant p < \infty$ one has*

$$\Big\|\sup_{t\in T}\Big\|\sum_{i=1}^{n} [{}^{\tau_{i-1}}f_t^{\tau_i}]x_i\Big\|_X\Big\|_{L_p} \leqslant c\bar{\psi}^{-1}(p)\Big\|\sup_{1\leqslant i\leqslant n} A_i({}^{\tau_{i-1}}f^{\tau_i})\Big\|_{L_p}\Big\|\sum_{i=1}^{n} \tilde{H}_i x_i\Big\|_\psi.$$

(ii) *There is a $c > 0$ depending on d only, such that for $f \in E$, elements $x_1, \ldots, x_n$ of a Banach space X, and $1 \leqslant p < \infty$ one has*

$$\Big\|\sup_{t\in T}\Big\|\sum_{i=1}^{n} [{}^{\tau_{i-1}}f_t^{\tau_i}]x_i\Big\|_X\Big\|_{L_p} \leqslant cp\Big\|\sup_{1\leqslant i\leqslant n} A_i({}^{\tau_{i-1}}f^{\tau_i})\Big\|_{L_p}\Big\|\sum_{i=1}^{n} \tilde{H}_i x_i\Big\|_{L_1^X}.$$

Before we verify Theorem 3.4 some lemmas are needed. The first one follows directly from the induction argument given by Kwapień and Woyczyński in [12], Theorem 5.1.1.

LEMMA 3.5. *Let* $1 \leqslant r < \infty$, *let* $(\mathcal{G}_i)_{i=0}^n$ *be a filtration, and let* $(\Delta_i)_{i=1}^n \subset L_r(\Omega, P)$ *be adapted with respect to* $(\mathcal{G}_i)_{i=1}^n$. *Assume independent* $H_1, \ldots, H_n \in L_r(M, \nu)$ *and a Banach space* X *be such that for all* $1 \leqslant i \leqslant n$ *and* $x, y \in X$ *one has*

$$E(\|x+\Delta_i y\|^r \mid \mathcal{G}_{i-1}) \leqslant E\|x+H_i y\|^r \text{ a.s.}$$

Then, for all $x_1, \ldots, x_n \in X$, *one has*

$$E(\|\sum_{i=1}^n \Delta_i x_i\|^r \mid \mathcal{G}_0) \leqslant E\|\sum_{i=1}^n H_i x_i\|^r \text{ a.s.}$$

LEMMA 3.6. *Let* $\psi \in \mathcal{D}$ *and* $\Delta \in L_1(\Omega, P)$ *of mean zero. If for* $\lambda \geqslant 1$ *one has*

$$P(|\Delta| > \lambda) \leqslant e^{1-\psi(\lambda)}$$

and if $H(t) := \psi^{-1}(1+\log t^{-1}) \in L_0(0, 1]$, *then*

$$\int_\Omega \|x+\Delta y\|^r \, dP \leqslant \frac{1}{2}\left[\int_0^1 \|x+cH(t)y\|^r \, dt + \int_0^1 \|x-cH(t)y\|^r \, dt\right]$$

for all $1 \leqslant r < \infty$, *all elements* x, y *of a Banach space* X, *and* $c := 4\psi^{-1}(3)$.

Proof. First we remark that [7], Lemma 4.4, implies $\psi(\lambda) \geqslant \lambda/c_\psi$ for some $c_\psi \geqslant 1$ and all $\lambda \geqslant 1$ so that $e^{1-\psi(\lambda)} \leqslant \exp(1-(\lambda/c_\psi))$ for $\lambda \geqslant 1$ and $\Delta \in L_r(\Omega, \mathcal{F}, P)$ as well as $H \in L_r(0, 1]$.

(a) We show that

$$2\exp\left(1-\psi\left(\frac{\lambda}{2} \vee 1\right)\right) \wedge 1 \leqslant |\{cH > \lambda\}| = \exp\left(1-\psi\left(\frac{\lambda}{c} \vee 1\right)\right) \quad \text{for } \lambda \geqslant 0.$$

Since

$$2\exp\left(1-\psi\left(\frac{\lambda}{2} \vee 1\right)\right) \leqslant 1 \quad \text{for } \lambda \geqslant c,$$

it remains to check that

$$2\exp\left(1-\psi\left(\frac{\lambda}{2}\right)\right) \leqslant \exp\left(1-\psi\left(\frac{\lambda}{c}\right)\right) \quad \text{for } \lambda \geqslant c.$$

Setting $\mu_0 := \psi^{-1}(1+\log 2)$ we get

$$\psi\left(\frac{\lambda}{c}\right)+\log 2 = \psi\left(\frac{\lambda}{c}\right)+\psi(\mu_0)-1 \leqslant \psi\left(\frac{\lambda}{c}+\mu_0\right) \leqslant \psi\left(\frac{\lambda}{2}\right) \quad \text{for } \lambda \geqslant c,$$

which implies the desired estimate.

(b) Now let Δ' be an independent copy of Δ. Then, for $\lambda \geqslant 0$,

$$\boldsymbol{P}(|\Delta-\Delta'| > \lambda) \leqslant 2\boldsymbol{P}\left(|\Delta| > \frac{\lambda}{2}\right) \wedge 1 \leqslant 2\exp\left(1-\psi\left(\frac{\lambda}{2} \vee 1\right)\right) \wedge 1 \leqslant |\{cH > \lambda\}|$$

and

$$\int_{\Omega} \|x+\Delta y\|^r \, d\boldsymbol{P} \leqslant \int_{\Omega} \|x+(\Delta-\Delta')\, y\|^r \, d\boldsymbol{P}$$
$$\leqslant \frac{1}{2}\Big[\int_0^1 \|x+cH(t)\, y\|^r \, dt + \int_0^1 \|x-cH(t)\, y\|^r \, dt\Big];$$

cf. [13], Lemma 4.6. ∎

The last lemma we need is Lemma 7.1 of [2]:

Lemma 3.7. *Let $f, g \in L_0^+(\Omega, \mathscr{F}, \boldsymbol{P})$ and $0 < p < \infty$ be such that for some $\beta > 1$ and $\delta, \varepsilon > 0$ with $\beta^p \varepsilon < 1$ one has*

$$\boldsymbol{P}(f > \beta\lambda,\, g \leqslant \delta\lambda) \leqslant \varepsilon \boldsymbol{P}(f > \lambda) \quad \textit{for all } \lambda > 0.$$

Then

$$\|f\|_p \leqslant \frac{\beta}{\delta} \frac{1}{\sqrt[p]{1-\beta^p \varepsilon}} \|g\|_p.$$

Proof of Theorem 3.4. For fixed $0 < c < \infty$ we introduce $U, V\colon E \to L_0^+(\Omega, \mathscr{F}, \boldsymbol{P})$ as

$$Uf(\omega) := \Big\|\sum_{i=1}^{n} [{}^{\tau_{i-1}}f_c^{\tau_i}(\omega)]x_i\Big\|_X \quad \text{and} \quad Vf(\omega) := \sup_{1 \leqslant i \leqslant n} A_i({}^{\tau_{i-1}}f^{\tau_i})(\omega).$$

The constant c is introduced for convenience to define $Uf(\omega)$ uniquely without the closure f_∞. Note that $(f_t^U)_{t\in T} := \big(U(f^t)\big)_{t\in T} \in \mathscr{CL}\big((\mathscr{F}_t)_{t\in T}\big)$.

(a) Let $\tau\colon \Omega \to [0, \infty]$ be a stopping time and $B \in \mathscr{F}_\tau$ of positive measure. Then for $1 \leqslant r < \infty$ and $\sigma := \tau \wedge \tau_n \wedge c$ we obtain

$$\|f_\infty^U - f_{\tau-}^U\|_{L_r(B,\boldsymbol{P}_B)} \leqslant \|f_\sigma^U - f_{\sigma-}^U\|_{L_r(B,\boldsymbol{P}_B)} + \|f_\infty^U - f_\tau^U\|_{L_r(B,\boldsymbol{P}_B)}$$
$$\leqslant \|f_\sigma - f_{\sigma-}\|_{L_r(B,\boldsymbol{P}_B)} \sup_i \|x_i\|$$
$$+ \Big\|\sum_{i=1}^{n} [{}^{\tau_{i-1}}f_c^{\tau_i} - {}^{\tau_{i-1}}f_{\tau\wedge c}^{\tau_i}]\, x_i\Big\|_{L_r^X(B,\boldsymbol{P}_B)} =: S_1 + S_2.$$

The first summand can be estimated via

$$S_1 = \Big\|\sum_{i=1}^{n} \chi_{\{\tau_{i-1} < \sigma \leqslant \tau_i\}} [{}^{\tau_{i-1}}f_\sigma^{\tau_i} - {}^{\tau_{i-1}}f_{\sigma-}^{\tau_i}]\Big\|_{L_r(B,\boldsymbol{P}_B)} \sup_i \|x_i\|$$

$$\leqslant \sup_i \|{}^{\tau_{i-1}}f_\sigma^{\tau_i} - {}^{\tau_{i-1}}f_{\sigma-}^{\tau_i}\|_{L_\infty(\Omega,P)} \sup_i \|x_i\| \leqslant \sup_i \|{}^{\tau_{i-1}}f^{\tau_i}\|_{BMO^*_{\psi_i}} \sup_i \|x_i\|$$

$$\leqslant \|Vf\|_{L_\infty(\Omega,P)} \Big\| \sum_{i=1}^n \tilde{H}_i x_i \Big\|_{L_1(M,\nu)}.$$

To estimate S_2 we let $\mathscr{G}_i := \mathscr{F}_{\tau_i \vee \tau}$ and $\Delta_i := {}^{\tau_{i-1}}f_c^{\tau_i} - {}^{\tau_{i-1}}f_{\tau \wedge c}^{\tau_i}$ for $i = 1, \ldots, n$ and obtain a martingale difference sequence $(\Delta_i)_{i=1}^n \subset L_1(\Omega, \mathscr{F}, P)$ with respect to $(\mathscr{G}_i)_{i=1}^n$. Assuming $B_{i-1} \in \mathscr{G}_{i-1}$ of positive measure, $\|V(f^c)\|_{L_\infty(\Omega,P)} < \beta$, and $\lambda \geqslant 1$, our assumption implies $\|{}^{\tau_{i-1}}(f^c)^{\tau_i}\|_{BMO^*_{\psi_i}} < \beta$ and

$$P_{B_{i-1}}(\beta^{-1}|\Delta_i| > \lambda) = P_{B_{i-1}}(\beta^{-1}|{}^{\tau_{i-1}}(f^c)_\infty^{\tau_i} - {}^{\tau_{i-1}}(f^c)_{\tau_{i-1}\vee\tau}^{\tau_i}| > \lambda) \leqslant \exp\big(1-\psi_i(\lambda)\big).$$

Applying Lemmas 3.6 and 3.5 yields

$$\Big\| \sum_{i=1}^n \Delta_i x_i \Big\|_{L_r^X(B,P_B)} \leqslant \beta \Big\| \sum_{i=1}^n \tilde{H}_i x_i \Big\|_{L_r^X(M,\nu)}$$

so that

$$\Big\| \sum_{i=1}^n \Delta_i x_i \Big\|_{L_r^X(B,P_B)} \leqslant d\|Vf\|_{L_\infty(\Omega,P)} \Big\| \sum_{i=1}^n \tilde{H}_i x_i \Big\|_{L_r^X(M,\nu)}$$

(note that property (S) gives $\|V(f^c)\|_{L_\infty(\Omega,P)} \leqslant d\,\|Vf\|_{L_\infty(\Omega,P)}$) and

$$\|f_\infty^U - f_{\tau-}^U\|_{L_r(B,P_B)} \leqslant (1+d) \Big\| \sum_{i=1}^n \tilde{H}_i x_i \Big\|_{L_r^X(M,\nu)} \|Vf\|_{L_\infty(\Omega,P)}. \tag{3}$$

(b) For the proof of assertion (i) we continue by dividing (3) by $\psi^{-1}(r)$ and by taking the supremum over $1 \leqslant r < \infty$ so that, by Theorem 2.4 and Remark 3.3,

$$\frac{1}{4\psi^{-1}(3)} \big\|(U(f^t))_{t\in T}\big\|_{BMO_\psi^*} \leqslant \big\|(U(f^t))_{t\in T}\big\|_{BMO_\psi} \leqslant (1+d)\,e \Big\| \sum_{i=1}^n \tilde{H}_i x_i \Big\|_\psi \|Vf\|_{L_\infty(\Omega,P)}. \tag{4}$$

(c) For the proof of assertion (ii) we use $\psi(\lambda) := 1 + \log \lambda$ and obtain from (3) ($r = 1$) and Theorem 2.4

$$\frac{1}{4\psi^{-1}(3)} \big\|(U(f^t))_{t\in T}\big\|_{BMO_\psi^*} \leqslant \big\|(U(f^t))_{t\in T}\big\|_{BMO_\psi} \leqslant (1+d) \Big\| \sum_{i=1}^n \tilde{H}_i x_i \Big\|_{L_1^X(M,\nu)} \|Vf\|_{L_\infty(\Omega,P)}. \tag{5}$$

(d) Now let us fix $f \in E$, $0 < \delta \leqslant 1$, and $\lambda > 0$. Property (S) implies the existence of stopping times $\varrho_1, \ldots, \varrho_n$ such that

$$\varrho_i = \infty \text{ a.s. on } \{A_i({}^{\tau_{i-1}}f^{\tau_i}) \leqslant \delta\lambda\} \quad \text{and} \quad \big\|A_i\big(({}^{\tau_{i-1}}f^{\tau_i})^{\varrho_i}\big)\big\|_{L_\infty(\Omega,P)} \leqslant d\delta\lambda.$$

Setting $\varrho := \inf_{i=1,\dots,n} \varrho_i$ we get

$$\varrho = \infty \text{ a.s. on } \{Vf \leqslant \delta\lambda\} \quad \text{and} \quad \|V(f^{\varrho})\|_{L_\infty(\Omega,P)} \leqslant d^2\,\delta\lambda.$$

Hence, for $g := f^{\varrho}$,

$$\chi_{\{Vf \leqslant \delta\lambda\}} U^*f \leqslant U^*g \leqslant U^*f \text{ a.s.},$$

where $U^*h(\omega) := \sup_{t\in T} U(h^t)(\omega)$. Let $v > 0$ be such that

$$v > 4\psi^{-1}(3)(1+d)e \left\|\sum_{i=1}^{n} \tilde{H}_i x_i\right\|_{\psi}$$

in the case of assertion (i) and $v > 4\psi^{-1}(3)(1+d)\|\sum_{i=1}^{n} \tilde{H}_i x_i\|_{L_1^X(M,v)}$ in the case of assertion (ii). Inequalities (4) and (5) yield

$$\left\|(U(g^t))_{t\in T}\right\|_{BMO_{\bar{\psi}}^*} \leqslant vd^2\,\delta\lambda,$$

so that, by Theorem 2.4,

$$\begin{aligned}
P\big(U^*f > (1+d^2)v\lambda,\ Vf \leqslant \delta\lambda\big) &\leqslant P\big(U^*g > (1+d^2)v\lambda\big)\\
&\leqslant \exp\left(1-\bar{\psi}\left(\frac{vd^2\lambda}{vd^2\,\delta\lambda}\right)\right) P(U^*g > v\lambda) = \exp\big(1-\bar{\psi}(1/\delta)\big) P(U^*g > v\lambda)\\
&\leqslant \exp\big(1-\bar{\psi}(1/\delta)\big) P(U^*f > v\lambda)
\end{aligned}$$

and

$$P\big(U^*f > (1+d^2)\lambda,\ vVf \leqslant \delta\lambda\big) \leqslant \exp\big(1-\bar{\psi}(1/\delta)\big) P(U^*f > \lambda)$$

for $\lambda > 0$ and $0 < \delta \leqslant 1$. Choosing $\kappa \geqslant 1$ with

$$e^{1-\kappa}(1+d^2) \leqslant \tfrac{1}{2} \quad \text{and} \quad 1/\delta := \bar{\psi}^{-1}(\kappa p) \text{ for } 1 \leqslant p < \infty,$$

from Lemma 3.7 we derive that

$$\|U^*f\|_{L_p} \leqslant 2(1+d^2)v\bar{\psi}^{-1}(\kappa p)\,\|Vf\|_{L_p}.$$

Exploiting $\bar{\psi}^{-1}(\kappa p) \leqslant c_\kappa \bar{\psi}^{-1}(p)$ ([7], Lemma 4.4) and $\bar{\psi}^{-1}(p) \leqslant ep$ if $\psi(\lambda) = 1+\log\lambda$ ([7], Example 4.3) we arrive at our assertion by $c \to \infty$ chosen in the beginning. ∎

4. OPERATORS WITH TAIL BEHAVIOUR $\exp(-\psi(\lambda))$, WHERE $\lim_{\lambda\to\infty} \lambda^2/\psi(\lambda) = 0$

In [9], Lemma 3.5, it is shown that

$$P\left(|f_N| > \lambda \left\|\|(df_k)_{k=1}^N\|_{l_{\beta,\infty}^N}\right\|_{L_\infty(\Omega,P)}\right) \leqslant \exp\big(1-(\lambda/c_\beta)^\alpha\big) \tag{6}$$

for $f \in \mathscr{P}\big((\mathscr{G}_k)_{k=0}^N\big)$ and $\lambda \geqslant c_\beta$, where $1 < \beta < 2 < \alpha < \infty$, $1 = 1/\alpha + 1/\beta$, and where $l_{\beta,\infty}^N$ is the usual Lorenz sequence space. Therefore, in order to obtain $BMO_{\psi_i}^*$-L_∞ estimates needed for Theorem 3.4, we introduce the following operators.

DEFINITION 4.1. Let $\gamma = (\gamma_k)_{k=1}^{\infty}$ be a sequence with $0 < \gamma_1 \leqslant \gamma_2 \leqslant \ldots < \infty$ and $\lim_{k\to\infty} \gamma_k = \infty$.

(i) We let

$$l^{\gamma,\infty} := \{(\xi_k)_{k=1}^{\infty} \mid \|(\xi_k)_{k=1}^{\infty}\|_{l^{\gamma,\infty}} := \sup_{N\geqslant 1} \sup_{1\leqslant k\leqslant N} \gamma_k \xi_{k,N}^{*} < \infty\},$$

where $(\xi_{k,N}^{*})_{k=1}^{N}$ is a non-increasing rearrangement of $(|\xi_k|)_{k=1}^{N}$.

(ii) We let $S_\gamma\colon \mathscr{P}\big((\mathscr{G}_k)_{k=0}^{\infty}\big) \supseteq E_\gamma \to L_0^{+}(\Omega, \mathscr{G}, P)$ by given by

$$S_\gamma f(\omega) := \big\|(df_k(\omega))_{k=1}^{\infty}\big\|_{l^{\gamma,\infty}}$$

on $\{\|(df_k)_{k=1}^{\infty}\|_{l^{\gamma,\infty}} < \infty\}$ and $S_\gamma f(\omega) := 0$ otherwise, where

$$E_\gamma := \{f \in \mathscr{P}\big((\mathscr{G}_k)_{k=0}^{\infty}\big) \mid (df_k)_{k=1}^{\infty} \in l^{\gamma,\infty} \text{ a.s.}\}.$$

We are going to replace in Theorem 4.7 the quantity $\|(df_k)_{k=1}^{N}\|_{l_{\beta,\infty}^{N}}$ of (6) by $S_\gamma f$ with $\gamma \in \mathscr{S}_1^2$ and $\exp\big(1-(\lambda/c_\beta)^\alpha\big)$ of (6) by $\exp\big(1-\psi(\lambda/c)\big)$ with $\psi \in \mathscr{D}_{\mathscr{W}}^2$ ($\mathscr{S}_1^2$ and $\mathscr{D}_{\mathscr{W}}^2$ are given by Definition 4.2). It turns out that there is a complete interplay between $\mathscr{D}_{\mathscr{W}}^2$ and $\mathscr{S}_1^2$ which will be described in Proposition 4.3.

4.1. The sets $\mathscr{D}_{\mathscr{W}}^2$ and $\mathscr{S}_1^2$.

DEFINITION 4.2. (i) The set of all convex decreasing bijections $W\colon [1, \infty) \to (0, 1]$ is denoted by $\mathscr{W}$. An increasing bijection $\psi\colon [1, \infty) \to [1, \infty)$ *belongs to* $\mathscr{D}_{\mathscr{W}}^2$ provided that there is some $W \in \mathscr{W}$ such that

$$\psi(\lambda) = \frac{\lambda^2}{W(\lambda)}.$$

(ii) We let $\gamma = (\gamma_k)_{k=1}^{\infty} \in \mathscr{S}_1^2$ if $1 < \gamma_1 \leqslant \gamma_2 \leqslant \ldots < \infty$,

$$\sum_{k=1}^{\infty} \frac{1}{\gamma_k^2} = 1, \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{\gamma_k} = \infty.$$

One easily sees that $\mathscr{D}_{\mathscr{W}}^2 \subset \bar{\mathscr{D}}$. Now we describe the interplay between $\mathscr{W}$ and $\mathscr{S}_1^2$, and therefore between $\mathscr{D}_{\mathscr{W}}^2$ and $\mathscr{S}_1^2$.

PROPOSITION 4.3. (i) *Let $\gamma = (\gamma_k)_{k=1}^{\infty} \in \mathscr{S}_1^2$ and define $W_\gamma\colon [1, \infty) \to (0, 1]$ by*

$$W_\gamma(1) := 1, \quad W_\gamma\Big(1+\sum_{l=1}^{k} \frac{1}{\gamma_l}\Big) := 1-\sum_{l=1}^{k} \frac{1}{\gamma_l^2} \text{ for } k \geqslant 1,$$

and piecewise linear otherwise. Then $W_\gamma \in \mathscr{W}$.

(ii) *Let $W \in \mathscr{W}$. Then there is a unique sequence $\gamma_W = (\gamma_{k,W})_{k=1}^{\infty} \subset (1, \infty)$ such that*

$$W\Big(1+\sum_{l=1}^{k} \frac{1}{\gamma_{l,W}}\Big) = 1-\sum_{l=1}^{k} \frac{1}{\gamma_{l,W}^2} \quad \text{for } k \geqslant 1.$$

Moreover, $\gamma_W \in \mathscr{S}_1^2$.

(iii) *For* $W_1, W_2 \in \mathscr{W}$ *with* $\gamma_{W_1} = \gamma_{W_2}$ *and* $\lambda \geqslant 1$ *one has*

$$W_1(2\lambda) \leqslant W_2(\lambda) \quad \textit{and} \quad W_2(2\lambda) \leqslant W_1(\lambda).$$

Proof. (i) The function W_γ is strictly decreasing and satisfies $\lim_{\lambda\to\infty} W_\gamma(\lambda) = 0$. Moreover, W_γ is convex because of

$$W_\gamma(x_k) \leqslant (1-\theta_k)\, W_\gamma(x_{k-1}) + \theta_k\, W_\gamma(x_{k+1}) \tag{7}$$

for $k = 1, 2, \ldots, x_0 := 1, x_l := 1+\sum_{j=1}^{l}(1/\gamma_j)$ for $l \geqslant 1$, and $\theta_k \in (0, 1)$ chosen such that $x_k = (1-\theta_k)\, x_{k-1} + \theta_k\, x_{k+1}$. Consequently, $W_\gamma \in \mathscr{W}$.

(ii) Since $\beta_1 \to W(1+\beta_1)$ is convex, $\beta_1 \to 1-\beta_1^2$ concave, $\lim_{\beta_1\to\infty} W(1+\beta_1) = 0$, and $\lim_{\beta_1\to\infty}(1-\beta_1^2) = -\infty$, there is exactly one $0 < \beta_1 < 1$ such that

$$W(1+\beta_1) = 1-\beta_1^2.$$

Now assume that we have $0 < \beta_1, \ldots, \beta_k < 1$ such that

$$W\left(1+\sum_{l=1}^{k} \beta_l\right) = 1-\sum_{l=1}^{k} \beta_l^2.$$

Using the argument from the first step we find exactly one $0 < \beta_{k+1} < 1$ such that this equality is satisfied for $k+1$ instead of for k. Setting $\gamma_{k,W} := 1/\beta_k$ we have found the unique sequence γ_W. It remains to show that $\gamma_W \in \mathscr{S}_1^2$. Since W is convex, one can deduce from (7) with $\gamma_{j,W}$ instead of γ_j in the definition of the x_l and W instead of W_γ the inequality $\gamma_{k,W} \leqslant \gamma_{k+1,W}$. Finally, we verify that

$$\sum_{k=1}^{\infty} \frac{1}{\gamma_{k,W}} = \infty.$$

Assuming

$$\sigma := \sum_{k=1}^{\infty} \frac{1}{\gamma_{k,W}} < \infty$$

and observing that

$$\gamma_{k+1,W}\left[W\left(1+\sum_{l=1}^{k+1}\frac{1}{\gamma_{l,W}}\right) - W\left(1+\sum_{l=1}^{k}\frac{1}{\gamma_{l,W}}\right)\right] = -\frac{1}{\gamma_{k+1,W}} \to 0 \quad \text{as } k\to\infty$$

we obtain a contradiction to the fact that there is some $\varepsilon > 0$ such that for all $1 \leqslant a < b \leqslant \sigma$ one has

$$\frac{W(b)-W(a)}{b-a} < -\varepsilon.$$

(iii) Let $\gamma_j = \gamma_{j,W_1} = \gamma_{j,W_2}$ and $(x_l)_{l=0}^{\infty}$ be given as in the proof of (i). Assuming $\lambda \geqslant 1$ with $x_{l-1} \leqslant 2\lambda \leqslant x_l$ for some $l \geqslant 1$, we can conclude

$$W_1(2\lambda) \leqslant W_1(x_{l-1}) = W_2(x_{l-1}) \leqslant W_2(x_l/2) \leqslant W_2(\lambda). \quad ■$$

DEFINITION 4.4. The pair $(\psi, \gamma) \in \mathscr{D}_W^2 \times \mathscr{S}_1^2$ is called *related* provided that for $\psi(\lambda) = \lambda^2/W(\lambda)$ one has $\gamma_W = \gamma$, where γ_W is the sequence from Proposition 4.3 (ii).

EXAMPLE 4.5. *Let $1 < \beta < 2 < \alpha < \infty$, $1 = 1/\beta + 1/\alpha$, $u \in \mathbf{R}$, and $A, B > 0$. Define*

$$\gamma_k := \kappa k^{1/\beta} [A + \log k]^u \quad \textit{and} \quad \psi_{\alpha,u}(\lambda) := \lambda^\alpha [1 + B^{-1} \log \lambda]^{\alpha u} \textit{ for } \lambda \geqslant 1$$

with $\kappa^2 := \sum_{k=1}^\infty k^{-2/\beta} [A + \log k]^{-2u}$. Then one has the following:

(i) *$\gamma := (\gamma_k)_{k=1}^\infty \in \mathscr{S}_1^2$ and $\psi_{\alpha,u} \in \mathscr{D}_W^2$ for $A, B \geqslant c(\alpha, u) > 0$.*

(ii) *If $A, B \geqslant c(\alpha, u) > 0$ and if $\psi \in \mathscr{D}_W^2$ and γ are related, then*

$$\psi(\lambda) \leqslant \psi_{\alpha,u}(c\lambda) \quad \textit{and} \quad \psi_{\alpha,u}(\lambda) \leqslant \psi(c\lambda)$$

for $\lambda \geqslant 1$, where $c \geqslant 1$ depends on α, u, A, and B only.

Proof. (i) follows from a simple computation. For example, for $\gamma \in \mathscr{S}_1^2$ one can check the monotonicity of the function $t \to t^{1/\beta} [A + \log t]^u$ for $t \geqslant 1$.

(ii) Since for $h \in \mathscr{D}_W^2$ and $\lambda, \mu \geqslant 1$ one has $\mu^2 h(\lambda) \leqslant h(\mu\lambda)$, it suffices to show that

$$\psi(\lambda) \sim_{c'} \psi_{\alpha,u}(\lambda) \quad \text{for } \lambda \geqslant 1 \text{ and some } c' \geqslant 1,$$

or

$$0 < \inf_{\lambda \geqslant \lambda_0} \frac{\psi(\lambda)}{\psi_{\alpha,u}(\lambda)} \leqslant \sup_{\lambda \geqslant \lambda_0} \frac{\psi(\lambda)}{\psi_{\alpha,u}(\lambda)} < \infty \tag{8}$$

for some $\lambda_0 \geqslant 1$. Setting $x_k := 1 + \sum_{l=1}^k (1/\gamma_l)$ for $k \geqslant 1$ it is known that

$$x_k \sim_d \frac{k^{1/\alpha}}{[1 + \log k]^u} \quad \text{and} \quad \sum_{l=k+1}^\infty \frac{1}{\gamma_l^2} \sim_d \frac{k^{1-(2/\beta)}}{[1 + \log k]^{2u}},$$

where $d \geqslant 1$ depends on β, u, and A only. Hence

$$\psi(x_k) = \frac{x_k^2}{W(x_k)} \sim_{d^3} \frac{k^{2/\alpha}/[1 + \log k]^{2u}}{k^{1-(2/\beta)}/[1 + \log k]^{2u}} = k$$

(note that $W(x_k) = 1 - \sum_{l=1}^k (1/\gamma_l^2) = \sum_{l=k+1}^\infty (1/\gamma_l^2)$) and, for $x_k \leqslant \lambda \leqslant x_{k+1}$,

$$\frac{1}{2d^3} \frac{k+1}{\psi_{\alpha,u}(x_{k+1})} \leqslant \frac{\psi(x_k)}{\psi_{\alpha,u}(x_{k+1})} \leqslant \frac{\psi(\lambda)}{\psi_{\alpha,u}(\lambda)} \leqslant \frac{\psi(x_{k+1})}{\psi_{\alpha,u}(x_k)} \leqslant 2d^3 \frac{k}{\psi_{\alpha,u}(x_k)}.$$

Moreover, there are $k_0 \in \{1, 2, \ldots\}$ and $d' \geqslant 1$, both depending at most on α, u, B, and d such that

$$\frac{1}{d'} \leqslant \frac{k}{\psi_{\alpha,u}\left(d \dfrac{k^{1/\alpha}}{[1 + \log k]^u}\right)} \leqslant \frac{k}{\psi_{\alpha,u}(x_k)} \leqslant \frac{k}{\psi_{\alpha,u}\left(\dfrac{1}{d} \dfrac{k^{1/\alpha}}{[1 + \log k]^u}\right)} \leqslant d'$$

for $k \geqslant k_0$. Hence we have (8) and are done. ∎

Remark 4.6. Let ψ and $\psi_{\alpha,u}$ be from Example 4.5 (ii) and take symmetric i.i.d. $H_1, \ldots, H_n \in L_0(M, \nu)$ and symmetric i.i.d. $H_1^{\alpha,u}, \ldots, H_n^{\alpha,u} \in L_0(M, \nu)$ with

$$\nu(|H_i| > \lambda) = e^{1-\psi(\lambda)} \quad \text{and} \quad \nu(|H_i^{\alpha,u}| > \lambda) = \exp(1-\psi_{\alpha,u}(\lambda))$$

for $\lambda \geqslant 1$. Then it can be easily seen that

$$\Big\|\sum_{i=1}^n H_i x_i\Big\|_{L_r^X} \sim_c \Big\|\sum_{i=1}^n H_i^{\alpha,u} x_i\Big\|_{L_r^X} \quad \text{for } 1 \leqslant r < \infty,$$

where $c > 0$ is taken from Example 4.5 (ii).

4.2. The tail behaviour generated by the operators S_γ. In the sequel we use the Lebesgue measure $|\cdot|$ on $[0, 1)$ and the dyadic σ-algebras on $[0, 1)$ given by

$$\mathscr{G}_0^{\mathrm{dyad}} := \{\emptyset, [0, 1)\}, \quad \mathscr{G}_k^{\mathrm{dyad}} := \sigma\{r_1, \ldots, r_k\}, \quad \text{and} \quad \mathscr{G}_\infty^{\mathrm{dyad}} := \bigvee_{k=0}^{\infty} \mathscr{G}_k^{\mathrm{dyad}},$$

where $(r_k)_{k=1}^\infty \subset L_\infty[0, 1)$ is the sequence of Rademacher functions.

THEOREM 4.7. *There exists an absolute constant $c \geqslant 1$ such that for all related pairs $(\psi, \gamma) \in \mathscr{D}_{\mathscr{W}}^2 \times \mathscr{S}_1^2$ the following is satisfied*:

(i) *For $f \in E_\gamma \subseteq \mathscr{P}((\mathscr{G}_k)_{k=0}^\infty)$ one has*

$$P(|f_\infty| > \lambda \|S_\gamma f\|_{L_\infty(\Omega,P)}) \leqslant \exp(1-\psi(\lambda/c)) \quad \textit{for } \lambda \geqslant c.$$

(ii) *For $f_\infty := \sum_{k=1}^\infty (r_k/\gamma_k) \in L_1[0, 1)$ and $f := \big(E(f_\infty \mid \mathscr{G}_k^{\mathrm{dyad}})\big)_{k=0}^\infty$ one has*

$$S_\gamma f(s) = 1 \textit{ for } s \in [0, 1) \quad \textit{and} \quad |\{|f_\infty| > \lambda\}| \geqslant e^{1-\psi(c\lambda)} \textit{ for } \lambda \geqslant c.$$

Proof. (i) Because of $f_N \to f_\infty$ with respect to the L_1-norm and because of $\|S_\gamma f^N\|_{L_\infty} \leqslant \|S_\gamma f\|_{L_\infty}$ it is sufficient to prove the first assertion for f^N with $N > 1$ instead of for f itself. For $\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_N \geqslant 0$ and $\lambda \geqslant 1$ we first show that

$$K\big((\xi_k)_{k=1}^N, \psi(\lambda)^{1/2}; l_1^N, l_2^N\big) \leqslant 3\lambda \sup_{1 \leqslant k \leqslant N} \gamma_k \xi_k. \tag{9}$$

By an extreme point argument it is enough to consider $\gamma_k \xi_k = 1$ for $k = 1, \ldots, N$. Since the case $\sum_{k=1}^N (1/\gamma_k) \leqslant \lambda$ is trivial (here we obtain $\|(\xi_k)_{k=1}^N\|_{l_1^N} \leqslant \lambda$), we consider $\sum_{k=1}^N (1/\gamma_k) > 1$ and choose $1 \leqslant N_0 < N$ with

$$\sum_{k=1}^{N_0} \frac{1}{\gamma_k} \leqslant \lambda < \sum_{k=1}^{N_0+1} \frac{1}{\gamma_k}.$$

We obtain

$$\begin{aligned}\psi(\lambda)\left(\sum_{k=N_0+1}^{N} \frac{1}{\gamma_k^2}\right) &\leqslant \psi\left(1+\sum_{k=1}^{N_0} \frac{1}{\gamma_k}\right)\left(\sum_{k=N_0+1}^{\infty} \frac{1}{\gamma_k^2}\right) \\ &= \psi\left(1+\sum_{k=1}^{N_0} \frac{1}{\gamma_k}\right) W\left(1+\sum_{k=1}^{N_0} \frac{1}{\gamma_k}\right) = \left(1+\sum_{k=1}^{N_0} \frac{1}{\gamma_k}\right)^2 \leqslant (1+\lambda)^2 \leqslant 4\lambda^2,\end{aligned}$$

so that $\psi(\lambda)^{1/2}\left(\sum_{k=N_0+1}^{N}\xi_k^2\right)^{1/2}\leqslant 2\lambda$. Since also

$$\sum_{k=1}^{N_0}\xi_k=\sum_{k=1}^{N_0}(1/\gamma_k)\leqslant\lambda,$$

inequality (9) follows by using the decomposition

$$(\xi_k)_{k=1}^N=(\xi_1,\ldots,\xi_{k_0},0,\ldots,0)+(0,\ldots,0,\xi_{k_0+1},\ldots,\xi_N).$$

Now, according to Theorem 4.1 in [11] (see also the proof of Lemma 3.5 in [9]) one has

$$\text{(10)}\qquad \boldsymbol{P}\left(|f_N|>c\left\|K\left((df_k)_{k=1}^N,\mu;l_1^N,l_2^N\right)\right\|_{L_\infty(\Omega,\boldsymbol{P})}\right)\leqslant\exp(1-\mu^2)$$

for $\mu\geqslant 1$ and $f\in\mathscr{P}\left((\mathscr{G}_k)_{k=0}^N\right)$, where $c>0$ is an absolute constant. Combining (10) for $\mu=\psi(\lambda)^{1/2}$ with

$$K\left((df_k(\omega))_{k=1}^N,\mu;l_1^N,l_2^N\right)=K\left((df_k^*(\omega))_{k=1}^N,\mu;l_1^N,l_2^N\right),$$

where $(df_k^*(\omega))_{k=1}^N$ is a non-increasing rearrangement of $(df_k(\omega))_{k=1}^N$, and (9) yields assertion (i).

(ii) We apply [13], Lemma 4.9, and get an absolute constant $c\geqslant 1$ such that for $\lambda\geqslant c$ with

$$\sum_{k=1}^{n-1}\frac{1}{\gamma_k}\leqslant\lambda<\sum_{k=1}^{n}\frac{1}{\gamma_k}\quad\text{and}\quad n\in\{2,3,\ldots\}$$

we can conclude that

$$\begin{aligned}\left|\left\{\left|\sum_{k=1}^{\infty}\frac{r_k}{\gamma_k}\right|>\lambda\right\}\right|&\geqslant\frac{1}{2}\exp\left(-c\frac{\lambda^2}{\sum_{k=n}^{\infty}\gamma_k^{-2}}\right)=\frac{1}{2}\exp\left(-c\frac{\lambda^2}{W\left(1+\sum_{k=1}^{n-1}\gamma_k^{-1}\right)}\right)\\&\geqslant\frac{1}{2}\exp\left(-c\frac{\lambda^2}{W(1+\lambda)}\right)\geqslant\frac{1}{2}\exp\left(-c\frac{\lambda^2}{W(2\lambda)}\right)=\frac{1}{2}\exp\left(-\frac{c}{4}\psi(2\lambda)\right)\\&\geqslant\exp\left(1-\psi(2d\lambda)\right)\end{aligned}$$

with $d:=\sqrt{1+c/4+\log 2}$ (we use $d^2\psi(2\lambda)\leqslant\psi(2d\lambda)$ in the last step). ∎

5. OPERATORS WITH TAIL BEHAVIOUR $\exp(-\psi(\lambda))$, WHERE $\psi(\lambda)=\lambda^\alpha$ AND $\alpha\in[1,2)$

The situation of this section differs basically from the situation of Section 4 which can be illustrated by the following example:

EXAMPLE 5.1. *For some $N\geqslant 1$ let $F\colon \boldsymbol{R}^N\to[0,\infty)$ be Borel-measurable with*

$$F(\xi_1,\ldots,\xi_N)=F(\theta_1\xi_1,\ldots,\theta_N\xi_N)\quad\textit{for all }\theta_k\in\{-1,1\}.$$

Let E be the set of mean-zero dyadic martingales $f=(f_k)_{k=0}^N\subset L_1[0,1)$ and *assume that*

(11) $\quad A\colon E \to L_0^+[0,1),\quad$ *given by* $\quad Af(\omega) := F(df_1(\omega), \ldots, df_N(\omega)),$

satisfies for some fixed $0 < p < \infty$ *and all* $\lambda \geqslant 1$ *the tail estimate*

(12) $$|\{|f_N| > \lambda \|Af\|_{L_\infty[0,1)}\}| \leqslant 1/\lambda^p.$$

Then there is a $c_p > 0$, *depending on* p *only, such that for all* $f \in E$ *one has*

(13) $$|\{|f_N| > \lambda c_p \|Af\|_{L_\infty[0,1)}\}| \leqslant 2\exp(-\lambda^2/2).$$

Proof. For $a = (\alpha_k)_{k=1}^N \in \boldsymbol{R}^N$ and $f^{(a)} := \left(\sum_{l=1}^N \chi_{\{l \leqslant k\}} \alpha_l r_l\right)_{k=0}^N \in E$ we obtain by Khintchine–Kahane's inequality for the Rademacher functions

$$\|a\|_{l_2^N} \leqslant c_p \|f_N^{(a)}\|_{L_{p,\infty}[0,1)} \leqslant c_p \|Af^{(a)}\|_{L_\infty[0,1)} = c_p F(\alpha_1, \ldots, \alpha_N).$$

But now Azuma's inequality (see [4], [16], [10]) implies for all $f \in E$ and all $\lambda > 0$

$$|\{|f_N| > \lambda c_p \|Af\|_{L_\infty[0,1)}\}| \leqslant \left|\left\{|f_N| > \lambda \left\|\|(df_k)_{k=1}^N\|_{l_2^N}\right\|_{L_\infty[0,1)}\right\}\right| \leqslant 2\exp(-\lambda^2/2). \quad \blacksquare$$

Consequently, the 'mild' tail behaviour of (12) already implies the sub-Gaussian tail behaviour of (13). This means, in order to find operators which describe the tail behaviour $\exp(-\psi(\lambda))$, where $\psi(\lambda) = \lambda^\alpha$ and $\alpha < 2$, in a proper way we have to look for operators which are not generated as in (11).

Definition 5.2. For $\theta = (\theta_k)_{k=1}^\infty$ with $\theta_k \in \{-1, 1\}$ and $2 < \varrho \leqslant \infty$ we let

$$S_{\varrho,\theta}\colon \mathscr{M}\left((\mathscr{G}_k)_{k=0}^\infty\right) \supseteq E_{\varrho,\theta} \to L_0^+(\Omega, \mathscr{G}, \boldsymbol{P})$$

be given by

$$(S_{\varrho,\theta} f)(\omega) := \sup_{N \geqslant 1} \left|\sum_{k=1}^N \theta_k k^{1/\varrho} df_k(\omega)\right|$$

on $\{\sup_{N \geqslant 1} |\sum_{k=1}^N \theta_k k^{1/\varrho} df_k| < \infty\}$ and $(S_{\varrho,\theta} f)(\omega) := 0$ otherwise, where

$$E_{\varrho,\theta} := \left\{f \in \mathscr{M}\left((\mathscr{G}_k)_{k=0}^\infty\right) \mid \sup_{N \geqslant 1} \left|\sum_{k=1}^N \theta_k k^{1/\varrho} df_k\right| < \infty \text{ a.s.}\right\}.$$

Theorem 5.3. *Let* $1 < \alpha < \sigma < 2 < \varrho < \beta < \infty$ *with* $1 = 1/\alpha + 1/\beta$ *and* $1 = 1/\sigma + 1/\varrho$ *or* $1 = \alpha = \sigma$ *and* $\varrho = \beta = \infty$.

(i) *There is some* $c > 0$ *depending on* ϱ *and* β *only, such that for all* $\theta \in \{-1, 1\}^N$ *and* $f \in E_{\varrho,\theta}$ *one has*

$$\boldsymbol{P}(|f_\infty| > \lambda \|S_{\varrho,\theta} f\|_\infty) \leqslant \exp\left(1 - (\lambda/c)^\alpha\right) \quad \text{for } \lambda \geqslant c.$$

(ii) *For* $f := \left(\boldsymbol{E}(f_\infty \mid \mathscr{G}_k^{\text{dyad}})\right)_{k=0}^\infty$ *and* $\theta_k := (-1)^k$ *with*

$$f_\infty := r_1 + \sum_{k=2}^\infty k^{-1/\varrho} r_k \prod_{u=1}^{k-1} \left[\frac{1+r_u}{2}\right] \in L_1[0,1)$$

one has

$$\|S_{\varrho,\theta}f\|_{L_\infty} \leqslant 2 \quad \textit{and} \quad |\{|f_\infty| > \lambda\}| \geqslant \tfrac{1}{8}\exp(-\lambda^\sigma) \textit{ for } \lambda \geqslant 0.$$

For the proof of Theorem 5.3 the following lemmas are used.

LEMMA 5.4. *There is some $c > 0$ such that for $2 \leqslant r < \infty$ and $f \in \mathcal{M}((\mathcal{G}_k)_{k=0}^N)$ one has*

$$\big\|\|(df_k)_{k=1}^N\|_{l_2^N}\big\|_{L_r} \leqslant c\sqrt{r}\,\|f_N\|_{L_r} \quad \textit{and} \quad \big\|\|(df_k)_{k=1}^N\|_{v_\infty^N(\theta)}\big\|_{L_r} \leqslant cr\,\|f_N\|_{L_r},$$

where $\|(\xi_k)_{k=1}^N\|_{v_\infty^N(\theta)} := \sup_{1\leqslant k\leqslant N}|\theta_1\xi_1+\ldots+\theta_k\xi_k|$ and $\theta = (\theta_k)_{k=1}^N \in \{-1, 1\}^N$.

Proof. One has to use Theorem II.1.1 of [6], Theorem 3.1 of [3], and Doob's maximal inequality. ∎

LEMMA 5.5. *For $0 < \eta < 1$, $\theta = (\theta_k)_{k=1}^N \in \{-1, 1\}^N$, and $x = (\xi_k)_{k=1}^N \in \boldsymbol{R}^N$ one has*

$$\sup_{1\leqslant k\leqslant N} k^{-(1-\eta)/2}|\theta_1\xi_1+\ldots+\theta_k\xi_k| \leqslant \|(\xi_k)_{k=1}^N\|_{(l_2^N, v_\infty^N(\theta))_{\eta,\infty}}.$$

Proof. We fix $1 \leqslant k_0 \leqslant N$ and set $t_0 := k_0^{-1/2}$. For $x = y+z$ we obtain

$$\begin{aligned} k_0^{-(1-\eta)/2}|\theta_1\xi_1+\ldots+\theta_{k_0}\xi_{k_0}| &\leqslant k_0^{-(1-\eta)/2}k_0^{1/2}\|y\|_{l_2^N}+k_0^{-(1-\eta)/2}\|z\|_{v_\infty^N(\theta)} \\ &= t_0^{-\eta}[\|y\|_{l_2^N}+t_0\|z\|_{v_\infty^N(\theta)}]. \end{aligned}$$

Hence $k_0^{-(1-\eta)/2}|\theta_1\xi_1+\ldots+\theta_{k_0}\xi_{k_0}| \leqslant t_0^{-\eta}K(x, t_0; l_2^N, v_\infty^N(\theta))$. ∎

LEMMA 5.6. *For $0 < \delta < \varepsilon < 1$, $N \geqslant 1$, and $(\xi_k)_{k=1}^N \in \boldsymbol{R}^N$ one has*

$$\sup_{k=1,\ldots,N}|1^{-\varepsilon}\xi_1+\ldots+k^{-\varepsilon}\xi_k| \leqslant c \sup_{k=1,\ldots,N} k^{-\delta}|\xi_1+\ldots+\xi_k|,$$

where $c > 0$ depends on δ and ε only.

Proof. We let $\eta_0 := 0$ and $\eta_k := \xi_1+\ldots+\xi_k$ for $1 \leqslant k \leqslant N$. Hence we have to show that

$$\sup_{k=1,\ldots,N}|1^{-\varepsilon}(\eta_1-\eta_0)+\ldots+k^{-\varepsilon}(\eta_k-\eta_{k-1})| \leqslant c \sup_{k=1,\ldots,N} k^{-\delta}|\eta_k|.$$

This can be rewritten as

$$\sup_{k=1,\ldots,N}|\eta_k k^{-\varepsilon}+\eta_{k-1}[(k-1)^{-\varepsilon}-k^{-\varepsilon}]+\ldots+\eta_1[1^{-\varepsilon}-2^{-\varepsilon}]| \leqslant c \sup_{k=1,\ldots,N} k^{-\delta}|\eta_k|.$$

To check this inequality it remains to consider $\eta_k = k^\delta$ so that we are done. ∎

The last lemma we need is known and completely standard.

LEMMA 5.7. *Let $1 \leqslant \alpha < \infty$ and $f \in L_0^+(\Omega, \boldsymbol{P})$ such that for all $2 \leqslant r < \infty$ one has $\|f\|_{L_r} \leqslant \sqrt[\alpha]{r}$. Then there is some constant $c > 0$, depending on α only,* *such that*

$$P(f > \lambda) \leqslant \exp\left(1-(\lambda/c)^{\alpha}\right) \quad \textit{for } \lambda \geqslant c.$$

Proof of Theorem 5.3. (i) For the same reason as in the proof of Theorem 4.7 (i) we can replace f by f^N for $N \in \{1, 2, \ldots\}$. The case $1 = \alpha = \sigma$ and $\varrho = \beta = \infty$ follows directly from the second inequality of Lemma 5.4 and Lemma 5.7. To consider the case $1 < \alpha < \sigma < 2 < \varrho < \beta < \infty$ we let $0 < \eta < 1$ with $(1-\eta)/2 = 1/\beta$. Then, for $2 \leqslant r < \infty$ from Lemmas 5.6, 5.5, and 5.4 one gets, for a martingale $(M_k)_{k=0}^N \subset L_1(\Omega, P)$ with $M_0 = 0$,

$$\begin{aligned}
&\|1^{-1/\varrho}\theta_1\, dM_1 + \ldots + N^{-1/\varrho}\theta_N\, dM_N\|_{L_r} \\
&\leqslant c_{(5.6)} \Big\| \sup_{k=1,\ldots,N} k^{-1/\beta} |\theta_1\, dM_1 + \ldots + \theta_k\, dM_k| \Big\|_{L_r} \\
&= c_{(5.6)} \Big\| \sup_{k=1,\ldots,N} k^{-(1-\eta)/2} |\theta_1\, dM_1 + \ldots + \theta_k\, dM_k| \Big\|_{L_r} \\
&\leqslant c_{(5.6)} \Big\| \|(dM_k)_{k=1}^N\|_{(l_2^N, v_\infty^N(\theta))_{\eta,\infty}} \Big\|_{L_r} \leqslant c_{(5.6)} \Big\| \|(dM_k)_{k=1}^N\|_{l_2^N}^{1-\eta} \|(dM_k)_{k=1}^N\|_{v_\infty^N(\theta)}^{\eta} \Big\|_{L_r} \\
&\leqslant c_{(5.6)} \Big\| \|(dM_k)_{k=1}^N\|_{l_2^N} \Big\|_{L_r}^{1-\eta} \Big\| \|(dM_k)_{k=1}^N\|_{v_\infty^N(\theta)} \Big\|_{L_r}^{\eta} \\
&\leqslant c_{(5.6)}\, c_{(5.4)}\, r^{(1-\eta)/2+\eta} \|M_N\|_{L_r} = c_{(5.6)}\, c_{(5.4)} \sqrt[\alpha]{r}\, \|M_N\|_{L_r}.
\end{aligned}$$

Consequently, $\|f_N\|_{L_r} \leqslant c_{(5.6)}\, c_{(5.4)} \sqrt[\alpha]{r}\, \|S_{\varrho,\theta} f^N\|_{L_r}$ for $2 \leqslant r < \infty$ so that we can use Lemma 5.7 and finish the proof of assertion (i).

(ii) One observes for $k \in \{1, 2, \ldots\}$ and $(k-1)^{1/\sigma} \leqslant \lambda < k^{1/\sigma}$ that

$$\begin{aligned}
\left|\{|f_\infty| > \lambda\}\right| &\geqslant \left|\{|f_\infty| > k^{1/\sigma}\}\right| \geqslant |\{r_1 = \ldots = r_{k+1} = 1,\ r_{k+2} = -1\}| \\
&= 1/2^{k+2} \geqslant \tfrac{1}{8}\exp(-\lambda^{\sigma}). \quad \blacksquare
\end{aligned}$$

PROBLEM 5.8. Is there a way to remove the gap between assertions (i) and (ii) of Theorem 5.3?

6. SPECIAL CASES OF THEOREM 3.4

In this section we replace in Theorem 3.4 the abstract operators A_i by the concrete operators examined in Sections 4 and 5.

DEFINITION 6.1. Let $(\mathscr{F}_t)_{t\in T}$ be a filtration on $[\Omega, \mathscr{F}, P]$ satisfying (C1), (C2), and (C3) such that for $t_{i,k} := i - 1/(k+1)$ with $i = 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$ one has

$$\mathscr{F}_s = \mathscr{F}_t \text{ whenever } t_{i,k} \leqslant s < t < t_{i,k+1} \quad \text{and} \quad \mathscr{F}_i = \bigvee_{0 \leqslant t < i} \mathscr{F}_t. \tag{14}$$

(i) We let $\mathscr{P}\left((\mathscr{F}_t)_{t\in T}\right)$ be the set of $f \in \mathscr{M}\left((\mathscr{F}_t)_{t\in T}\right)$ such that

(a) $|f_{t_{i,k}}-f_{t_{i,k-1}}|$ is $\mathscr{F}_{t_{i,k-1}}$-measurable for $i, k \geqslant 1$,

(b) $f_s(\omega) = f_t(\omega)$ for $\omega \in \Omega$ and $t_{i,k} \leqslant s < t < t_{i,k+1}$ with $i \geqslant 1$ and $k \geqslant 0$.

(ii) Let $\Gamma := (\gamma^{(i)})_{i=1}^{\infty} \subseteq \mathscr{S}_1^2$, $\Theta := (\theta^{(i)})_{i=1}^{\infty} = ((\theta_1^{(i)}, \theta_2^{(i)}, \ldots))_{i=1}^{\infty} \subset \{-1, 1\}^{N}$, and $2 < \varrho \leqslant \infty$. Given $f \in \mathscr{P}((\mathscr{F}_t)_{t\in T})$, we define

$$\mathscr{S}_{\gamma^i} f(\omega) := \|(f_{t_{i,k}}(\omega) - f_{t_{i,k-1}}(\omega))_{k=1}^{\infty}\|_{l^{\gamma^{(i)},\infty}},$$

$$\mathscr{S}_{\varrho,\theta^{(i)}} f(\omega) := \sup_{N\geqslant 1} \Big|\sum_{k=1}^{N} \theta_k^{(i)} k^{1/\varrho} [f_{t_{i,k}}(\omega) - f_{t_{i,k-1}}(\omega)]\Big|,$$

$$\begin{aligned}{}^{*}\mathscr{S}_{\varrho,\theta^{(i)}} f(\omega) := \sup_{N\geqslant 1} \Big[\Big|\sum_{k=1}^{N} \theta_k^{(i)} k^{1/\varrho} [f_{t_{i,k}}(\omega) - f_{t_{i,k-1}}(\omega)]\Big| \\ +(N+1)^{1/\varrho} |f_{t_{i,N+1}}(\omega) - f_{t_{i,N}}(\omega)|\Big],\end{aligned}$$

where we set these operators zero on those ω for which the corresponding right-hand sides are infinite, and the ranges of definition

$$\mathscr{E}_{\Gamma} := \{f \in \mathscr{P}((\mathscr{F}_t)_{t\in T}) \mid \sup_{i\geqslant 1} \|(f_{t_{i,k}} - f_{t_{i,k-1}})_{k=1}^{\infty}\|_{l^{\gamma^{(i)},\infty}} < \infty \text{ a.s.}\},$$

$$\mathscr{E}_{\varrho,\Theta} := \{f \in \mathscr{P}((\mathscr{F}_t)_{t\in T}) \mid \sup_{i,N\geqslant 1} \Big|\sum_{k=1}^{N} \theta_k^{(i)} k^{1/\varrho} [f_{t_{i,k}} - f_{t_{i,k-1}}]\Big| < \infty \text{ a.s.}\}.$$

Remark 6.2. In the definition above condition (C3) automatically follows from assumption (14).

Lemma 6.3. (i) $\mathscr{S}_{\varrho,\theta^{(i)}} f(\omega) \leqslant {}^{*}\mathscr{S}_{\varrho,\theta^{(i)}} f(\omega) \leqslant 3\mathscr{S}_{\varrho,\theta^{(i)}} f(\omega)$.

(ii) *The operators*

$$\mathscr{S}_{\gamma^{(i)}}: \mathscr{E}_{\Gamma} \to L_0^{+}(\Omega, \mathscr{F}, P) \quad \textit{and} \quad {}^{*}\mathscr{S}_{\varrho,\theta^{(i)}}: \mathscr{E}_{\varrho,\Theta} \to L_0^{+}(\Omega, \mathscr{F}, P)$$

satisfy property (S) *with constant* 1.

Proof. (i) and properties (S1) and (S2) of (ii) are standard. For example, to check that $|{}^{\sigma}f^{\tau}_{t_{i,k}} - {}^{\sigma}f^{\tau}_{t_{i,k-1}}|$ is $\mathscr{F}_{t_{i,k-1}}$-measurable for stopping times $\sigma \leqslant \tau$ and $f \in \mathscr{P}((\mathscr{F}_t)_{t\in T})$ one can use

$${}^{\sigma}f^{\tau}_{t_{i,k}} - {}^{\sigma}f^{\tau}_{t_{i,k-1}} = \chi_{\{\sigma < t_{i,k} \leqslant \tau\}} [f_{t_{i,k}} - f_{t_{i,k-1}}].$$

To show (S3) of (ii) we fix $i \geqslant 1$, $0 < \lambda < \infty$, and f from the corresponding range of definition. Then we can use the stopping time

$$\tilde{\varrho}(\omega) := \inf\{t_{i,k} \mid k \geqslant 0,\ \mathscr{S}_{\gamma^{(i)}}(f^{t_{i,k+1}})(\omega) > \lambda\}$$

in the first case and

$$\tilde{\varrho}(\omega) := \inf\{t_{i,k} \mid k \geqslant 0,\ ({}^{*}\mathscr{S}_{\varrho,\theta^{(i)}})(f^{t_{i,k+1}})(\omega) > \lambda\}$$

in the second case, where $\inf\emptyset := \infty$ (note that $\mathscr{S}_{\gamma^{(i)}}(f^{t_{i,k+1}})$ and $({}^{*}\mathscr{S}_{\varrho,\theta^{(i)}})(f^{t_{i,k+1}})$ are $\mathscr{F}_{t_{i,k}}$-measurable). ∎

COROLLARY 6.4. *For all* $\psi \in \mathscr{D}$ *there is a constant* $c > 0$ *such that for* $(\psi_i)_{i=1}^{\infty} \subset \mathscr{D}_{\Psi}^2$, $\Gamma = (\gamma^{(i)})_{i=1}^{\infty} \subset \mathscr{S}_1^2$, *where* ψ_i *and* $\gamma^{(i)}$ *are related*, $f \in \mathscr{E}_\Gamma$, $1 \leqslant p < \infty$, *and for all elements* $x_1, \ldots, x_n$ *of a Banach space* X *one has*

$$\Big\|\sup_{t\in T} \Big\| \sum_{i=1}^{n} [{}^{i-1}f_t^i]\, x_i\Big\|_X\Big\|_{L_p} \leqslant c\bar{\psi}^{-1}(p) \| \sup_{1\leqslant i\leqslant n} \mathscr{S}_{\gamma^{(i)}}({}^{i-1}f^i)\|_{L_p} \Big\| \sum_{i=1}^{n} H_i x_i\Big\|_{\psi},$$

where $H_1, \ldots, H_n \in L_1(M, \nu)$ *are independent and symmetric and satisfy*

$$(15) \qquad \nu(|H_i| > \lambda) = \exp(1-\psi_i(\lambda)) \quad \textit{for } \lambda \geqslant 1.$$

COROLLARY 6.5. *Let* $1 < \alpha < 2 < \varrho < \beta < \infty$ *with* $1 = 1/\alpha + 1/\beta$ *or* $\alpha = 1$ *and* $\varrho = \beta = \infty$. *Then there is a constant* $c > 0$, *depending on* ϱ *and* β *only, such that for all* $\Theta = (\theta^{(i)})_{i=1}^{\infty} \subset \{-1, 1\}^N$, $f \in \mathscr{E}_{\varrho,\Theta}$, $1 \leqslant p < \infty$, *and for all elements* $x_1, \ldots, x_n$ *of a Banach space* X *one has*

$$\Big\|\sup_{t\in T} \Big\| \sum_{i=1}^{n} [{}^{i-1}f_t^i]\, x_i\Big\|_X\Big\|_{L_p} \leqslant cp \| \sup_{1\leqslant i\leqslant n} \mathscr{S}_{\varrho,\theta^{(i)}}({}^{i-1}f^i)\|_{L_p} \Big\| \sum_{i=1}^{n} H_{i,\alpha} x_i\Big\|_{L_1^X},$$

where $H_{1,\alpha}, \ldots, H_{n,\alpha} \in L_1(M, \nu)$ *are independent and symmetric and satisfy*

$$(16) \qquad \nu(|H_{i,\alpha}| > \lambda) = \exp(1-\lambda^\alpha) \quad \textit{for } \lambda \geqslant 1.$$

Proof of Corollaries 6.4 and 6.5. Fix $i \in \{1, \ldots, n\}$, a stopping time τ, $B \in \mathscr{F}_\tau$ of positive measure, and $S \in \{\mathscr{S}_{\gamma^{(i)}}, {}^*\mathscr{S}_{\varrho,\theta^{(i)}}\}$. Let $c \geqslant 1$ be the constant from Theorem 4.7 if $S = \mathscr{S}_{\gamma^{(i)}}$, and from Theorem 5.3 (i) if $S = {}^*\mathscr{S}_{\varrho,\theta^{(i)}}$. First we observe that $\mathscr{F}_i = \bigvee_{0\leqslant t<i} \mathscr{F}_t$ implies that

$$(17) \qquad \lim_{t\to i, t<i} f_t = f_i \text{ a.s. and in the } L_1\text{-norm.}$$

Moreover,

$$(18) \qquad \begin{aligned} \|S({}^{t_{i,k}}f^i)\|_{L_\infty(\Omega,P)} &\leqslant \|S({}^{i-1}f^{t_{i,k}})\|_{L_\infty(\Omega,P)} + \|S({}^{i-1}f^i)\|_{L_\infty(\Omega,P)} \\ &\leqslant 2\|S({}^{i-1}f^i)\|_{L_\infty(\Omega,P)} \end{aligned}$$

and

$$\|({}^{i-1}f^i)_\tau - ({}^{i-1}f^i)_{\tau-}\|_{L_\infty(\Omega,P)} \leqslant \|S({}^{i-1}f^i)\|_{L_\infty(\Omega,P)},$$

where we also use (17). Then, for $\lambda \geqslant 1$,

$$\begin{aligned} &P_B(|{}^{i-1}f_\infty^i - {}^{i-1}f_{\tau-}^i| > \lambda\, 3c\, \|S({}^{i-1}f^i)\|_{L_\infty(\Omega,P)}) \\ &\leqslant P_B(|{}^{i-1}f_\infty^i - {}^{i-1}f_\tau^i| > \lambda\, 2c\, \|S({}^{i-1}f^i)\|_{L_\infty(\Omega,P)}) \\ &= P_B(|f_i - f_{[(i-1)\vee\tau]\wedge i}| > \lambda\, 2c\, \|S({}^{i-1}f^i)\|_{L_\infty(\Omega,P)}) \\ &= P_B(\tau < i-1)\, P_{B\cap\{\tau<i-1\}}(|f_i - f_{i-1}| > \lambda\, 2c\, \|S({}^{i-1}f^i)\|_{L_\infty(\Omega,P)}) \\ &\quad + \sum_{k=0}^{\infty} P_B(t_{i,k} \leqslant \tau < t_{i,k+1})\, P_{B\cap\{t_{i,k}\leqslant\tau<t_{i,k+1}\}}(|f_i - f_{t_{i,k}}| > \lambda\, 2c\|S({}^{i-1}f^i)\|_{L_\infty(\Omega,P)}). \end{aligned}$$

Because of

$$B\cap\{\tau < i-1\}\in\mathcal{F}_{i-1} \quad \text{and} \quad B\cap\{t_{i,k}\leqslant\tau < t_{i,k+1}\}\in\mathcal{F}_{t_{i,k}}$$

and because of (17) and (18) we can apply Theorems 4.7 and 5.3 (to ${}^{i-1}f^i$ restricted to $B\cap\{\tau < i-1\}\in\mathcal{F}_i$ and ${}^{t_{i,k}}f^i$ restricted to $B\cap\{t_{i,k}\leqslant\tau < t_{i,k+1}\}\in\mathcal{F}_{t_{i,k}}$) to derive

$$P_B\left(|{}^{i-1}f^i_\infty - {}^{i-1}f^i_{\tau-}| > \lambda\, 3\, c_{(4.7)}\, \|\mathcal{S}_{\gamma^{(i)}}({}^{i-1}f^i)\|_{L_\infty(\Omega,P)}\right)\leqslant \exp\left(1-\psi_i(\lambda)\right)$$

and

$$P_B\left(|{}^{i-1}f^i_\infty - {}^{i-1}f^i_{\tau-}| > \lambda\, 3\, c_{(5.3)}\, \|\mathcal{S}_{\varrho,\theta^{(i)}}({}^{i-1}f^i)\|_{L_\infty(\Omega,P)}\right)\leqslant \exp\left(1-\lambda^\alpha\right).$$

Consequently, by Theorem 2.4,

$$\|{}^{i-1}f^i\|_{BMO^*_{\psi_i}}\leqslant 4\sqrt{3}\,\|{}^{i-1}f^i\|_{BMO_{\psi_i}}\leqslant 12\sqrt{3}\,c_{(4.7)}\,\|\mathcal{S}_{\gamma^{(i)}}({}^{i-1}f^i)\|_{L_\infty(\Omega,P)}$$

and

$$\|{}^{i-1}f^i\|_{BMO^*_{\psi^{(\alpha)}}}\leqslant 12\,\|{}^{i-1}f^i\|_{BMO_{\psi^{(\alpha)}}}\leqslant 36\,c_{(5.3)}\,\|{}^*\mathcal{S}_{\varrho,\theta^{(i)}}({}^{i-1}f^i)\|_{L_\infty(\Omega,P)}$$

with $\psi^{(\alpha)}(\lambda):=\lambda^\alpha$, where we used $\psi_i^{-1}(3)\leqslant\sqrt{3}$ and $(\psi^{(\alpha)})^{-1}(3)\leqslant 3$. Now we can apply Lemma 6.3 (ii), Theorem 3.4 with $\tau_i := i$ (observe that $\tilde{H}_i\leqslant 4\sqrt{3}\,H_i$ in the case of Corollary 6.4 and $\tilde{H}_{i,\alpha}\leqslant 12\,H_{i,\alpha}$ in the case of Corollary 6.5), and Lemma 6.3 (i). ∎

In particular, we obtain statement (b) from the table in the Introduction.

COROLLARY 6.6. *Let $1<\beta<2<\alpha<\infty$ with $1=1/\alpha+1/\beta$ and $\psi\in\mathcal{D}$. Then there is a constant $c>0$ depending on α and ψ only, such that for $f\in\mathcal{P}\left((\mathcal{G}_k)_{k=0}^N\right)$, $1\leqslant p<\infty$, elements $x_1,\ldots,x_n$ of a Banach space X, and stopping times $0=\tau_0\leqslant\tau_1\leqslant\ldots\leqslant\tau_n=N$ one has*

$$\Big\|\sup_{0\leqslant k\leqslant N}\Big\|\sum_{i=1}^n\Big[\sum_{l=\tau_{i-1}+1}^{\tau_i\wedge k} df_l\Big]x_i\Big\|_X\Big\|_{L_p}\leqslant c\bar{\psi}^{-1}(p)\,\|\sup_{1\leqslant i\leqslant n} S_\gamma({}^{\tau_{i-1}}f^{\tau_i})\|_{L_p}\,\Big\|\sum_{i=1}^n g_{\alpha,i}x_i\Big\|_\psi,$$

where $\gamma:=(k^{1/\beta})_{k=1}^\infty$, S_γ is defined as in Definition 4.1, and

$${}^{\tau_{i-1}}f^{\tau_i} := (\chi_{\{\tau_{i-1}<k\leqslant\tau_i\}}\,df_k)_{k=0}^N.$$

Proof. First we complete the filtration $(\mathcal{G}_k)_{k=0}^N$ and apply Lemma B.1 to come formally in the continuous time setting. Then we apply Corollary 6.4, Example 4.5, Remark 4.6, and

$$\exp\left(-(c_\alpha\lambda)^\alpha\right)\leqslant\nu(|g_{\alpha,i}|>\lambda)\quad\text{for } \lambda\geqslant\lambda_\alpha>0 \text{ and some } c_\alpha>0. \ ∎$$

In the same way there is an approach to statement (c), mentioned in the table of the Introduction, where we have to replace Theorem 4.7 by [9] (Lemma 3.5, $p=q=2$). Now we check that Corollary 6.4 is optimal whereas Corollary 6.5 is 'nearly' optimal. This is done in the following way.

Using $(\mathscr{G}_k^{\mathrm{dyad}})_{k=0}^{\infty}$ introduced in Section 4.2 we equip

$$\Omega^{\mathrm{dyad}} := \mathop{\times}_{i=1}^{\infty} [0, 1) \quad \text{with } P^{\mathrm{dyad}} := \mathop{\times}_{i=1}^{\infty} |\cdot|,$$

the product measure of the Lebesgue measure, and with the filtration

$$\mathscr{F}_s^{\mathrm{dyad},0} := \mathscr{G}_k^{\mathrm{dyad}} \times \Big[\mathop{\times}_{j=2}^{\infty} \mathscr{G}_0^{\mathrm{dyad}}\Big] \quad \text{for } t_{1,k} \leqslant s < t_{1,k+1},$$

$$\mathscr{F}_s^{\mathrm{dyad},0} := \Big[\mathop{\times}_{j=1}^{i-1} \mathscr{G}_\infty^{\mathrm{dyad}}\Big] \times \mathscr{G}_k^{\mathrm{dyad}} \times \Big[\mathop{\times}_{j=i+1}^{\infty} \mathscr{G}_0^{\mathrm{dyad}}\Big] \quad \text{for } t_{i,k} \leqslant s < t_{i,k+1}$$

and $i \geqslant 2$, and $\mathscr{F}_\infty^{\mathrm{dyad},0} := \times_{j=1}^{\infty} \mathscr{G}_\infty^{\mathrm{dyad}}$. This filtration can be completed to $(\mathscr{F}_t^{\mathrm{dyad}})_{0 \leqslant t \leqslant \infty}$ so that conditions (C1), (C2), (C3), and (14) are satisfied. The corresponding sets $\mathscr{E}_\Gamma$ and $\mathscr{E}_{\varrho,\Theta}$ from Definition 6.1 are denoted by $\mathscr{E}_\Gamma^{\mathrm{dyad}}$ and $\mathscr{E}_{\varrho,\Theta}^{\mathrm{dyad}}$, respectively.

Now Theorem 4.7 implies

PROPOSITION 6.7. *Let $(\psi_i)_{i=1}^{\infty} \subset \mathscr{D}_{\mathscr{W}}^2$ and $\Gamma = (\gamma^{(i)})_{i=1}^{\infty} \subset \mathscr{S}_1^2$ be such that ψ_i and $\gamma^{(i)}$ are related. For $i \geqslant 1$, $k \geqslant 0$, and $s = (s_i)_{i=1}^{\infty} \in \Omega^{\mathrm{dyad}}$ we let*

$$\Delta_i := \sum_{j=1}^{\infty} \frac{r_j}{\gamma_j^{(i)}} \in L_1[0, 1) \quad \text{and} \quad f_{t_{i,k}}(s) := \sum_{j=1}^{i-1} \Delta_j(s_j) + \sum_{l=1}^{k} \frac{r_l(s_i)}{\gamma_l^{(i)}},$$

where sums of type $\sum_{m=1}^{0}$ are treated as zero, and complete this to a càdlàg process $f = (f_t)_{t \in T}$ such that $f^n \in \mathscr{E}_\Gamma^{\mathrm{dyad}}$ for $n = 1, 2, \ldots$ Then the following holds:

(i) *$\mathscr{S}_{\gamma^{(i)}}({}^{i-1}f^i)(s) = 1$ for all $s \in \Omega^{\mathrm{dyad}}$.*

(ii) *$(f_i - f_{i-1})_{i=1}^{\infty}$ is a sequence of independent and symmetric random variables.*

(iii) *If $\lambda \geqslant 0$ and $\kappa := \sup_{i \geqslant 1} \exp\big(\psi_i(c_{(4.7)}^2) - 1\big) < \infty$, where $c_{(4.7)} \geqslant 1$ is the constant from Theorem 4.7, then one has*

$$P^{\mathrm{dyad}}\left(|f_i - f_{i-1}| > \frac{\lambda}{c_{(4.7)}}\right) \geqslant \frac{1}{\kappa} \exp\big(1 - \psi_i(\lambda \vee 1)\big).$$

In the same way Theorem 5.3 gives

PROPOSITION 6.8. *Assume that $1 \leqslant \sigma < 2 < \varrho \leqslant \infty$ with $1 = 1/\varrho + 1/\sigma$ and $\theta^{(i)} := \big((-1)^k\big)_{k=1}^{\infty}$. For $i \geqslant 1$, $k \geqslant 3$, and $s = (s_i)_{i=1}^{\infty} \in \Omega^{\mathrm{dyad}}$ we let*

$$\Delta := r_1 \left[2^{-1/\varrho} r_2 + \sum_{j=3}^{\infty} j^{-1/\varrho} r_j \prod_{u=2}^{j-1} \left[\frac{1 + r_u}{2} \right] \right] \in L_1[0, 1),$$

$$f_{t_{i,0}}(s) = f_{t_{i,1}}(s) := \sum_{j=1}^{i-1} \Delta(s_j), \quad f_{t_{i,2}}(s) := \sum_{j=1}^{i-1} \Delta(s_j) + r_1(s_i) 2^{-1/\varrho} r_2(s_i),$$

and

$$f_{t_{i,k}}(s) := \sum_{j=1}^{i-1} \Delta(s_j) + r_1(s_i)\left[2^{-1/\varrho} r_2(s_i) + \sum_{l=3}^{k} l^{-1/\varrho} r_l(s_i) \prod_{u=2}^{l-1} \left[\frac{1+r_u(s_i)}{2}\right]\right],$$

where the sums $\sum_{j=1}^{0}$ *are treated as zero, and complete this to a càdlàg process* $f = (f_t)_{t\in T}$ *such that* $f^n \in \mathscr{E}_{\varrho,\Theta}^{\mathrm{dyad}}$ *for* $n = 1, 2, \ldots$ *Then the following holds*:

(i) $\mathscr{S}_{\varrho,\theta(i)}({}^{i-1}f^i)(s) \leqslant 2$ *for all* $s \in \Omega^{\mathrm{dyad}}$.

(ii) $(f_i - f_{i-1})_{i=1}^{\infty}$ *is a sequence of independent and symmetric random variables.*

(iii) *For* $\lambda \geqslant 0$ *and* $1 = 1/\sigma + 1/\varrho$ *one has*

$$P^{\mathrm{dyad}}(|f_i - f_{i-1}| + 1 > \lambda) \geqslant \frac{1}{8e}\exp(1-\lambda^\sigma).$$

Let $(\Delta_i)_{i=1}^n \subset L_0(\Omega, P)$ and $(H_i)_{i=1}^n \subset L_0(M, \nu)$ be sequences of independent and symmetric random variables such that

$$\nu(|H_i| > \lambda) \leqslant \kappa P(|\Delta_i| > \lambda)$$

for some $\kappa \geqslant 1$ and all $\lambda \geqslant 0$. Then Lemma 4.6 of [13] gives

$$\Big\|\sum_{i=1}^n H_i x_i\Big\|_{L_p^X} \leqslant \kappa \Big\|\sum_{i=1}^n \Delta_i x_i\Big\|_{L_p^X}$$

for all Banach spaces X with elements $x_1, \ldots, x_n$ and all $1 \leqslant p < \infty$. Hence, for the process f considered in Proposition 6.7 and H_i as in (15) we obtain

$$\text{(19)}\quad \|\sup_{i\geqslant 1} \mathscr{S}_{\gamma(i)}({}^{i-1}f^i)\|_{L_\infty} \leqslant 1 \quad\text{and}\quad \Big\|\sum_{i=1}^n H_i x_i\Big\|_{L_p^X} \leqslant c_{(4.7)}\kappa \Big\|\sum_{i=1}^n [f_i - f_{i-1}] x_i\Big\|_{L_p^X}$$

for all $1 \leqslant p < \infty$. Analogously, for the process f considered in Proposition 6.8 and $H_{i,\sigma}$ as in (16) we obtain

$$\|\sup_{i\geqslant 1} \mathscr{S}_{\varrho,\theta(i)}({}^{i-1}f^i)\|_{L_\infty} \leqslant 2 \qquad\text{and}\qquad \Big\|\sum_{i=1}^n H_{i,\sigma} x_i\Big\|_{L_p^X} \leqslant c_\sigma \Big\|\sum_{i=1}^n [f_i - f_{i-1}] x_i\Big\|_{L_p^X}.$$

In this way we obtain a converse statement to Corollary 6.4 and nearly a converse statement to Corollary 6.5 ($H_{i,\alpha}$ is replaced by $H_{i,\sigma}$).

Remark 6.9. One can take advantage from the operators $\mathscr{S}_{\gamma(i)}$ and $\mathscr{S}_{\varrho,\theta(i)}$ in the factors of correction in Corollaries 6.4 and 6.5. For example, for the process f from Proposition 6.7 in the case $\psi_1 = \psi_2 = \ldots$ or for the process f from Proposition 6.8 one has, without the corresponding operators,

$$\sup_{n\geqslant 1} \Big\|\sup_{1\leqslant i\leqslant n} |f_i - f_{i-1}|\Big\|_{L_1} = \infty.$$

The optimality of Corollary 6.4 can be expressed in a more elegant way. As a direct consequence of (19) and Corollary 6.4 we obtain

Corollary 6.10. *Let $\psi \in \mathscr{D}$, $n \geqslant 1$, $(\psi_i)_{i=1}^n \subset \mathscr{D}_\Psi^2$, and $(H_i)_{i=1}^n \subset L_1(M, \nu)$ be independent and symmetric with*

$$\nu(|H_i| > \lambda) = \exp(1-\psi_i(\lambda)) \quad \textit{for } \lambda \geqslant 1.$$

Then, for all elements $x_1, \ldots, x_n$ of a Banach space X, one has

$$\Big\|\sum_{i=1}^n H_i x_i\Big\|_{\bar{\psi}} \leqslant c_{(4.7)} c_{(6.4)} \kappa \Big\|\sum_{i=1}^n H_i x_i\Big\|_{\psi},$$

where $c_{(4.7)} \geqslant 1$ is taken from Theorem 4.7, $\kappa := \sup_{1 \leqslant i \leqslant n} \exp(\psi_i(c_{(4.7)}^2) - 1)$, and the constant $c_{(6.4)} > 0$ is taken from Corollary 6.4.

Note that for $\psi \in \mathscr{D}$ one has $\psi(\lambda) \leqslant \bar{\psi}(\lambda)$ for $\lambda \geqslant 1$ and

$$\|F\|_\psi \leqslant \|F\|_{\bar{\psi}} \quad \text{for } F \in L_0^X(M, \nu),$$

where the converse with some multiplicative constant is not true in general (for instance, use $\psi(\lambda) := \sqrt{\lambda}$, $F(t) := (1+\log t^{-1})^2 \in L_1(0, 1]$, Remark 3.3, and $\bar{\psi}(\lambda) \geqslant \lambda/c$ from [7], Lemma 4.4).

Problem 6.11. Is it possible to replace in Corollary 6.5 in the case $2 < \varrho < \beta < \infty$ the variables $H_{i,\alpha}$ by $H_{i,\sigma}$, where $1 = 1/\sigma + 1/\varrho$?

7. AN APPLICATION TO SPACES OF TYPE α

By means of Corollary 6.5 we demonstrate, in Corollary 7.2, how one can apply the results from the previous sections. For this purpose we recall that $(h_k)_{k=0}^\infty \subset L_\infty[0, 1)$ is the normalized sequence of Haar functions and $(r_k)_{k=1}^\infty \subset L_\infty[0, 1)$ the sequence of Rademacher functions. Moreover, we use the operators $\mathscr{S}_{\varrho,\theta^{(i)}}\colon \mathscr{E}_{\varrho,\Theta} \to L_0^+(\Omega, \mathscr{F}, P)$ from Section 6 for $\varrho = \infty$.

Definition 7.1. For $1 < \alpha \leqslant 2$ a Banach space X is of *type* α provided that there is a constant $c > 0$ such that for all $n = 1, 2, \ldots$ and $x_1, \ldots, x_n \in X$ one has

$$\Big\|\sum_{i=1}^n r_i x_i\Big\|_{L_2^X} \leqslant c\Big(\sum_{i=1}^n \|x_i\|_X^\alpha\Big)^{1/\alpha}.$$

We let $t_\alpha(X) := \inf c$.

Corollary 7.2. *For a Banach space X and $1 < \alpha \leqslant 2$ the following assertions are equivalent*:

(i) *X is of type α.*

(ii) *There is a constant $c_2 > 0$ such that for all $n = 1, 2, \ldots$, $\Theta =$*

$=(\theta^{(i)})_{i=1}^n \subset \{-1, 1\}^N$, $f \in \mathscr{E}_{\infty,\Theta}$, $1 \leqslant p < \infty$, *and* $x_1, \ldots, x_n \in X$ *one has*

$$\Big\|\sup_{t\in T}\Big\|\sum_{i=1}^n [^{i-1}f_t^i]\,x_i\Big\|_X\Big\|_{L_p} \leqslant c_2\, p\,\Big\|\sup_{1\leqslant i\leqslant n} \mathscr{S}_{\infty,\theta^{(i)}}(^{i-1}f^i)\Big\|_{L_p}\Big(\sum_{i=1}^n \|x_i\|_X^\alpha\Big)^{1/\alpha}.$$

(iii) *There is a constant* $c_3 > 0$ *such that for all* $n, N = 1, 2, \ldots$, *all sequences of stopping times* $0 = \tau_0 \leqslant \tau_1 \leqslant \ldots \leqslant \tau_n = N$ *with respect to* $(\mathscr{F}_k^h)_{k=0}^N$, *all* $\xi_1, \ldots, \xi_N \in \boldsymbol{R}$, *and all* $x_1, \ldots, x_n \in X$ *one has*

$$\Big\|\sum_{i=1}^n \Big[\sum_{k=\tau_{i-1}+1}^{\tau_i} \xi_k h_k\Big]\,x_i\Big\|_{L_\alpha^X} \leqslant c_3 \sup_{1\leqslant i\leqslant n}\Big\|\sum_{k=\tau_{i-1}+1}^{\tau_i} \xi_k h_k\Big\|_{L_\infty}\Big(\sum_{i=1}^n \|x_i\|_X^\alpha\Big)^{1/\alpha}.$$

Proof. (i) $\Rightarrow$ (ii) follows from Corollary 6.5 and (with the notation of Corollary 6.5)

$$\Big\|\sum_{i=1}^n H_{i,1}\,x_i\Big\|_{L_1^X} \leqslant \Big(\int_M \Big\|\sum_{i=1}^n r_i H_{i,1}(t)\,x_i\Big\|_{L_2^X[0,1)}^\alpha dv(t)\Big)^{1/\alpha}$$
$$\leqslant t_\alpha(X)\Big(\int_M \sum_{i=1}^n \|H_{i,1}(t)\,x_i\|^\alpha dv(t)\Big)^{1/\alpha} = t_\alpha(X)\,\|H_{1,1}\|_{L_\alpha}\Big(\sum_{i=1}^n \|x_i\|^\alpha\Big)^{1/\alpha}.$$

(ii) $\Rightarrow$ (iii). We let $\theta^{(i)} := (1, 1, \ldots)$ and apply Lemma B.1 with $\mathscr{G}_k := \mathscr{F}_k^h$. Hence we have $c_3 \leqslant c_2\,\alpha$.

(iii) $\Rightarrow$ (i). Taking $N := 2^n - 1$, $\tau_i := 2^i - 1$, and $\xi_i := 1$ we get

$$\sup_{1\leqslant i\leqslant n}\Big\|\sum_{k=\tau_{i-1}+1}^{\tau_i} \xi_k h_k\Big\|_{L_\infty} = 1 \quad \text{and} \quad \Big\|\sum_{i=1}^n \Big[\sum_{k=\tau_{i-1}+1}^{\tau_i} \xi_k h_k\Big]\,x_i\Big\|_{L_\alpha^X} = \Big\|\sum_{i=1}^n r_i x_i\Big\|_{L_\alpha^X}.$$

Using (2) we obtain $t_\alpha(X) \leqslant c c_3$ with an absolute constant $c > 0$. ∎

Remark 7.3. (i) In the same way as described in Remark 6.9 one can take advantage from the factor $\|\sup_i \mathscr{S}_{\infty,\theta^{(i)}}(^{i-1}f^i)\|_{L_p}$ in Corollary 7.2 (ii).

(ii) The L_α-norm on the left-hand side of the inequality in Corollary 7.2 (iii) can be replaced by any L_p-norm with $1 \leqslant p < \infty$. To relate this assertion to Proposition 7.5 (ii) we have chosen the L_α-norm.

We conclude with Proposition 7.5 which provides a counterpart to Corollary 7.2 in Banach spaces having an equivalent norm with a modulus of smoothness of power type α. According to [14] those Banach spaces are characterized by the following 'martingale-type' property:

DEFINITION 7.4. Given $1 < \alpha \leqslant 2$, a Banach space X is said to be of *martingale-type* α provided that there is a constant $c > 0$ such that for all $n = 1, 2, \ldots$ and all martingale difference sequences $(df_i)_{i=1}^n \subset L_1^X$ one has

$$\Big\|\sum_{i=1}^n df_i\Big\|_{L_\alpha^X} \leqslant c\Big(\sum_{i=1}^n \|df_i\|_{L_\alpha^X}^\alpha\Big)^{1/\alpha}.$$

We let $M - t_\alpha(X) := \inf c$.

PROPOSITION 7.5. *For a Banach space X and $1 < \alpha \leqslant 2$ the following assertions are equivalent:*

(i) *X is of martingale-type α.*

(ii) *There is a constant $c_2 > 0$ such that for all $n, N = 1, 2, \ldots$, all sequences of stopping times $0 = \tau_0 \leqslant \tau_1 \leqslant \ldots \leqslant \tau_n = N$ with respect to $(\mathscr{F}_k^h)_{k=0}^N$, all $\xi_1, \ldots, \xi_N \in \boldsymbol{R}$, and all $x_1, \ldots, x_n \in X$ one has*

$$\Big\|\sum_{i=1}^{n}\Big[\sum_{k=\tau_{i-1}+1}^{\tau_i}\xi_k h_k\Big] x_i\Big\|_{L_\alpha^X} \leqslant c_2 \sup_{1\leqslant i\leqslant n}\Big\|\sum_{k=\tau_{i-1}+1}^{\tau_i}\xi_k h_k\Big\|_{L_\alpha}\Big(\sum_{i=1}^{n}\|x_i\|_X^\alpha\Big)^{1/\alpha}.$$

Proof. (i) $\Rightarrow$ (ii) follows from

$$\Big\|\sum_{i=1}^{n}\Big[\sum_{k=\tau_{i-1}+1}^{\tau_i}\xi_k h_k\Big] x_i\Big\|_{L_\alpha^X} \leqslant M\!-\!t_\alpha(X)\Big(\sum_{i=1}^{n}\Big\|\Big[\sum_{k=\tau_{i-1}+1}^{\tau_i}\xi_k h_k\Big] x_i\Big\|_{L_\alpha^X}^\alpha\Big)^{1/\alpha}$$
$$\leqslant M\!-\!t_\alpha(X)\sup_{1\leqslant i\leqslant n}\Big\|\sum_{k=\tau_{i-1}+1}^{\tau_i}\xi_k h_k\Big\|_{L_\alpha}\Big(\sum_{i=1}^{n}\|x_i\|_X^\alpha\Big)^{1/\alpha}.$$

(ii) $\Rightarrow$ (i). Choosing $N = n = 2^L - 1$ for $L \geqslant 1$ and $\tau_i := i$ we obtain

$$\Big\|\sum_{l=1}^{L}\Big[\sum_{i=2^{l-1}}^{2^l-1} h_i x_i\Big]\Big\|_{L_\alpha^X} = \Big\|\sum_{i=1}^{n}\frac{h_i}{\|h_i\|_{L_\alpha}}\|h_i\|_{L_\alpha} x_i\Big\|_{L_\alpha^X}$$
$$\leqslant c_2\Big(\sum_{i=1}^{n}\|h_i\|_{L_\alpha}^\alpha\|x_i\|_X^\alpha\Big)^{1/\alpha} = c_2\Big(\sum_{l=1}^{L}\Big\|\sum_{i=2^{l-1}}^{2^l-1} h_i x_i\Big\|_{L_\alpha^X}^\alpha\Big)^{1/\alpha}$$

so that we are done according to [14] (Theorem 3.1 and Proposition 2.4); note that $\sum_{i=2^{l-1}}^{2^l-1} h_i x_i$ are the martingale differences of a dyadic martingale. ∎

APPENDIX A. PROOF OF THEOREM 2.4

Given $(f_t)_{t\in T}$, we fix $\Omega_0 \subseteq \Omega$ of measure one such that $(f_t)_{t\in T}$ has right--continuous paths with finite left limits on Ω_0 and $f_\infty \in L_0(\Omega, \mathscr{F}, \boldsymbol{P})$ such that $f_\infty = \lim_{t\to\infty} f_t$ a.s. Moreover, we fix a stopping time $\tau\colon \Omega \to [0, \infty]$ and $B \in \mathscr{F}_\tau$ of positive measure. Now for $v > 0$ we define the stopping times

$$\varrho_v := \inf\{t \geqslant \tau \mid |f_t - f_{\tau-}| > v\}$$

and we can follow the proof of Theorem 4.6 in [7].

(a) If $\|f\|_{BMO_\psi^*} = 1$, $\lambda > 0$, and $\mu \geqslant 1$, then we get

$$\boldsymbol{P}_B\big(\sup_{t\geqslant\tau}|f_t - f_{\tau-}| > \lambda+\mu\big) = \boldsymbol{P}_B(\varrho_{\lambda+\mu} < \infty,\ \varrho_\lambda < \infty)$$
$$\leqslant \boldsymbol{P}_B(|f_{\varrho_{\lambda+\mu}} - f_{\tau-}| \geqslant \lambda+\mu,\ \varrho_\lambda < \infty)$$
$$= \boldsymbol{P}_{B\cap\{\varrho_\lambda<\infty\}}(|f_{\varrho_{\lambda+\mu}} - f_{\tau-}| \geqslant \lambda+\mu)\,\boldsymbol{P}_B(\varrho_\lambda < \infty)$$
$$\leqslant \boldsymbol{P}_{B\cap\{\varrho_\lambda<\infty\}}(|f_{\varrho_{\lambda+\mu}} - f_{\tau-}| \geqslant |f_{\varrho_{\lambda}-} - f_{\tau-}| + \mu)\,\boldsymbol{P}_B(\varrho_\lambda < \infty)$$

$$\leqslant P_{B\cap\{\varrho_\lambda<\infty\}}(|f_{\varrho_\lambda+\mu}-f_{\varrho_\lambda-}|\geqslant\mu)P_B(\varrho_\lambda<\infty)$$
$$\leqslant e^{1-\psi(\mu)}P_B(\varrho_\lambda<\infty)=e^{1-\psi(\mu)}P_B(\sup_{t\geqslant\tau}|f_t-f_{\tau-}|>\lambda).$$

For $\mu_i\geqslant 1$ with $\mu=\sum_{i=1}^M\mu_i$ the iteration gives

$$P_B(\sup_{t\geqslant\tau}|f_t-f_{\tau-}|>\lambda+\mu)\leqslant\Big[\prod_{i=1}^M\exp\big(1-\psi(\mu_i)\big)\Big]P_B(\sup_{t\geqslant\tau}|f_t-f_{\tau-}|>\lambda)$$

so that assertion (ii) ($\tau=0$, $B=\Omega$) and $\|\cdot\|_{BMO_{\bar\psi}^*}=\|\cdot\|_{BMO_\psi^*}$ of (i) follow.

(b) For $\|f\|_{BMO_\psi}=1$, $\mu>\mu-\varepsilon\geqslant 2$, $B\subseteq\Omega_0$ ($B\in\mathscr{F}_\tau$ was fixed above) we get

$$P_B(\sup_{t\geqslant\tau}|f_t-f_{\tau-}|>\mu)\leqslant P_B(|f_{\varrho_\mu}-f_{\tau-}|\geqslant\mu)$$
$$\leqslant P_B\left(|f_\infty-f_{\tau-}|\geqslant\frac{\mu}{2}\right)+P_B\left(|f_\infty-f_{\varrho_\mu}|>\frac{\mu}{2}\right)$$
$$\leqslant\exp\left(1-\psi\left(\frac{\mu-\varepsilon}{2}\right)\right)+P_B\left(\lim_{n\to\infty}|f_\infty-f_{(\varrho_\mu+1/n)-}|>\frac{\mu}{2}\right)$$
$$\leqslant\exp\left(1-\psi\left(\frac{\mu-\varepsilon}{2}\right)\right)+\liminf_{n\to\infty}P_B\left(|f_\infty-f_{(\varrho_\mu+1/n)-}|>\frac{\mu}{2}\right)$$
$$\leqslant\exp\left(1-\psi\left(\frac{\mu-\varepsilon}{2}\right)\right)+\exp\left(1-\psi\left(\frac{\mu}{2}\right)\right),$$

which implies

$$P_B(\sup_{t\geqslant\tau}|f_t-f_{\tau-}|>\mu)\leqslant 2\exp\big(1-\psi(\mu/2)\big)\quad\text{ for }\mu\geqslant 2.$$

Applying the iteration argument carried out in (a) for $M=2$ and $e^{1-\psi(\mu)}$ replaced by $2e^{1-\psi(\mu/2)}$ we obtain for $\mu=\mu_1+\mu_2$ with $\mu_i\geqslant 2$ and $\lambda>0$ the inequality

$$P_B(\sup_{t\geqslant\tau}|f_t-f_{\tau-}|>\lambda+\mu)$$
$$\leqslant 2\exp\big(1-\psi(\mu_1/2)\big)2\exp\big(1-\psi(\mu_2/2)\big)P_B(\sup_{t\geqslant\tau}|f_t-f_{\tau-}|>\lambda).$$

Now one checks that

$$\big[2\exp\big(1-\psi(\mu/4)\big)\big]^2\leqslant\exp\big(1-\psi(\mu/c)\big)\quad\text{ for }\mu\geqslant c:=4\psi^{-1}(3)$$

and obtains the remaining part of assertion (i). ∎

APPENDIX B. A RESCALING ARGUMENT

LEMMA B.1. *Let* $t_{i,k}:=i-1/(k+1)$ *for* $i=1,2,\ldots$ *and* $k=0,1,2,\ldots$ *Assume stopping times* $0=\tau_0\leqslant\tau_1\leqslant\ldots\leqslant\tau_n=N$ *with respect to a filtration* $(\mathscr{G}_k)_{k=0}^N$ *and* $f\in\mathscr{P}\big((\mathscr{G}_k)_{k=0}^N\big)$. *Let*

$$\mathscr{F}_t:=\mathscr{G}_{(\tau_{i-1}+k)\wedge\tau_i}\quad\text{and}\quad\tilde f_t:=f_{(\tau_{i-1}+k)\wedge\tau_i}\quad\text{for } t_{i,k}\leqslant t<t_{i,k+1},\ 1\leqslant i\leqslant n,$$
$$\mathscr{F}_t:=\mathscr{G}_{\tau_n}=\mathscr{G}_N\quad\text{and}\quad\tilde f_t:=f_{\tau_n}=f_N\quad\text{for } n\leqslant t<\infty.$$

Then the following holds:

(i) $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$ *for all* $0 \leqslant t < \infty$ *and* $\mathcal{F}_{t_{i,N}} = \mathcal{F}_i$ *for all* $i \geqslant 1$.

(ii) $|\tilde{f}_{t_{i,k}} - \tilde{f}_{t_{i,k-1}}|$ *is* $\mathcal{F}_{t_{i,k-1}}$*-measurable for* $i, k \geqslant 1$.

(iii) $\tilde{f}_i - \tilde{f}_{i-1} = \sum_{k=\tau_{i-1}+1}^{\tau_i} df_k$ *for* $i = 1, \ldots, n$.

Proof. Assertions (i) and (iii) are evident. To prove (ii) it is sufficient to observe that $|f_\tau - f_\sigma|$ is $\mathcal{G}_\sigma$-measurable whenever $f \in \mathcal{P}\left((\mathcal{G}_k)_{k=0}^N\right)$ and the stopping times $0 \leqslant \sigma, \tau \leqslant N$ with respect to $(\mathcal{G}_k)_{k=0}^N$ satisfy $\sigma \leqslant \tau \leqslant \sigma + 1$. ∎

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