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REFLECTED BSDE WITH SUPERLINEAR QUADRATIC COEFFICIENT

BY

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Abstract. In this paper, we provide existence of a reflected solution of the one-dimensional backward stochastic differential equation when the coefficient is continuous, has a superlinear growth in y and quadratic growth in z. We also give a characterization of the solution as the value function of an optimal stopping time problem. We also study the links between the solution of the quadratic RBSDE and the corresponding obstacle problem. Then we give an application of quadratic RBSDE's to the pricing of American contingent claims in an incomplete market.

Key words and phrases: Backward stochastic differential equations, reflexion, viscosity solution, American option.

1. INTRODUCTION

Nonlinear backward stochastic differential equations (BSDE's) were first introduced in 1990 by Pardoux and Peng [11]. Recall that the solution of a BSDE consists of a pair of adapted processes (Y, Z) satisfying

$$-dY_t = f(t, Y_t, Z_t)\,dt - Z_t\,dW_t, \qquad Y_T = \xi, \tag{1}$$

where f is called the *coefficient* and ξ the *terminal condition*. Under the Lipschitz assumption on f, the authors stated the first existence and uniqueness result. The interest in BSDE's comes from their connections with PDE's, stochastic control and mathematical finance. In particular, as shown in 1997 by El Karoui, Peng and Quenez, BSDE's are a useful tool in the pricing of a European option, which consists of a contract which pays the amount ξ at time T. In a complete market (with eventually some nonlinear constraints), the price process Y of ξ is a solution of a BSDE such as (1). In 1997, El Karoui et al. proved

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in [1], still under Lipschitz assumptions, the existence and uniqueness of an adapted solution for a reflected BSDE (in short, RBSDE) for which the solution is constrained to stay above an "obstacle" $\{\xi_t, 0 \leqslant t \leqslant T\}$. In this case, the solution of the RBSDE associated with obstacle ξ and coefficient f consists of a triple (Y, Z, K), where K is an increasing process satisfying

$$(2) \qquad -dY_t = f(t, Y_t, Z_t)\,dt + dK_t - Z_t\,dW_t, \qquad Y_T = \xi_T,$$

with $Y_t \geqslant \xi_t$ and $(Y_t - \xi_t)\,dK_t = 0$. As shown in [1] or [3], RBSDE's are a useful tool for the pricing of American options. In a complete market (with eventually some nonlinear constraints), the price process Y of an American option associated with payoff $\{\xi_t, 0 \leqslant t \leqslant T\}$ is a solution of the RBSDE such as (2).

Recall that many assumptions have been made to relax the assumption on the coefficient f; for instance, in [8] Lepeltier and San Martín have proved the existence of a solution for BSDE's with a coefficient which is only continuous with linear growth. Moreover, Matoussi [10] established the existence of a solution for RBSDE's with continuous and linear growth coefficient.

In [6] Kobylanski studied the case of BSDE's without reflexion and proved an existence result in the case when the coefficient is only linear growth in y, and quadratic in z. In [9] Lepeltier and San Martín generalized the result to a superlinear case in y. Recall that such BSDE's appear in risk-sensitive control and also in finance. More precisely, in [4], El Karoui and Rouge studied the problem of pricing the European options via exponential utility. In the case of an incomplete market, they stated that the price of such an option is a solution of a quadratic BSDE. Thus, if we are concerned with American options instead of European options, we are naturally led to the study of reflected quadratic BSDE's.

In this paper, under assumptions similar to those in [9], we state the existence of a maximal solution of a quadratic RBSDE, by adapting some techniques of [6] and [9] to the reflected case. We also state the existence of a minimal solution of the RBSDE, which cannot be derived directly from the existence of a maximal solution because of the reflexion. Also, under some stronger assumptions (as in [6]), we state that, as in the case of a Lipschitz coefficient, the solution can be characterized as the value function of an optimal stopping time problem. From this property we derive both the uniqueness of the solution and a comparison theorem. We also show that, even if there is no uniqueness, the maximal (respectively, minimal) solution coincides with the value function of an optimal stopping time problem. In this paper we also give a stability result: more precisely, if we are given a sequence of obstacles (ξ^n) which converges a.s. to ξ and a sequence of coefficients (f^n) which converges to f locally uniformly, then the solutions (Y^n) of RBSDE's associated with (ξ^n, f^n) converge uniformy a.s. to Y, the solution of the RBSDE associated with (ξ, f). Note that this stability result can be used to obtain some continuity properties of the solutions of RBSDE's with respect to parameters. In this paper we also study the links between RBSDE's and obstacle problems, generalizing the results of [1] to the quadratic case.

The outline of the paper is organized as follows. In Section 2, we show the existence of maximal and minimal solutions of the RBSDE. In Section 3, we study the links between solutions of RBSDE's and some value functions of optimal stopping time problems. Section 4 provides a stability result. Section 5 provides connections between the solutions of quadratic RBSDE's and solutions of related obstacle problems. We give a direct proof that the solution of the RBSDE is a viscosity solution of the associated obstacle problem. We also prove a uniqueness result for the viscosity solution of this obstacle problem. In Section 6, we give an application of the previous result to the pricing of American contingent claims in an incomplete market via utility maximization. In this case, the fair price of an American contingent claim can be characterized as the unique solution of a quadratic RBSDE, where the obstacle corresponds to the payoff of the option.

2. EXISTENCE OF MAXIMAL AND MINIMAL SOLUTIONS

Let $(W_t)_{0\leqslant t\leqslant T}$ be a standard d-dimensional Brownian motion defined on a probability space $(\Omega, \mathscr{F}, P)$. Let $(\mathscr{F}_t, 0\leqslant t\leqslant T)$ denote the natural filtration of (W_t), where $\mathscr{F}_0$ contains all the P-null sets of $\mathscr{F}$. Let us assume the following:

H1. A bounded adapted process $\{\xi_t, 0\leqslant t\leqslant T\}$, called "obstacle", is continuous on $[0, T[$ and such that $\overline{\lim}_{t\to T}\xi_t\leqslant \xi_T$ a.s.

H2. A "coefficient" $f\colon [0, T]\times\Omega\times \boldsymbol{R}\times \boldsymbol{R}^d\to \boldsymbol{R}$ is $\mathscr{P}\otimes\mathscr{B}(\boldsymbol{R})\otimes\mathscr{B}(\boldsymbol{R}^d)$-measurable, where $\mathscr{P}$ denotes the predictable σ-algebra.

We introduce the following notation:

$$H^2(\boldsymbol{R}^d) = \Big\{X\colon [0, T]\times\Omega\to \boldsymbol{R}^d,\ X\in\mathscr{P},\ \|X\|^2 = E\Big(\int_0^T |X_s|^2\,ds\Big)<\infty\Big\}.$$

DEFINITION 1. We say that (Y, Z, K) is a *solution of the RBSDE associated with obstacle* (ξ_t) *and coefficient* f, which is denoted by $\mathrm{Eq}(\xi, f)$, if

(a) it satisfies

$$Y_t = \xi_T+\int_t^T f(s, Y_s, Z_s)\,ds+K_T-K_t-\int_t^T Z_s\,dW_s \text{ a.s.},\qquad 0\leqslant t\leqslant T; \tag{3}$$

(b) $Y\in H^2(\boldsymbol{R})$ and $Z\in H^2(\boldsymbol{R}^d)$;

(c) $Y_t\geqslant \xi_t$ a.s., $0\leqslant t\leqslant T$;

(d) K is a continuous increasing process with $K_0 = 0$ and $\int_0^T (Y_t-\xi_t)\,dK_t = 0$ (K is increasing only on $Y = \xi$).

In the sequel, we will suppose that the coefficient f satisfies the following assumption:

H3. For all (t, ω), $f(t, \omega, \cdot, \cdot)$ is continuous and there exists a function l strictly positive such that

$$\int_0^{\infty} \frac{dx}{l(x)} = +\infty \quad \text{with } |f(t, \omega, y, z)| \leqslant l(y) + C|z|^2, \; dt \otimes dP\text{-a.s.}$$

2.1. Existence of a maximal solution. We remember ([9], Lemma 1) that the assumption on l implies that for all $b \geqslant 0$ the ODE

$$U_t = b + \int_t^T l(U_s)\, ds \tag{4}$$

has a unique solution on $[0, T]$. Our main result is the following

THEOREM 1. *Under the assumptions* H1–H3 *on* (ξ, f), *the RBSDE* Eq(ξ, f) (3) *has a maximal bounded solution* (Y^*, Z^*, K^*). *Moreover, for all t we have* $Y_t^* \leqslant U_t \leqslant U_0$ *a.s., where U is the unique solution of the equation* (4) *with*

$$b = \operatorname{Max}\{\|\xi_T\|_\infty, \operatorname*{ess\,sup}_{(t,\omega)} \xi_t(\omega)\}.$$

If we transform the equation (4) by an exponential function, that is, if we define $V_t = \exp(2CU_t)$, we infer that V_t is the unique solution of the ODE

$$V_t = \beta + \int_t^T g(V_s)\, ds, \qquad 1 \leqslant \beta = e^{2Cb}, \tag{5}$$

where

$$g(x) = 2Cxl\left(\frac{\ln(x)}{2C}\right) \quad \text{when } x > 0.$$

Then we have $\beta \leqslant V_t \leqslant \exp(2CU_0) = V_0$.

Remark 1. Let $\varrho: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ be a smooth function such that

$$\varrho(x) = \begin{cases} x, & x \in [r, R], \\ r/2, & 0 < x < r/2, \\ 2R, & x > 2R. \end{cases} \tag{6}$$

Assume also that $V_0 < R$, and $\beta > r > 0$. It is direct to verify that the unique solution of the equation

$$V_t^{\varrho} = \beta + \int_t^T g(\varrho(V_s^{\varrho}))\, ds \tag{7}$$

is $V^{\varrho} = V$.

A useful tool in our considerations will be the following two lemmas.

LEMMA 2.1 (Comparison). *Let f^1 and f^2 defined on $[0, T]\times\Omega\times\boldsymbol{R}\times\boldsymbol{R}^d$, which are $\mathscr{P}\otimes\mathscr{B}(\boldsymbol{R}^{d+1})$-measurable, satisfy*

$$f^2(t, y, z)\geqslant f^1(t, y, z)\ dt\otimes dP\text{-a.s.},\quad y\in\boldsymbol{R},\ z\in\boldsymbol{R}^d,\ 0\leqslant t\leqslant T. \tag{8}$$

Assume that, for all (t, ω), $f^2(t, \omega, y, z)$ is continuous and linear increasing, i.e., there exists a constant $k<\infty$ such that

$$|f^2(t, \omega, y, z)|\leqslant k(1+|y|+|z|).$$

Let (ξ_t^1) and (ξ_t^2) be two adapted real processes belonging to H^2, continuous on $[0, T)$, such that $\limsup_{t\to T}\xi_t^i\leqslant\xi_T^i$ for $i=1, 2$ and satisfying

$$\xi_t^2\geqslant\xi_t^1\ \text{a.s.},\quad 0\leqslant t\leqslant T. \tag{9}$$

Let (Y^1, Z^1, K^1) be a solution of Eq(ξ^1, f^1) *and let (Y^2, Z^2, K^2) be the maximal solution of* Eq(ξ^2, f^2). *Then $Y^1\leqslant Y^2$ a.s.*

If in addition f^1 is also continuous and linear increasing, if $\xi^1=\xi^2$, and if (Y^1, Z^1, K^1) is the maximal solution of Eq(ξ^1, f^1), *then $dK^1\geqslant dK^2$ a.s.*

Proof. The existence of a maximal solution in the linear growth case provides from the result of Matoussi [10]. Moreover, if for $n>K$, $(Y^{2,n}, Z^{2,n}, K^{2,n})$ denotes the unique solution of Eq(ξ^2, f_n^2), where (f_n^2) is the sequence of Lipschitz functions approximating f^2 from above (see [8] or [10]), from the usual comparison theorem for RBSDE [1], we infer that, for all $n>K$, $Y^1\leqslant Y^{2,n}$, and since $Y^{2,n}\downarrow Y^2$, we obtain $Y^1\leqslant Y^2$ a.s.

Now we assume $\xi^1=\xi^2$. In the same way as previously, if (f_n^1) denotes the sequence of Lipschitz functions approximating f^1 from above and $(Y^{1,n}, Z^{1,n}, K^{1,n})$ is the unique solution of Eq(ξ^1, f_n^1), we have, for all $n>k$, $f_n^1\leqslant f_n^2$, which implies, by [5], $dK^{1,n}\geqslant dK^{2,n}$. Finally, since $K^{1,n}\downarrow K^1$ and $K^{2,n}\downarrow K^2$, we obtain $dK^1\geqslant dK^2$ P-a.s. ∎

LEMMA 2.2. *Let $f_1\colon\boldsymbol{R}\to\boldsymbol{R}_+$ be a continuous function with linear growth. If the ODE*

$$J_t=a+\int_t^T f_1(J_s)\,ds,\qquad a\geqslant 0,$$

has a unique solution J, then the RBSDE Eq(a, f_1)

$$X_t=a+\int_t^T f_1(X_s)\,ds+K_T-K_t-\int_t^T Z_s\,dW_s \tag{10}$$

has a unique solution given by $Z=0$, $K=0$ and $X=J$.

Proof. As in [8] we approximate f_1 by a Lipschitz function from above. Let us call (f_1^n) such an approximation and consider the unique solution of Eq(a, f_1^n):

$$X_t^n=a+\int_t^T f_1^n(X_s^n)\,ds+K_T^n-K_t^n-\int_t^T Z_s^n\,dW_s.$$

From the uniqueness theorem we infer that X_t^n is the solution of the ODE

$$X_t^n = a + \int_t^T f_1(X_s^n)\,ds, \qquad Z^n = 0,\ K^n = 0$$

(since $X^n \geqslant a$, $K_0 = 0$).

We know that

$$(X^n, Z^n) \xrightarrow{H^2(\mathbb{R}^{d+1})} (\bar{X}, \bar{Z}) \quad \text{and} \quad K^n \uparrow \bar{K} \text{ a.s.},$$

where $(\bar{X}, \bar{Z}, \bar{K})$ satisfies $\mathrm{Eq}(a, f_1)$ (see [10]), and therefore $\bar{Z} = 0$, $\bar{K} = 0$ and $\bar{X} = J$. The approximation can be done from above or from below to obtain either a maximal or a minimal solution of $\mathrm{Eq}(a, f_1)$. Consequently, $(J, 0, 0)$ is a maximal and a minimal solution of this equation. Finally, let (J, Z, K) be another solution of $\mathrm{Eq}(a, f_1)$. For all $t \leqslant T$ we obtain $\int_0^t Z_s\,dW_s = K_t$. Since the left-hand side is an increasing continuous process and the right-hand side is a continuous martingale, we obtain $Z = K = 0$. ∎

2.2. Proof of Theorem 1. Making the change of variable $\theta_t = \exp(2CY_t)$ in $\mathrm{Eq}(\xi, f)$ (3) we are led to solving the following RBSDE $\mathrm{Eq}(\eta, F)$:

$$\theta_t = \eta_T + \int_t^T F(s, \theta_s, \Lambda_s)\,ds + J_T - J_t - \int_t^T \Lambda_s\,dW_s, \tag{11}$$

with $\eta_s = \exp(2C\xi_s)$, $\Lambda_s = 2CZ_s\theta_s$, $dJ_s = 2C\exp(2CY_s)\,dK_s$, and

$$F(s, \omega, x, \lambda) = 2Cx\left[f\left(s, \omega, \frac{\ln(x)}{2C}, \frac{\lambda}{2Cx}\right) - \frac{|\lambda|^2}{4Cx^2}\right]. \tag{12}$$

Since $Y_t \geqslant \xi_t$ and K increases only on $(Y_t = \xi_t)$, we have $\exp(2CY_t) = \theta_t \geqslant \exp(2C\xi_t) = \eta_t$, and

$$\int_0^T (\theta_t - \eta_t)\,dJ_t = \int_0^T (\exp(2CY_t) - \exp(2C\xi_t))\,2C\exp(2CY_t)\,dK_t = 0.$$

Consider now

$$g(x) = 2Cx\,l\left(\frac{\ln(x)}{2C}\right),$$

where l is given by H3. Taking

$$b = \mathrm{Max}\,\{\|\xi_T\|_\infty,\ \operatorname*{ess\,sup}_{(t,\omega)} \xi_t(\omega)\} \quad \text{and} \quad \beta = e^{2Cb}$$

we obtain $\beta \leqslant V_t \leqslant V_0$, where V_0 is given by (5). Finally, consider

$$m = \operatorname*{ess\,inf}_{(t,\omega)} \xi_t(\omega) \quad \text{and} \quad F_\psi(s, \omega, x, \lambda) = \psi(x)F(s, \omega, x, \lambda),$$

where ψ is any smooth function satisfying

(13)
$$\psi(x)\in[0,1] \quad \text{for all } x,$$
$$\psi(x)=\begin{cases}0 & \text{for } x\notin[K_1, K_2],\\ 1 & \text{for } x\in[2K_1, K_2/2],\end{cases}$$

where $0<2K_1<e^{2Cm}$, $K_2/2>V_0$. Then for all s, ω, $x>0$, and λ we have

$$-g(x)-\frac{1}{K_1}|\lambda|^2\leqslant F_\psi(s,\omega,x,\lambda)\leqslant g(x).$$

Assume that for any ψ there exists $(\theta^\psi, \Lambda^\psi, J^\psi)$ being a maximal solution of $\mathrm{Eq}(\eta, F_\psi)$ such that $e^{2Cm}\leqslant\eta_t\leqslant\theta_t^\psi\leqslant V_t\leqslant V_0$. Then we claim that Theorem 1 follows. In fact,

$$Y^\psi=\frac{\ln(\theta^\psi)}{2C},\qquad Z^\psi=\frac{\Lambda^\psi}{2C\theta^\psi},\qquad dK^\psi=\frac{dJ^\psi}{2C\theta^\psi}$$

is a bounded solution of (3) $\mathrm{Eq}(\xi, f, L)$. On the other hand, let $(\hat{Y}, \hat{Z}, \hat{K})$ be a bounded solution of $\mathrm{Eq}(\xi, f)$ with $\hat{Y}\leqslant h$ (bound). We can consider $\hat{K}_1$, $\hat{K}_2$ and $\hat{\psi}$ satisfying (13), and such that

$$\hat{\psi}=\begin{cases}0 & \text{outside } [\hat{K}_1, \hat{K}_2],\\ 1 & \text{in } [2\hat{K}_1, \hat{K}_2/2],\end{cases}$$

where $0<2\hat{K}_1\leqslant e^{2Cm}$, $\hat{K}_2/2>h$. Then $\hat{\theta}=e^{2C\hat{Y}}$, $\hat{\Lambda}=2C\hat{Z}\hat{\theta}$ and $d\hat{J}=2C\hat{\theta}d\hat{K}$ is a solution of $\mathrm{Eq}(\eta, F_{\hat{\psi}})$. Therefore $\hat{\theta}\leqslant\theta^{\hat{\psi}}$, where $(\theta^{\hat{\psi}}, \Lambda^{\hat{\psi}}, J^{\hat{\psi}})\in\mathrm{Eq}(\eta, F_{\hat{\psi}})$ is a maximal bounded solution such that $e^{2Cm}\leqslant\theta^{\hat{\psi}}\leqslant V_t\leqslant V_0$, and then $\theta_t^{\hat{\psi}}\in\mathrm{Eq}(\eta, F_\psi)$. Consequently, $\hat{Y}\leqslant Y^{\hat{\psi}}\leqslant Y^\psi$. This argument also shows that $\theta^{\hat{\psi}}$ does not depend on $\hat{\psi}$. Summing up, to prove Theorem 1 we are led to proving the following result:

THEOREM 2. *Assume that*

$$G=\{g\colon (0,\infty)\to\boldsymbol{R}_+,\ \textit{for all } \beta\geqslant 1,\ V_t=\beta+\int_t^T g(V_s)\,ds \textit{ has a unique solution}\}.$$

Let $g\in G$ *and let* η *be a bounded obstacle process which satisfies* $\alpha\leqslant\eta\leqslant\beta$, $\beta>1$, $0<\alpha\leqslant 1$. *Suppose that* $F\colon[0,T]\times\Omega\times\boldsymbol{R}_+\times\boldsymbol{R}^d\to\boldsymbol{R}$ *satisfies for some constant* $C>0$ *the following condition*:

$$\forall t, \omega, x\geqslant 0, \lambda,\quad -g(x)-C|\lambda|^2\leqslant F(t,\omega,x,\lambda)\leqslant g(x).$$

Assume also that F *is* $\mathscr{P}\otimes\mathscr{B}(\boldsymbol{R}^{d+1})$*-measurable and* $F(t,\omega,\cdot,\cdot)$ *is continuous for each* (t,ω). *Then the RBSDE* $\mathrm{Eq}(\eta, F)$

(14)
$$\theta_t=\eta_T+\int_t^T F(s,\omega,\theta_s,\Lambda_s)\,ds+J_T-J_t-\int_t^T\Lambda_s\,dW_s$$

has a maximal solution $(\theta^*, \Lambda^*, J^*)$ *which satisfies* $\underline{m} \leqslant \eta_t \leqslant \theta_t^* \leqslant V_t \leqslant V_0$, *where* $\underline{m} = \operatorname{ess\,inf}_{(t,\omega)}(\eta_t(\omega)) > 0$ *and where* V_t *is the unique solution of*

$$V_t = \beta + \int_t^T g(V_s)\, ds.$$

Proof. For any $0 < r < \underline{m}$, $V_0 < R$ (where V_0 and $\underline{m}$ are defined as in Theorem 2) we consider a smooth function ϱ such that

$$\varrho(x) = \begin{cases} x, & x \in [r, R], \\ r/2, & 0 < x < r/2, \\ 2R, & x > 2R. \end{cases}$$

We shall prove that the RBSDE $\mathrm{Eq}(\eta, F_\varrho)$

$$\theta_t^\varrho = \eta + \int_t^T F(s, \omega, \varrho(\theta_s^\varrho), \Lambda_s^\varrho)\, ds + J_T^\varrho - J_t^\varrho - \int_t^T \Lambda_s^\varrho\, dW_s \tag{15}$$

has a unique solution which satisfies $\underline{m} \leqslant \theta_t \leqslant V_0$, and therefore we can choose it independent of (r, R, ϱ). This shows that $(\theta^\varrho, \Lambda^\varrho, J^\varrho)$ is a solution of $\mathrm{Eq}(\eta, F)$ (14). We proceed in the way like in the proof of Theorem 2 of [9] defining $\tilde{F}\colon [0, T] \times \Omega \times \boldsymbol{R} \times \boldsymbol{R}^d \to \boldsymbol{R}$ by $\tilde{F}(t, \omega, \theta, \lambda) = F(t, \omega, \varrho(\theta), \lambda)$, and taking $\kappa_p\colon \boldsymbol{R}^d \to \boldsymbol{R}$ to be a sequence of smooth functions such that $0 \leqslant \kappa_p \leqslant 1$, $\kappa_p(\lambda) = 1$ if $|\lambda| \leqslant p$, and $\kappa_p(\lambda) = 0$ if $|\lambda| \geqslant p+1$, we define

$$\tilde{F}_p(t, \omega, \theta, \lambda) = g(\varrho(\theta))(1 - \kappa_p(\lambda)) + \kappa_p(\lambda)\tilde{F}(t, \omega, \theta, \lambda).$$

Then $\tilde{F}_p \downarrow F$, $\tilde{F}_p$ is a bounded and continuous function of (θ, λ). Therefore by [10] we have proved the existence of a maximal solution $(\theta^p, \Lambda^p, J^p)$ for $\mathrm{Eq}(\eta, \tilde{F}_p)$. Since

$$-g(\varrho(\theta)) - C|\lambda|^2 \kappa_p(\lambda) \leqslant \tilde{F}_p \leqslant g(\varrho(\theta)),$$

we infer from the comparison theorem that $\underline{m} \leqslant \theta^p \leqslant H$, where H is the maximal solution of $\mathrm{Eq}(\beta, g \circ \varrho)$ (5). From Lemma 2.2 and Remark 1 we have $H = V$. Since (θ^p) is a decreasing and bounded sequence, we have proved the existence of $\theta \in H^2(\boldsymbol{R})$ such that, for all $t \leqslant T$, $\theta^p \downarrow \theta$ $\boldsymbol{P}$-a.s., and $\theta^p \xrightarrow{H^2(\boldsymbol{R})} \theta$. Moreover, θ satisfies, for all $t \leqslant T$, $\underline{m} \leqslant \theta_t \leqslant V_0$ $\boldsymbol{P}$-a.s.

We now claim that (Λ^p) has a convergent subsequence in H^2. First we prove that $(\|\Lambda^p\|_2)_p$ is bounded. Applying Itô's formula to $\phi(\theta_t^p)$, where $\phi(x) = e^{-3Cx}$, and taking the expected value we get

$$\begin{aligned} &\boldsymbol{E}(\phi(\theta_0)) + \frac{1}{2}\boldsymbol{E}\Big(\int_0^T \phi''(\theta_s^p)|\Lambda_s^p|^2\, ds\Big) \\ &\qquad = \boldsymbol{E}(\phi(\eta_T)) + \boldsymbol{E}\Big(\int_0^T \phi'(\theta_s^p)\tilde{F}_p(s, \theta_s^p, \Lambda_s^p)\, ds\Big) + \boldsymbol{E}\Big(\int_0^T \phi'(\theta_s^p)\, dJ_s^p\Big). \end{aligned}$$

Since $\phi'(x) = -3Ce^{-3Cx} < 0$ and J^p is increasing, the last term is negative. Then

$$E(\phi(\theta_0)) + \frac{1}{2} E\Big(\int_0^T \phi''(\theta_s^p)\,|\Lambda_s^p|^2\,ds\Big) \leqslant E(\phi(\eta_T)) + E\Big(\int_0^T \phi'(\theta_s^p)\,\tilde{F}_p(s, \theta_s^p, \Lambda_s^p)\,ds\Big).$$

Using the relations $\underline{m} \leqslant \theta_s^p \leqslant V_0$ and the facts that

$$\forall s, \omega, \forall \theta \in [m, V_0], \ \forall \lambda, \ \tilde{F}_p(s, \omega, \theta, \lambda) \geqslant -A - C|\lambda|^2,$$

where

$$A = \max_{\underline{m} \leqslant \theta \leqslant V_0} g(\theta) \quad \text{and} \quad \tfrac{1}{2}\phi'' + C\phi' = \tfrac{3}{2} C^2 \phi,$$

we get

$$\tfrac{3}{2} C^2 e^{-3CM} \|\Lambda^p\|_2^2 \leqslant E(\phi(\eta_T)) + 3Ce^{-3Cm} AT.$$

Consequently, $\|\Lambda^p\|_2 \leqslant K < \infty$, where the constant K is independent of p.

Next take a weak convergent subsequence which still will be denoted by (Λ^p), and $\Lambda \in H^2$ the weak limit. Applying Itô's formula to

$$\phi(x) = \frac{e^{12Cx} - 1}{12C} - x$$

like in [9], and taking the expected value we get for $p < q$

$$\begin{aligned} &E(\phi(\theta_0^p - \theta_0^q)) + \frac{1}{2} E\Big(\int_0^T \phi''(\theta_s^p - \theta_s^q)\,|\Lambda_s^p - \Lambda_s^q|^2\,ds\Big) \\ &= E\Big(\int_0^T \phi'(\theta_s^p - \theta_s^q)\{\tilde{F}_p(s, \theta_s^p, \Lambda_s^p) - \tilde{F}_q(s, \theta_s^q, \Lambda_s^q)\}\,ds\Big) + E\Big(\int_0^T \phi'(\theta_s^p - \theta_s^q)\,d(J_s^p - J_s^q)\Big). \end{aligned}$$

Since $\phi'(\theta_s^p - \theta_s^q) \geqslant 0$ and $dJ_s^p \leqslant dJ_s^q$, we see that the last term is negative. Now $\phi(\theta_t^p - \theta_t^q) \geqslant 0$ and

$$\begin{aligned} &\tilde{F}_p(s, \omega, \theta_s^p, \Lambda_s^p) - \tilde{F}_q(s, \omega, \theta_s^q, \Lambda_s^q) \\ &\qquad \leqslant 2A + C|\Lambda_s^q|^2 \leqslant 2A + 3C\{|\Lambda_s^p - \Lambda_s^q|^2 + |\Lambda_s^p - \Lambda_s|^2 + |\Lambda_s|^2\}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} &E\Big(\int_0^T ([\tfrac{1}{2}\phi'' - 3C\phi'](\theta_s^p - \theta_s^q))\,|\Lambda_s^p - \Lambda_s^q|^2\,ds\Big) \\ &\qquad \leqslant E\Big(\int_0^T \phi'(\theta_s^p - \theta_s^q)\{2A + 3C\{|\Lambda_s^p - \Lambda_s|^2 + |\Lambda_s|^2\}\}\,ds\Big). \end{aligned}$$

The convergence of (θ^p) to θ being pointwise, (θ^p) being bounded and (Λ^p) being bounded in H^2, we have, by Lebesgue's theorem, the following results: For each sequence (q_j) such that

$$\lim_{j\to\infty} E\Big(\int_0^T ([\tfrac{1}{2}\phi''-3C\phi'](\theta_s^p-\theta_s^{q_j}))\,|\Lambda_s^p-\Lambda_s^{q_j}|^2\,ds\Big)$$

exists, this limit coincides with

$$\lim_{j\to\infty} E\Big(\int_0^T ([\tfrac{1}{2}\phi''-3C\phi'](\theta_s^p-\theta_s))\,|\Lambda_s^p-\Lambda_s^{q_j}|^2\,ds\Big).$$

Consequently,

$$\overline{\lim_{q\to\infty}}\, E\int_0^T [\tfrac{1}{2}\phi''-3C\phi'](\theta_s^p-\theta_s)\,|\Lambda_s^p-\Lambda_s^q|^2\,ds$$

$$= \overline{\lim_{q\to\infty}}\, E\int_0^T [\tfrac{1}{2}\phi''-3C\phi'](\theta_s^p-\theta_s^q)\,|\Lambda_s^p-\Lambda_s^q|^2\,ds$$

$$\leqslant \overline{\lim_{q\to\infty}}\, E\int_0^T \phi'(\theta_s^p-\theta_s^q)\,\{2A+3C\,\{|\Lambda_s^p-\Lambda_s|^2+|\Lambda_s|^2\}\}\,ds.$$

Now, Lebesgue's theorem gives

$$\lim_{q\to\infty} E\Big(\int_0^T \phi'(\theta_s^p-\theta_s^q)\,\{2A+3C\,\{|\Lambda_s^p-\Lambda_s|^2+|\Lambda_s|^2\}\}\,ds\Big)$$

$$= \lim_{q\to\infty} E\Big(\int_0^T \phi'(\theta_s^p-\theta_s)\,\{2A+3C\,\{|\Lambda_s^p-\Lambda_s|^2+|\Lambda_s|^2\}\}\,ds\Big).$$

Also, since the functional

$$\Lambda \mapsto E\Big(\int_0^T \phi'(\theta_s^p-\theta_s)\,|\Lambda_s^p-\Lambda_s|^2\,ds\Big)$$

is convex and l.s.c., it follows that

$$E\Big(\int_0^T \phi'(\theta_s^p-\theta_s)\,|\Lambda_s^p-\Lambda_s|^2\,ds\Big) \leqslant \underline{\lim}_{q\to\infty}\, E\Big(\int_0^T \phi'(\theta_s^p-\theta_s)\,|\Lambda_s^p-\Lambda_s^q|^2\,ds\Big).$$

Once more we deduce that

$$\overline{\lim_{q\to\infty}}\, E\Big(\int_0^T ([\phi''/2-6C\phi'](\theta_s^p-\theta_s))\,|\Lambda_s^p-\Lambda_s^q|^2\,ds\Big) \leqslant E\Big(\int_0^T \phi'(\theta_s^p-\theta_s)\,\{2A+3C\,|\Lambda_s|^2\}\,ds\Big).$$

Finally, since $\phi''/2-6C\phi' = 6C$, and since by the convexity of the l.s.c. functional

$$\Lambda \mapsto E\Big(\int_0^T |\Lambda_s^p-\Lambda_s|^2\,ds\Big),$$

we have

$$E\Big(\int_0^T |\Lambda_s^p - \Lambda_s|^2\, ds\Big) \leqslant \varliminf_{q\to\infty} E\Big(\int_0^T |\Lambda_s^p - \Lambda_s^q|^2\, ds\Big).$$

It follows clearly that

$$E\Big(\int_0^T |\Lambda_s^p - \Lambda_s|^2\, ds\Big) \to 0 \quad \text{as } p \to \infty.$$

We consider now the sequence (J^p). This sequence is increasing. Then there exists $\{J_t\colon 0 \leqslant t \leqslant T\}$ valued in $\bar{R}_+$ such that, for all t, $J_t^p \uparrow J_t$. We now claim that $J_T < \infty$ a.s. In fact, we have

$$J_T^p = \theta_0^p - \eta_T - \int_0^T \tilde{F}_p(s, \theta_s^p, \Lambda_s^p)\, ds + \int_0^T \Lambda_s^p\, dW_s.$$

Then

$$E(J_T^p) = \theta_0^p - E(\eta_T) - E\Big(\int_0^T \tilde{F}_p(s, \theta_s^p, \Lambda_s^p)\, ds\Big) \leqslant \theta_0^p - E(\eta_T) + AT + CE\Big(\int_0^T |\Lambda_s^p|^2\, ds\Big).$$

Since $(\|\Lambda^p\|_2)_p$ is bounded by a constant K, we deduce that

$$\sup_{p\in N} E(J_T^p) \leqslant \theta_0^p - E(\eta_T) + AT + K^2\, CT.$$

Since

$$E(J_T) = \lim_{p\to\infty} E(J_T^p) \leqslant \theta_0^p - E(\eta_T) + AT + K^2\, CT,$$

we conclude that J_T is integrable, and consequently a.s. finite. Thus $\{J_t\colon 0 \leqslant t \leqslant T\}$ is clearly an increasing process and $J_0 = 0$. Finally, it remains to verify (d) of Definition 1. Let $p < q$. Then we have

$$\theta_t^p - \theta_t^q = \int_t^T \big(\tilde{F}_p(s, \theta_s^p, \Lambda_s^p) - \tilde{F}_q(s, \theta_s^q, \Lambda_s^q)\big)\, ds - \int_t^T (\Lambda_s^p - \Lambda_s^q)\, dW_s + \int_t^T (dJ_s^p - dJ_s^q),$$

the last term of the right-hand side being less than or equal to 0. Consequently,

$$\begin{aligned} &E\big(\sup_{0\leqslant t\leqslant T} |\theta_t^p - \theta_t^q|\big) \\ &\quad \leqslant E\Big(\int_0^T |\tilde{F}_p(s, \theta_s^p, \Lambda_s^p) - \tilde{F}_q(s, \theta_s^q, \Lambda_s^q)|\, ds\Big) + E\Big(\sup_{0\leqslant t\leqslant T} \Big|\int_0^T (\Lambda_s^p - \Lambda_s^q)\, dW_s\Big|\Big). \end{aligned}$$

Extracting if necessary a subsequence still denoted by (Λ^n), we may assume without loss of generality that (Λ^n) converges a.s. to Λ and $\tilde{\Lambda} = \sup_n |\Lambda^n| \in H^2$.

Since $\tilde{F}_p \downarrow F$ is continuous, by Dini's theorem we see that $\tilde{F}_p \to F$ uniformly on compact sets. Then taking a subsequence we get

$$\tilde{F}_p(s, \theta_s^p, \Lambda_s^p) \to F(s, \theta_s, \Lambda_s) \quad \text{as } p \to \infty,$$

$$\tilde{F}_q(s, \theta_s^q, \Lambda_s^q) \to F(s, \theta_s, \Lambda_s) \quad \text{as } q \to \infty.$$

Since $|\tilde{F}_p(s, \theta_s^p, \Lambda_s^p)| \leqslant A + C|\Lambda_s^p|^2 \leqslant A + C\tilde{\Lambda}_s^2$, it follows by the dominated convergence theorem that

$$E\Big(\int_0^T |\tilde{F}_p(s, \theta_s^p, \Lambda_s^p) - \tilde{F}_q(s, \theta_s^q, \Lambda_s^q)|\, ds\Big) \to 0 \quad \text{as } p, q \to \infty.$$

Now, by the Burkholder–Davis–Gundy inequality, we get

$$E\Big(\sup_{0 \leqslant t \leqslant T} \Big|\int_0^T (\Lambda_s^p - \Lambda_s^q)\, dW_s\Big|\Big)^2 \leqslant DE\Big(\int_0^T |\Lambda_s^p - \Lambda_s^q|^2\, ds\Big) \to 0 \quad \text{as } p, q \to \infty.$$

Consequently,

$$E\Big(\sup_{0 \leqslant t \leqslant T} |\theta_t^p - \theta_t^q|\Big) \to 0 \;\; \boldsymbol{P}\text{-a.s.} \quad \text{as } p, q \to \infty.$$

This implies that $\theta^p \to \theta$ uniformly in t, where θ is a continuous process, and finally J is also a continuous process. It remains to show that

$$\int_0^T (\theta_t - \eta_t)\, dJ_t = 0.$$

Since

$$\int_0^T (\theta_t - \eta_t)\, dJ_t = \int_0^T (\theta_t - \theta_t^p)\, dJ_t + \int_0^T (\theta_t^p - \eta_t)\, dJ_t,$$

$$\theta^p \downarrow \theta, \quad \text{and} \quad \int_0^T (\theta_t - \theta_t^p)\, dJ_t \leqslant 0,$$

we obtain

$$\int_0^T (\theta_t - \eta_t)\, dJ_t \leqslant \int_0^T (\theta_t^p - \eta_t)\, dJ_t = \int_0^T (\theta_t^p - \eta_t)(dJ_t - dJ_t^p)$$

and

$$\Big|E\Big(\int_0^T (\theta_t^p - \eta_t)(dJ_t - dJ_t^p)\Big)\Big| \leqslant KE(J_T - J_T^p) \quad \text{for all } p \in N.$$

The dominated convergence theorem gives

$$E\Big(\int_0^T (\theta_t^p - \eta_t)(dJ_t - dJ_t^p)\Big) \to 0 \quad \text{as } p \to \infty.$$

Then

$$\int_0^T (\theta_t - \eta_t)\, dJ_t = 0 \text{ a.s.},$$

which completes the proof. ■

2.3. Existence of a minimal solution.

THEOREM 3. *Under the assumptions* H1–H3 *on* (ξ, f), *the RBSDE* $\mathrm{Eq}(\xi, f)$ (3) *has a minimal bounded solution* (Y_*, Z_*, K_*). *Moreover, for all t we have* $Y_*(t) \leqslant U_t \leqslant U_0$ *a.s., where U is the unique solution of the equation* (4) *with*

$$b = \operatorname{Max}\{\|\xi_T\|_\infty, \operatorname*{ess\,sup}_{(t,\omega)} \xi_t(\omega)\}.$$

If we define $Q_t = \exp(-2CU_t)$, we infer that (Q_t) is the unique solution of the ODE

$$(16) \qquad Q_t = \alpha - \int_t^T \bar{g}(Q_s)\, ds, \qquad \alpha = e^{-2Cb} \leqslant 1,$$

where

$$\bar{g}(x) = 2Cxl\left(\frac{-\ln(x)}{2C}\right) \quad \text{when } x > 0.$$

Then we have $Q_0 = \exp(-2CU_0) \leqslant Q_t \leqslant \alpha$.

Sketch of the proof of Theorem 3. It should be noted that the existence of a minimal solution of the RBSDE $\mathrm{Eq}(\xi, f)$ (3) cannot be directly derived from the existence of a minimal solution because the problem is not symmetric. However, the proof of the existence of a minimal solution is based more or less on similar arguments to those in the case of the maximal solution. In the following, we give the main arguments of this proof.

Making the change of variable $\bar{\theta}_t = \exp(-2CY_t)$ in $\mathrm{Eq}(\xi, f)$ (3) we are led to solving the following RBSDE with upper obstacle $\bar{\eta}$ and coefficient $\bar{F}$:

$$(17) \qquad \bar{\theta}_t = \bar{\eta}_T + \int_t^T \bar{F}(s, \bar{\theta}_s, \bar{\Lambda}_s)\, ds - (\bar{J}_T - \bar{J}_t) - \int_t^T \bar{\Lambda}_s\, dW_s,$$

with $\bar{\eta}_s = \exp(-2C\xi_s)$, $\bar{\Lambda}_s = -2CZ_s\bar{\theta}_s$, $d\bar{J}_s = -2C\exp(-2CY_s)\, dK_s$, and

$$(18) \qquad \bar{F}(s, \omega, x, \lambda) = 2Cx\left[-f\left(s, \omega, \frac{-\ln(x)}{2C}, \frac{-\lambda}{2Cx}\right) - \frac{|\lambda|^2}{4Cx^2}\right].$$

Since $Y_t \geqslant \xi_t$ and K increases only on $(Y_t = \xi_t)$, we have

$$\exp(-2CY_t) = \bar{\theta}_t \leqslant \exp(-2C\xi_t) = \bar{\eta}_t \quad \text{and} \quad \int_0^T (\bar{\theta}_t - \bar{\eta}_t)\, d\bar{J}_t = 0.$$

Consider now

$$\bar{g}(x) = 2Cxl\left(\frac{-\ln(x)}{2C}\right),$$

where l is given by **H3**. Taking

$$b = \operatorname{Max}\{\|\xi_T\|_\infty, \operatorname*{ess\,sup}_{(t,\omega)} \xi_t(\omega)\} \quad \text{and} \quad \alpha = e^{-2Cb},$$

we obtain $Q_0 \leqslant Q_t \leqslant \alpha$, where Q_t is given by (16).

Finally, consider

$$m = \operatorname*{ess\,inf}_{(t,\omega)} \xi_t(\omega) \quad \text{and} \quad \bar{F}_\psi(s, \omega, x, \lambda) = \psi(x)\bar{F}(s, \omega, x, \lambda),$$

where ψ is any smooth function satisfying (13) where $0 < 2K_1 < Q_0$, $K_2/2 > e^{-2Cm}$. Then for all $s, \omega, x > 0$ and λ we have

$$-\bar{g}(x) - \frac{1}{K_1}|\lambda|^2 \leqslant \bar{F}_\psi(s, \omega, x, \lambda) \leqslant \bar{g}(x).$$

Assume that for any ψ there exists $(\bar{\theta}^\psi, \bar{\Lambda}^\psi, \bar{J}^\psi)$ being a maximal solution of the RBSDE associated with coefficient $\bar{F}_\psi$ and upper obstacle $\bar{\eta}$ such that

$$Q_0 \leqslant Q_t \leqslant \bar{\theta}_t^\psi \leqslant \bar{\eta}_t \leqslant e^{-2Cm}.$$

Then we claim that, by arguments as in the proof of Theorem 1, Theorem 3 holds. Thus, to prove Theorem 3 we are led to prove the following result.

PROPOSITION 2.3. *Assume that*

$$\bar{G} = \{\bar{g}\colon (0, \infty) \to \mathbf{R}_+, \textit{for all } \alpha \in]0, 1[, Q_t = \alpha - \int_t^T \bar{g}(Q_s)\, ds \textit{ has a unique solution}\}.$$

Let $\bar{g} \in \bar{G}$ and let $\bar{\eta}$ be a bounded predictable process, continuous on $[0, T[$, such that $\underline{\lim}_{t \to T} \bar{\eta}_t \geqslant \bar{\eta}_T$, which satisfies $\alpha \leqslant \bar{\eta} \leqslant \beta$, $\beta > 1$, $0 < \alpha \leqslant 1$. Suppose that $\bar{F}\colon [0, T] \times \Omega \times \mathbf{R}_+ \times \mathbf{R}^d \to \mathbf{R}$ satisfies for some constant $C > 0$ the following condition:

$$\forall t, \omega, x \geqslant 0, \lambda, \quad -\bar{g}(x) - C|\lambda|^2 \leqslant \bar{F}(t, \omega, x, \lambda) \leqslant \bar{g}(x).$$

Assume also that $\bar{F}$ is $\mathscr{P} \otimes \mathscr{B}(\mathbf{R}^{d+1})$-measurable and $\bar{F}(t, \omega, \cdot, \cdot)$ is continuous for all (t, ω). Then the RBSDE associated with coefficient $\bar{F}$ and upper obstacle $\bar{\eta}$

$$\bar{\theta}_t = \bar{\eta}_T + \int_t^T \bar{F}(s, \omega, \theta_s, \bar{\Lambda}_s)\, ds - (\bar{J}_T - \bar{J}_t) - \int_t^T \bar{\Lambda}_s\, dW_s \tag{19}$$

has a maximal solution $(\bar{\theta}^*, \bar{\Lambda}^*, \bar{J}^*)$ which satisfies $Q_0 \leqslant Q_t \leqslant \bar{\theta}_t^* \leqslant \bar{\eta}_t \leqslant \bar{m}$, where $\bar{m} = \operatorname{ess\,sup}_{(t,\omega)}(\bar{\eta}_t(\omega))$ and where (Q_t) is the unique solution of

$$Q_t = \alpha - \int_t^T \bar{g}(Q_s)\, ds.$$

Proof. We consider for any $0 < r < Q_0, \bar{m} < R$, a smooth function ϱ satisfying (6). We have to prove that the RBSDE associated with upper obstacle $\bar{\eta}$ and coefficient $\bar{\bar{F}}$ defined by $\bar{\bar{F}}(t, \omega, \theta, \lambda) = \bar{F}(t, \omega, \varrho(\theta), \lambda)$ has a unique solution which satisfies $Q_0 \leqslant \bar{\theta}_t \leqslant \bar{m}$, and therefore we can choose it independent of (r, R, ϱ). We proceed in the same way as in the proof of Theorem 2. We define

$$\bar{\bar{F}}_p(t, \omega, \theta, \lambda) = \bar{g}(\varrho(\theta))(1-\kappa_p(\lambda))+\kappa_p(\lambda)\bar{\bar{F}}(t, \omega, \theta, \lambda).$$

Then $\bar{\bar{F}}_p \downarrow \bar{\bar{F}}$ and, for each p, $\bar{\bar{F}}_p$ is a bounded and continuous function of (θ, λ). Therefore, it can be shown, using similar arguments to those used in [10], that there exists a maximal solution $(\bar{\theta}^p, \bar{\Lambda}^p, \bar{J}^p)$ of the RBSDE, associated with upper obstacle $\bar{\eta}$ and coefficient $\bar{\bar{F}}_p$, and we infer from the comparison theorem that $Q \leqslant \bar{\theta}^p \leqslant \bar{m}$.

Since $(\bar{\theta}^p)$ is a decreasing and bounded sequence, we have proved the existence of $\bar{\theta} \in H^2(\boldsymbol{R})$ such that, for all $t \leqslant T$, $\bar{\theta}_t^p \downarrow \bar{\theta}_t$ a.s. Moreover, $\bar{\theta}$ satisfies, for all $t \leqslant T$, $Q_0 \leqslant \bar{\theta} \leqslant \bar{m}$ a.s.

The boundedness of $(\|\bar{\Lambda}^p\|_2)_p$ is obtained by the same argument as before by applying Itô's formula to $\phi(\bar{\theta}_t^p)$, where $\phi(x) = e^{+3Cx}$. Next take a weak convergent subsequence which still will be denoted by $(\bar{\Lambda}^p)$ and let $\bar{\Lambda} \in H^2$ be the weak limit. Applying Itô's formula to

$$\phi(x) = \frac{e^{12Cx}-1}{12C} - x$$

and using the fact that, for $p < q$, $d\bar{J}^p \geqslant d\bar{J}^q$, and then the same arguments as in the proof of Theorem 2, we derive

$$E\Big(\int_0^T |\bar{\Lambda}_s^p - \bar{\Lambda}_s|^2\, ds\Big) \to 0 \quad \text{as } p \to \infty.$$

We consider now the sequence $(\bar{J}^p)$. This sequence is decreasing. Then there exists $\{\bar{J}_t\colon 0 \leqslant t \leqslant T\}$ valued in $\boldsymbol{R}_+$ such that, for all t, $\bar{J}_t^p \downarrow \bar{J}_t$. We have $\bar{J}_T \leqslant \bar{J}_T^0$ a.s., and hence $\bar{J}_T \in L^2$. Now, for $p < q$, we have

$$\bar{\theta}_t^p - \bar{\theta}_t^q = \int_t^T (\bar{\bar{F}}_p(s, \bar{\theta}_s^p, \bar{\Lambda}_s^p) - \bar{\bar{F}}_q(s, \bar{\theta}_s^q, \bar{\Lambda}_s^q))\, ds - \int_t^T (\bar{\Lambda}_s^p - \bar{\Lambda}_s^q)\, dW_s - \int_t^T (d\bar{J}_s^p - d\bar{J}_s^q).$$

Using the fact that, for $p < q$, $d\bar{J}^p \geqslant d\bar{J}^q$, and then applying the same arguments as in the proof of Theorem 2, we obtain

$$E\big(\sup_{0 \leqslant t \leqslant T} |\bar{\theta}_t^p - \bar{\theta}_t^q|\big) \to 0 \ \ \boldsymbol{P}\text{-a.s.} \quad \text{as } p, q \to \infty.$$

This implies that $\bar{\theta}^p \to \bar{\theta}$ uniformly in t, where $\bar{\theta}$ is a continuous process, and finally $\bar{J}$ is also a continuous process. It remains to show that

$$\int_0^T (\bar{\theta}_t - \bar{\eta}_t)\, d\bar{J}_t = 0.$$

This can be obtained by using the fact that (see [13], p. 465), at least for a subsequence,

$$0 = \int_0^T (\bar{\theta}_t^p - \bar{\eta}_t)\, d\bar{J}_t^p \to \int_0^T (\bar{\theta}_t - \bar{\eta}_t)\, d\bar{J}_t \quad \text{as } p \to \infty.$$

The proof is completed. ∎

3. UNIQUENESS AND CHARACTERIZATION OF THE SOLUTION OF THE RBSDE AS THE VALUE FUNCTION OF AN OPTIMAL STOPPING PROBLEM

3.1. Uniqueness, characterization, comparison. In this section, we will suppose as in [6] that the coefficient f is locally Lipschitz continuous and has a quadratic growth in z in a strong sense, that means, the partial derivatives of f have a linear growth. More precisely, the assumption (H3) will be replaced by the following assumption:

H4. For each constant $M > 0$, there exists a constant $C > 0$ satisfying for all $(t, y, z) \in [0, T] \times [-M, M] \times \boldsymbol{R}^d$:

(i) $$|f(t, y, z)| \leqslant C(1+|z|^2) \text{ a.s.},$$

(ii) $$\left|\frac{\partial f}{\partial z}(t, y, z)\right| \leqslant C(1+|z|) \text{ a.s.},$$

(iii) $\forall \varepsilon > 0$, $\exists C_\varepsilon$ such that

$$\frac{\partial f}{\partial y}(t, y, z) \leqslant C_\varepsilon + \varepsilon |z|^2 \text{ a.s.},$$

where the partial derivatives are taken in the sense of distributions.

Remark 2. Recall that H4 (i), (ii), (iii) are the assumptions under which the comparison theorem and the uniqueness of the solution hold for a non-reflected BSDE with quadratic growth (see [6]).

Remark 3. Note that if

$$\frac{\partial f}{\partial y}(t, y, z) \leqslant a(1+|z|), \quad \text{where } a > 0,$$

then the assumption (iii) is satisfied.

In the following, we show that under the assumptions H1, H2 and H4, a solution (Y_t) of the RBSDE Eq(ξ, f) corresponds to the value function of an optimal stopping time problem. This result generalizes the result obtained in [6] in the case of a Lipschitz coefficient.

For each $t \in [0, T]$, let us denote by T_t the set of stopping times τ such that $\tau \in [t, T]$ a.s.

For each $\tau \in T_t$, we will denote by $\big(X_s(\tau, \xi_\tau), \pi_s(\tau, \xi_\tau), t \leqslant s \leqslant \tau\big)$ the (unique) solution of the BSDE associated with terminal time τ, terminal condition ξ_τ and coefficient f. We derive easily the following property:

PROPOSITION 3.1 (Characterization). *Suppose that the assumptions* H1, H2 *and* H4 *are satisfied. Suppose that* (Y, Z, K) *is a solution of the reflected BSDE* $\mathrm{Eq}(\xi, f)$. *Then, for each* $t \in [0, T]$,

$$Y_t = X_t(D_t, \xi_{D_t}) = \operatorname*{ess\,sup}_{\tau \in T_t} X_t(\tau, \xi_\tau), \tag{20}$$

where $D_t = \inf\{u \geqslant t;\ Y_u = \xi_u\}$.

Proof. The proof is based on the same arguments as in the case of a Lipschitz coefficient. Since $Y_{D_t} = \xi_{D_t}$ and since the process K is constant on $[t, D_t]$, we derive easily that $(Y_s, t \leqslant s \leqslant D_t)$ is a solution of the BSDE associated with terminal time D_t, terminal condition ξ_{D_t} and coefficient f, that is

$$Y_t = X_t(D_t, \xi_{D_t}).$$

It remains now to show that $Y_t \geqslant X_t(\tau, \xi_\tau)$ for each $\tau \in T_t$. Fix $\tau \in T_t$. Note that on the interval $[t, \tau]$ the pair (Y_s, Z_s) satisfies

$$-dY_s = f(s, Y_s, Z_s)\,ds + dK_s - Z_s\,dW_s. \tag{21}$$

In other words, the pair $(Y_s, Z_s, t \leqslant s \leqslant D_t)$ is a solution of the BSDE associated with terminal time τ, terminal condition Y_τ and coefficient $f(s, y, z) + dK_s$. Since $f(s, y, z) + dK_s \geqslant f(s, y, z)$ and since $Y_\tau \geqslant \xi_\tau$, the comparison theorem for quadratic BSDE's gives

$$Y_t \geqslant X(\tau, \xi_\tau),$$

and the proof is completed. ∎

Note that Proposition 3.1 gives clearly the uniqueness of the solution:

COROLLARY 1 (Uniqueness). *Suppose that the assumptions* H1, H2, H3 *and* H4 *are satisfied. Then there exists a unique solution of the RBSDE* $\mathrm{Eq}(\xi, f)$.

Moreover, from Proposition 3.1 we derive easily the following comparison theorem for quadratic RBSDE's:

PROPOSITION 3.2 (Comparison). *Let* ξ^1, ξ^2 *be two obstacle processes (satisfying* H1) *and let* f^1, f^2 *be two coefficients (satisfying* H2 *and* H3) *such that* f^1 *or* f^2 *satisfies* H4. *Let* (Y^1, Z^1, K^1) *(respectively,* (Y^2, Z^2, K^2)) *be a solution of the RBSDE* $\mathrm{Eq}(\xi^1, f^1)$ *(respectively,* $\mathrm{Eq}(\xi^2, f^2)$) *and assume that*

(1) $\xi^1 \leqslant \xi^2$ *a.s.*,

(2) $f^1(t, y, z) \leqslant f^2(t, y, z)$, $t \in [0, T]$, $(y, z) \in \boldsymbol{R} \times \boldsymbol{R}^d$.

Then $Y_t^1 \leqslant Y_t^2$ *for all* $t \in [0, T]$ *a.s.*

Proof. Case 1. Suppose that f^2 satisfies H4. The proof is based on Proposition 3.1 and on the comparison theorem for quadratic BSDE's.

For each $\tau \in T_t$, let us denote by $X^2(\tau, \xi_\tau^2)$ the unique solution on $[t, \tau]$ of the BSDE associated with (τ, ξ_τ^2, f^2).

Let us introduce the stopping time $D_t^1 = \inf\{u \geqslant t;\ Y_u^1 = \xi_u^1\}$. Recall that since the process K^1 is constant on $[t, D_t^1]$, we derive easily that

$$Y_t^1 = X_t^1(D_t^1, \xi_{D_t}^1),$$

where $X_t^1(D_t^1, \xi_{D_t}^1)$ is a solution of the BSDE associated with $(D_t^1, \xi_{D_t^1}^1, f^1)$. The classical comparison theorem for quadratic BSDE's gives

$$X_t^1(D_t^1, \xi_{D_t^1}^1) \leqslant X_t^2(D_t^1, \xi_{D_t^1}^2) \text{ a.s.}$$

Hence, by Proposition 3.1,

$$Y_t^1 \leqslant \operatorname*{ess\,sup}_{\tau \in T_t} X_t^2(\tau, \xi_\tau^2) = Y_t^2 \text{ a.s.},$$

and the proof is completed.

Case 2. If f^1 (but not f^2) satisfies H4, the proof is very similar and uses the characterization of the minimal solution given in Proposition 3.3. ■

3.2. Characterization of the minimal and maximal solutions. In the following proposition, we state that under the weaker assumption H3, Proposition 3.1 still holds but for the maximal (respectively, minimal) solutions of the RBSDE's. More precisely:

PROPOSITION 3.3. *Let ξ and f be, respectively, an obstacle process and a coefficient (satisfying* H1, H2, *and* H3). *Let (Y^*, Z^*, K^*) (respectively, (Y_*, Z_*, K_*)) be the maximal (respectively, minimal) solution of the RBSDE* Eq(ξ, f). *For each $\tau \in T_t$, let us denote by $X^*(\tau, \xi_\tau)$ (respectively, $X_*(\tau, \xi_\tau)$) the maximal (respectively, minimal) solution of the quadratic BSDE associated with terminal time τ, terminal condition ξ_τ, and coefficient f. Then, for each $t \in [0, T]$,*

$$Y_t^* = X_t^*(D_t^*, \xi_{D_t^*}) = \operatorname*{ess\,sup}_{\tau \in T_t} X^*(\tau, \xi_\tau) \tag{22}$$

and

$$Y_*(t) = \operatorname*{ess\,sup}_{\tau \in T_t} X_*(\tau, \xi_\tau)(t), \tag{23}$$

where $D_t^* = \inf\{u \geqslant t;\ Y_u^* = \xi_u\}$.

Remark 4. Let $D_*(t) = \inf\{u \geqslant t;\ Y_*(u) = \xi_u\}$. It can be noted that $Y_*(t) = X_t(D_*(t), \xi_{D_*(t)})$, where $X(D_*(t), \xi_{D_*(t)})$ is a solution (but not necessarily the minimal solution) of the BSDE associated with terminal time $D_*(t)$, terminal condition $\xi_{D_*(t)}$, and coefficient f.

Before giving the proof, let us state the following lemma:

LEMMA 3.4. *Let f be defined on $[0, T] \times \Omega \times \mathbf{R} \times \mathbf{R}^d$, which is $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^{d+1})$--measurable, continuous with respect to y, z, linear increasing, i.e., there exists a constant $K < \infty$ such that*

$$|f(t, \omega, y, z)| \leqslant K(1+|y|+|z|).$$

Let (ξ_t) be an adapted real process belonging to H^2, continuous on $[0, T[$, such that $\limsup_{t\to T} \xi_t \leqslant \xi_T$. Let (Y^*, Z^*, K^*) (respectively, (Y_*, Z_*, K_*)) be the maximal (respectively, minimal) solution of RBSDE Eq(ξ, f). *Then the equalities* (22) *and* (23) *hold a.s.*

Proof of Lemma 3.4. First, since the process K^* (respectively, K_*) is constant on $[t, D_t^*]$ (respectively, $[t, D_*(t)]$), we have clearly

$$Y_t^* = X_t(D_t^*, \xi_{D_t^*}), \qquad Y_*(t) = X_t\big(D_*(t), \xi_{D_*(t)}\big), \tag{24}$$

where $X(D_t^*, \xi_{D_t^*})$ (respectively, $X\big(D_*(t), \xi_{D_*(t)}\big)$) is a solution of the BSDE associated with terminal time D_t^* (respectively, $D_*(t)$), terminal condition $\xi_{D_t^*}$ (respectively, $\xi_{D_*(t)}$), and coefficient f.

Let us show now that, for any fixed $t \in [0, T]$, the equalities (22) hold.

Let (f_n) be a sequence of Lipschitz functions approximating f from above. Let (Y^n, Z^n, K^n) be the unique solution associated with the RBSDE Eq(ξ, f_n) and for each $\tau \in T_t$ let us denote by $X^n(\tau, \xi_\tau)$ a solution of the quadratic BSDE associated with terminal time τ, terminal condition ξ_τ, and coefficient f_n.

By the comparison theorem for quadratic BSDE's, for each $n \in \mathbf{N}$, for each $\tau \in T_t$,

$$X_t^n(\tau, \xi_\tau) \geqslant X_t^*(\tau, \xi_\tau) \text{ a.s.},$$

and hence, by taking the supremum over T_t, for each $n \in \mathbf{N}$,

$$Y_t^n \geqslant \operatorname*{ess\,sup}_{\tau\in T_t} X^*(\tau, \xi_\tau)(t) \text{ a.s.}$$

Consequently, since $Y_t^* = \lim_{n\to\infty} \downarrow Y_t^n$, we have

$$Y_t^* \geqslant \operatorname*{ess\,sup}_{\tau\in T_t} X^*(\tau, \xi_\tau)(t) \text{ a.s.}$$

Moreover, since $Y_t^* = X_t(D_t^*, \xi_{D_t^*}) \leqslant X_t^*(D_t^*, \xi_{D_t^*})$, the equalities (22) follow clearly.

Let us show now that, for any fixed $t \in [0, T]$, the equality (23) holds.

Let $(\bar{f}_n)$ be the sequence of Lipschitz functions approximating f from below (see [10]). Let $(\bar{Y}^n, \bar{Z}^n, \bar{K}^n)$ be the unique solution associated with the RBSDE Eq$(\xi, \bar{f}_n)$, and for each $\tau \in T_t$ let us denote by $\bar{X}^n(\tau, \xi_\tau)$ the solution of the quadratic BSDE associated with terminal time τ, terminal condition ξ_τ, and coefficient $\bar{f}^n$.

By the comparison theorem for quadratic BSDE's, for each $n \in N$, for each $\tau \in T_t$,

$$\bar{X}_t^n(\tau, \xi_\tau) \leqslant X_*(\tau, \xi_\tau)(t) \text{ a.s.,}$$

and hence, by taking the supremum over T_t, for each $n \in N$,

$$\bar{Y}_t^n \leqslant \operatorname*{ess\,sup}_{\tau \in T_t} X_*(\tau, \xi_\tau)(t) \text{ a.s.}$$

Consequently, since $Y_*(t) = \lim_{n \to \infty} \uparrow \bar{Y}_t^n$, we have

$$Y_*(t) \leqslant \operatorname*{ess\,sup}_{\tau \in T_t} X_*(\tau, \xi_\tau)(t) \text{ a.s.}$$

It remains now to show the converse inequality. Suppose that we have showed that there exists a sequence $(\tau_p, p \in N)$ such that a.s.

$$\operatorname*{ess\,sup}_{\tau \in T_t} X_*(\tau, \xi_\tau)(t) = \lim_{p \to \infty} \uparrow X_*(\tau_p, \xi_{\tau_p})(t). \tag{25}$$

Then, we infer clearly that, for all n and p,

$$Y_*(t) \geqslant \bar{Y}_t^n \geqslant \bar{X}_t^n(\tau_p, \xi_{\tau_p}).$$

Hence, by letting n tend to ∞, we derive that, for all p,

$$Y_*(t) \geqslant \lim_{n \to \infty} \bar{X}_t^n(\tau_p, \xi_{\tau_p}) = X_*(\tau_p, \xi_{\tau_p})(t).$$

Then, by letting p tend to ∞, we obtain

$$Y_*(t) \geqslant \operatorname*{ess\,sup}_{\tau \in T_t} X_*(\tau, \xi_\tau)(t).$$

It remains now to show that there exists a sequence $(\tau_p, p \in N)$ such that the equality (25) holds.

First, note that the family of random variables $\{X_*(\tau, \xi_\tau)(t), \tau \in T_t\}$ is stable by supremum. More precisely, for any $\tau_1, \tau_2 \in T_t$, there exists $\tau \in T_t$ such that

$$X_*(\tau, \xi_\tau)(t) = X_*(\tau_1, \xi_{\tau_1})(t) \vee X_*(\tau_2, \xi_{\tau_2})(t)$$

(just consider $\tau := \tau_1$ on $\{X_*(\tau_1, \xi_{\tau_1})(t) \geqslant X_*(\tau_2, \xi_{\tau_2})(t)\}$ and $\tau := \tau_2$ otherwise). Using this property, we derive that there exists a sequence $(\tau_p, p \in N)$ such that the equality (25) holds. ∎

Proof of Proposition 3.3. Let us show that the equalities (22) and (23) hold. By making the change of variables $\theta_t = \exp(2CY_t^*)$ as in the proof of Theorem 1 we are led to state a similar property concerning the process θ. Then, applying Lemma 3.4 to θ_p for $p \in N$, and applying similar arguments to those used in the proof of Lemma 3.4 to the decreasing sequences $(\tilde{F}^p)$ and (θ_p), we obtain the desired result. Furthermore, the equality (23) can also be obtained by using the change of variable and the approximation introduced in the proof of Theorem 3. ∎

4. STABILITY OF RBSDE's

We state the following theorem:

THEOREM 4. *Let $(\xi^p)_{p\in N}$, ξ be a family of obstacles and $(f^p)_{p\in N}, f$ a family of coefficients satisfying the assumptions* H1, H2, H3, H4 *and such that*:

(a) *there exists a constant $b>0$ such that, for each p,*

$$|\xi_t^p| \leqslant b \text{ a.s.}, \quad t\in[0, T];$$

(b) *there exists a function l of the form $l(y) = a(1+|y|)$ with $a>0$ and a constant C such that, for each p,*

$$|f^p(t, y, z)| \leqslant l(y) + C|z|^2 \text{ a.s.}, \qquad 0\leqslant t\leqslant T,\ y\in \boldsymbol{R},\ z\in \boldsymbol{R}^d;$$

(c) *the sequence (f^p) converges to f locally uniformly on $[0, T]\times \boldsymbol{R}\times \boldsymbol{R}^d$ and the sequence (ξ^p) converges a.s. to ξ.*

For each p, let (Y^p, Z^p, K^p) be the unique solution of the RBSDE $\mathrm{Eq}(\xi^p, f^p)$. *Then the sequence (Y^p) converges to Y uniformly on $[0, T]$, (Z^p) converges to Z in H^2, and (K^p) converges to K uniformly on $[0, T]$, where (Y, Z, K) is the unique solution of the RBSDE* $\mathrm{Eq}(\xi, f)$.

Proof. Step 1. Suppose first that the sequences (f^p) and (ξ^p) are increasing. By the comparison theorem, this implies that the sequence (Y^p) is also increasing. Note that, since the obstacles are not the same, the sequence (K^p) is not monotonic, which makes the problem more difficult than in the case of the same obstacle, studied more or less in the proofs of Theorems 1 and 3.

The idea is to make an exponential change of variable as in the proof of Theorem 1 or Theorem 3. In fact, the good one is $\bar{\theta} = e^{-2Cy}$ since, as we will see, it makes the associated increasing processes sufficiently integrable. Thus, making the change of variable $\bar{\theta}_t^p = \exp(-2CY_t^p)$ in $\mathrm{Eq}(\xi^p, f^p)$, we are led to solving the following RBSDE with upper obstacle $\bar{\eta}^p$ and coefficient $\bar{F}^p$:

$$\bar{\theta}_t^p = \bar{\eta}_T^p + \int_t^T \bar{F}^p(s, \bar{\theta}_s^p, \bar{\Lambda}_s^p)\,ds - (\bar{J}_T^p - \bar{J}_t^p) - \int_t^T \bar{\Lambda}_s^p\,dW_s, \tag{26}$$

with $\bar{\eta}_s^p = \exp(-2C\xi_s^p)$, $\bar{\Lambda}_s^p = -2CZ_s^p\bar{\theta}_s^p$, $d\bar{J}_s^p = -2C\exp(-2CY_s^p)\,dK_s^p$, and

$$\bar{F}^p(s, \omega, x, \lambda) = 2Cx\left[-f^p\left(s, \omega, \frac{-\ln(x)}{2C}, \frac{-\lambda}{2Cx}\right) - \frac{|\lambda|^2}{4Cx^2}\right]$$

with $\bar{\theta}_t^p \leqslant \bar{\eta}_t^p$, and $\int_0^T(\bar{\theta}_t^p - \bar{\eta}_t^p)\,d\bar{J}_t^p = 0$.

Consider now

$$\bar{g}(x) = 2Cxl\left(\frac{-\ln(x)}{2C}\right), \qquad m = \operatorname{ess}\inf_{(t,\omega,p)} \xi_t^p(\omega), \qquad \alpha = e^{-2Cb},$$

and Q_t given by (16). Note that, for each p, $Q_0 \leqslant \bar{\theta}_t^p \leqslant e^{-2Cm}$ a.s. Then for each p, for all s, ω, $x \in [Q_0, e^{-2Cm}]$, and λ we have

$$-\bar{g}(x) - C|\lambda|^2 \leqslant \bar{F}^p(s, \omega, x, \lambda) \leqslant \bar{g}(x),$$

where C is a constant (independent of p). Since $(\bar{\theta}^p)$ is a decreasing and bounded sequence, we have proved the existence of $\bar{\theta} \in H^2(\boldsymbol{R})$ such that, for all $t \leqslant T$, $\bar{\theta}_t^p \downarrow \bar{\theta}_t$ a.s. Moreover, $\bar{\theta}$ satisfies, for all $t \leqslant T$, $Q_0 \leqslant \bar{\theta} \leqslant e^{-2Cm}$ a.s.

LEMMA 4.1. *There exists $\bar{\Lambda} \in H^2(\boldsymbol{R}^d)$ and a bounded adapted process $\bar{J}$ such that $\bar{\theta}^p$ converges to $\bar{\theta}$ a.s. uniformly on $[0, T]$, $\bar{J}^p$ converges to $\bar{J}$ a.s. uniformly on $[0, T]$, $\bar{\Lambda}^p$ converges to $\bar{\Lambda} \in H^2(\boldsymbol{R}^d)$, and $(\bar{\theta}, \bar{\Lambda}, \bar{J})$ is a solution of the RBSDE associated with upper obstacle $\bar{\eta} := e^{-2C\xi}$ and coefficient $\bar{F}$ defined by the formula* (18).

End of the proof of Step 1. The desired result follows clearly by considering

$$Y_t = \frac{-\operatorname{Log}\bar{\theta}_t}{2C}, \qquad Z_t = \frac{-\bar{\Lambda}_t}{2C\bar{\theta}_t} \qquad \text{and} \qquad dK_t = \frac{-\exp(2CY_t)}{2C}\, d\bar{J}_t. \ \blacksquare$$

Proof of Lemma 4.1. The boundedness of $(\|\bar{\Lambda}^p\|_2)_p$ is obtained by the same argument as in the proof of Proposition 2.3. Next take a weak convergent subsequence which still will be denoted by $(\bar{\Lambda}^p)$ and let $\bar{\Lambda} \in H^2$ be the weak limit. Applying Itô's formula to

$$\phi(x) = \frac{e^{12Cx} - 1}{12C} - x$$

and taking the expected value, we get for $p < q$:

$$\begin{aligned} &E\big(\phi(\bar{\theta}_0^p - \bar{\theta}_0^q)\big) + \frac{1}{2} E\Big(\int_0^T \phi''(\bar{\theta}_s^p - \bar{\theta}_s^q)\,|\bar{\Lambda}_s^p - \bar{\Lambda}_s^q|^2\, ds\Big) \\ &\quad = E\big(\phi(\bar{\eta}_T^p - \bar{\eta}_T^q)\big) + E\Big(\int_0^T \phi'(\bar{\theta}_s^p - \bar{\theta}_s^q)\{\bar{F}^p(s, \bar{\theta}_s^p, \bar{\Lambda}_s^p) - \bar{F}^q(s, \bar{\theta}_s^q, \bar{\Lambda}_s^q)\}\, ds\Big) \\ &\qquad - E\Big(\int_0^T \phi'(\bar{\theta}_s^p - \bar{\theta}_s^q)\, d(\bar{J}_s^p - \bar{J}_s^q)\Big). \end{aligned}$$

Note that here the sequence of increasing processes $(\bar{J}^p)$ is no longer monotonic (because the obstacles are not equal). However, we have

$$(27) \qquad -\int_0^T \phi'(\bar{\theta}_s^p - \bar{\theta}_s^q)\, d(\bar{J}_s^p - \bar{J}_s^q) \leqslant \int_0^T \phi'(\bar{\theta}_s^p - \bar{\theta}_s^q)\, d\bar{J}_s^q \leqslant \int_0^T \phi'(\bar{\eta}_s^p - \bar{\eta}_s^q)\, d\bar{J}_s^q,$$

because $(\bar{\theta}_s^q - \bar{\eta}_s^q)\, d\bar{J}_s^q = 0$, $\bar{\theta}_s^p \leqslant \bar{\eta}_s^p$ and ϕ' and ϕ'' are positive on $\boldsymbol{R}^+$. Then, by Cauchy–Schwartz's inequality, we derive

$$-E\Big(\int_0^T \phi'(\bar{\theta}_s^p - \bar{\theta}_s^q)\, d(\bar{J}_s^p - \bar{J}_s^q)\Big) \leqslant E\Big(\sup_{0 \leqslant s \leqslant T} \phi'(\bar{\eta}_s^p - \bar{\eta}_s^q)^2\Big)^{1/2} E\big((\bar{J}_T^q)^2\big)^{1/2}.$$

We now claim that the sequence $(\bar{J}_T^p)$ is bounded in L^2. In fact, we have

$$\bar{J}_T^p = -\bar{\theta}_0^p + \bar{\eta}_T^p + \int_0^T \bar{F}^p(s, \bar{\theta}_s^p, \bar{\Lambda}_s^p)\, ds - \int_0^T \bar{\Lambda}_s^p\, dW_s, \tag{28}$$

which implies that

$$0 \leqslant \bar{J}_T^p \leqslant \bar{\eta}_T^p + AT - \int_0^T \bar{\Lambda}_s^p\, dW_s, \quad \text{where } A = \max_{\varrho_0 \leqslant \theta \leqslant e^{-2Cm}} \bar{g}(\theta).$$

Consequently, there exists a constant K such that, for each p, $E(\bar{J}_T^p)^2 \leqslant K$. Then, this with the same arguments as in the proof of Theorem 2 and Proposition 2.3 shows that

$$E\Big(\int_0^T |\bar{\Lambda}_s^p - \bar{\Lambda}_s|^2\, ds\Big) \to 0 \quad \text{as } p \to \infty.$$

We will now prove that

$$E\big(\sup_{0 \leqslant t \leqslant T} (\bar{\theta}_t^p - \bar{\theta}_t^q)^2\big) \to 0 \quad \text{as } p, q \to \infty. \tag{29}$$

Applying Itô's formula to $(\bar{\theta}_t^p - \bar{\theta}_t^q)^2$, using the same argument as for the inequality (27) and then taking the supremum over t, we get for $p < q$

$$\begin{aligned}\sup_t (\bar{\theta}_t^p - \bar{\theta}_t^q)^2 \leqslant{}& (\bar{\eta}_T^p - \bar{\eta}_T^q)^2 + 2\int_0^T (\bar{\theta}_s^p - \bar{\theta}_s^q)\, |\bar{F}^p(s, \bar{\theta}_s^p, \bar{\Lambda}_s^p) - \bar{F}^q(s, \bar{\theta}_s^q, \bar{\Lambda}_s^q)|\, ds \\ &+ 2\int_0^T (\bar{\eta}_s^p - \bar{\eta}_s^q)\, d\bar{J}_s^q + 2\sup_t \Big|\int_t^T (\bar{\theta}_s^p - \bar{\theta}_s^q)(\bar{\Lambda}_s^p - \bar{\Lambda}_s^q)\, dW_s\Big|.\end{aligned}$$

Taking the expectation, using the fact that the sequence $\bar{\theta}^p$ is bounded and applying Cauchy–Schwartz's inequality and Burkholder–Davis–Gundy's inequality, we can derive that

$$\begin{aligned}E\big(\sup_t (\bar{\theta}_t^p - \bar{\theta}_t^q)^2\big) \leqslant{}& E(\bar{\eta}_T^p - \bar{\eta}_T^q)^2 + CE\Big(\int_0^T |\bar{F}^p(s, \bar{\theta}_s^p, \bar{\Lambda}_s^p) - \bar{F}^q(s, \bar{\theta}_s^q, \bar{\Lambda}_s^q)|\, ds\Big) \\ &+ 2E\big(\sup_t (\bar{\eta}_t^p - \bar{\eta}_t^q)^2\big)^{1/2} K^{1/2} + CE\Big(\int_0^T |\bar{\Lambda}_s^p - \bar{\Lambda}_s|^2\, ds\Big)^{1/2}.\end{aligned}$$

Extracting if necessary a subsequence still denoted by $(\bar{\Lambda}^p)$, we may assume without loss of generality that $(\bar{\Lambda}^p)$ converges a.s. to $\bar{\Lambda}$, $\tilde{\Lambda} = \sup_p |\Lambda^p| \in H^2$

and $\int_0^t \bar{\Lambda}_s^p \, dW_s$ converges a.s. uniformly in t to $\int_0^t \bar{\Lambda}_s \, dW_s$. By using the fact that

$$|\bar{F}^p(s, \bar{\theta}_s^p, \bar{\Lambda}_s^p) - \bar{F}^q(s, \bar{\theta}_s^q, \bar{\Lambda}_s^q)| \leqslant 2(A + C\tilde{\Lambda}_s^2)$$

and Lebesgue's theorem, we infer easily that

$$E\Big(\int_0^T |\bar{F}^p(s, \bar{\theta}_s^p, \bar{\Lambda}_s^p) - \bar{F}^q(s, \bar{\theta}_s^q, \bar{\Lambda}_s^q)| \, ds\Big) \to 0 \quad \text{ as } p, q \to \infty.$$

Then it follows clearly that

$$E\big(\sup_t (\bar{\theta}_t^p - \bar{\theta}_t^q)^2\big) \to 0 \quad \text{ as } p, q \to \infty.$$

Consequently, extracting if necessary a subsequence, we see that $\bar{\theta}^p$ converges to $\bar{\theta}$ uniformly in t a.s. Also, by using the equation (28), we can easily derive that $\bar{J}^p$ converges to $\bar{J}$ uniformly in t a.s., where $\bar{J}$ is a continuous increasing process. Applying the same arguments as in the proof of Proposition 2.3, we can prove that

$$\int_0^T (\bar{\theta}_t - \bar{\eta}_t) \, d\bar{J}_t = 0,$$

from which the desired result follows. ∎

Step 2. Using exactly the same arguments as in Step 1, one can show that the result holds if the sequences (f^n) and (ξ^n) are supposed to be decreasing.

Step 3. We now consider the general case, that is, when the sequences (f^n) and (ξ^n) are not necessarily monotonic. In this case, the result can be derived by using exactly the same arguments as in [6] (see the proof of Theorem 2.8). ∎

5. RBSDE AND OBSTACLE PROBLEM

In this section, we will show that in the Markovian case the reflected BSDE is a solution of an obstacle problem for PDE's.

Let b: $[0, T] \times \boldsymbol{R}^d \to \boldsymbol{R}^d$ and σ: $[0, T] \times \boldsymbol{R}^d \to \boldsymbol{R}^{d \times d}$ be continuous mappings, which are Lipschitz with respect to their second variable, uniformly with respect to $t \in [0, T]$ and satisfying, for a positive constant K,

$$|b(t, x)| + |\sigma(t, x)| \leqslant K(1 + |x|).$$

For each $(t, x) \in [0, T] \times \boldsymbol{R}^d$, let $\{X_s^{t,x};\ t \leqslant s \leqslant T\}$ be the unique $\boldsymbol{R}^d$-valued solution of the SDE:

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) \, dr + \int_t^s \sigma(r, X_r^{t,x}) \, dW_r.$$

We suppose now that the data (ξ, f) of the RBSDE take the form:

$$\xi_T = g(X_T^{t,x}),$$

$$f(s, y, z) = f(s, X_s^{t,x}, y, z),$$

$$\xi_s = h(s, X_s^{t,x}), \qquad s < T,$$

where g, f and h are determined as follows:

$g \in C(\boldsymbol{R}^d)$ and is bounded.

f: $[0, T] \times \boldsymbol{R}^d \times \boldsymbol{R} \times \boldsymbol{R}^d \to \boldsymbol{R}$ is jointly continuous and satisfies the following assumptions (which correspond to H1–H4):

$$|f(t, x, y, z)| \leqslant l(y) + C|z|^2, \quad \text{where } l(y) = a(1+|y|) \text{ with } a > 0.$$

Moreover, for each constant $M > 0$ there exists a constant $C > 0$ such that for all $(t, x, y, z) \in [0, T] \times \boldsymbol{R}^n \times [-M, M] \times \boldsymbol{R}^d$:

$$|f(t, x, y, z)| \leqslant C(1+|z|^2), \qquad \left|\frac{\partial f}{\partial z}(t, x, y, z)\right| \leqslant C(1+|z|),$$

$$\forall \varepsilon > 0,\ \exists C_\varepsilon > 0,\ \frac{\partial f}{\partial y}(t, x, y, z) \leqslant C_\varepsilon + \varepsilon|z|^2,$$

where the partial derivatives are taken in the sense of distributions.

h: $[0, T] \times \boldsymbol{R}^d \to \boldsymbol{R}$ is bounded, jointly continuous in t and x. We also assume that $h(T, x) \leqslant g(x)$, $x \in \boldsymbol{R}^d$.

For each $t > 0$, we denote by $\{\mathscr{F}_s^t,\ t \leqslant s \leqslant T\}$ the natural filtration of the Brownian motion $\{W_s - W_t,\ t \leqslant s \leqslant T\}$, augmented by the P-null sets of $\mathscr{F}$. It follows from the results of the above sections that for each (t, x) there exists a unique triple $(Y^{t,x}, Z^{t,x}, K^{t,x})$ of $\{\mathscr{F}_s^t\}$-adapted processes, which solves the RBSDE $\mathrm{Eq}(\xi, f)$.

We now consider the related obstacle problem for a parabolic PDE. Roughly speaking, a solution of the obstacle problem is a function u: $[0, T] \times \boldsymbol{R}^d \to \boldsymbol{R}$ which satisfies:

$$\min\left(u(t, x) - h(t, x),\ -\frac{\partial u}{\partial t}(t, x) - L_t u(t, x) - f\big(t, x, u(t, x), (\nabla u \sigma)(t, x)\big)\right) = 0,$$

(30)

$$(t, x) \in (0, T) \times \boldsymbol{R}^d, \qquad u(T, x) = g(x),\ x \in \boldsymbol{R}^d,$$

where

$$L_t = \frac{1}{2}\sum_{i,j=1}^{d} \big(\sigma\sigma^*(t, x)\big)_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t, x)\frac{\partial}{\partial x_i}.$$

More precisely, we shall consider solutions of (30) in the viscosity sense.

DEFINITION 2. (a) A bounded function $u \in C([0, T] \times \boldsymbol{R}^d)$ is said to be a *viscosity subsolution* of (30) if $u(T, x) \leqslant g(x)$, $x \in \boldsymbol{R}^d$, and if for any point $(t_0, x_0) \in$

$\in (0, T) \times \boldsymbol{R}^d$ and any $\phi \in C^{1,2}([0, T] \times \boldsymbol{R}^d)$ such that $\phi(t_0, x_0) = u(t_0, x_0)$ and $\phi - u$ attains its minimum at (t_0, x_0), then

$$\min\Big(u(t_0, x_0) - h(t_0, x_0),\ -\frac{\partial \phi}{\partial t}(t_0, x_0) - L_t \phi(t_0, x_0) - f\big(t, x, u(t_0, x_0), (\nabla \phi \sigma)(t_0, x_0)\big)\Big) \leqslant 0.$$

In other words, if $u(t_0, x_0) > h(t_0, x_0)$, then

$$-\frac{\partial \phi}{\partial t}(t_0, x_0) - L_t \phi(t_0, x_0) - f\big(t, x, u(t_0, x_0), (\nabla \phi \sigma)(t_0, x_0)\big) \leqslant 0.$$

(b) A bounded function $u \in C([0, T] \times \boldsymbol{R}^d)$ is said to be a *viscosity supersolution* of (30) if $u(T, x) \geqslant g(x)$, $x \in \boldsymbol{R}^d$, and if for any point $(t_0, x_0) \in (0, T) \times \boldsymbol{R}^d$ and any $\phi \in C^{1,2}([0, T] \times \boldsymbol{R}^d)$ such that $\phi(t_0, x_0) = u(t_0, x_0)$ and $\phi - u$ attains its maximum at (t_0, x_0), then

$$\min\Big(u(t_0, x_0) - h(t_0, x_0),\ -\frac{\partial \phi}{\partial t}(t_0, x_0) - L_t \phi(t_0, x_0) - f\big(t, x, u(t_0, x_0), (\nabla \phi \sigma)(t_0, x_0)\big)\Big) \geqslant 0.$$

In other words, we have both $u(t_0, x_0) \geqslant h(t_0, x_0)$ and

$$-\frac{\partial \phi}{\partial t}(t_0, x_0) - L_t \phi(t_0, x_0) - f\big(t, x, u(t_0, x_0), (\nabla \phi \sigma)(t_0, x_0)\big) \geqslant 0.$$

(c) $u \in C([0, T] \times \boldsymbol{R}^d)$ is said to be a *viscosity solution* of (30) if it is both a viscosity subsolution and supersolution.

We now define

$$u(t, x) := Y_t^{t,x}, \qquad (t, x) \in [0, T] \times \boldsymbol{R}^d, \tag{31}$$

which is a deterministic quantity.

Lemma 5.1. $u \in C([0, T] \times \boldsymbol{R}^d)$.

Proof. Note first that for each (t, x) the solution $(Y_s^{t,x})$ can also be defined on the whole interval $[0, T]$ by putting $Y_s^{t,x} = Y_t^{t,x}$ for $s \leqslant t$. Note now that, for each sequence (t_n, x_n) which converges to (t, x), the sequence of coefficients $f(s, X_s^{t_n, x_n}, y, z)$ converges locally uniformly to $f(s, X_s^{t,x}, y, z)$. Applying Theorem 4, we derive that the sequence $(Y_s^{t_n, x_n})$ converges to $(Y_s^{t,x})$ uniformly on $[0, T]$, which gives the desired result. ∎

Theorem 5. *The function u, defined by* (31), *is a viscosity solution of the obstacle problem* (30).

Proof. We give a direct proof of this property. First, let us show that u is a subsolution of (30). Let $(t_0, x_0) \in (0, T) \times \boldsymbol{R}^d$ and $\phi \in C^{1,2}([0, T] \times \boldsymbol{R}^d)$ be such that $\phi(t_0, x_0) = u(t_0, x_0)$ and $\phi(t, x) \geqslant u(t, x)$ for all $(t, x) \in [0, T] \times \boldsymbol{R}^n$.

Suppose now that $u(t_0, x_0) > h(t_0, x_0)$ and that

$$-\frac{\partial \phi}{\partial t}(t_0, x_0) - L_t \phi(t_0, x_0) - f\big(t, x, \phi(t_0, x_0), (\nabla\phi\sigma)(t_0, x_0)\big) > 0.$$

Let us show that this leads to a contradiction. Note that, by continuity, we can suppose that there exists $\varepsilon > 0$ and there exists $\eta_\varepsilon > 0$ (small enough) such that: for each (t, x) such that $t_0 \leqslant t \leqslant t_0 + \eta_\varepsilon$ and $|x - x_0| \leqslant \eta_{\bar\varepsilon}$, we have $u(t, x) \geqslant h(t, x) + \varepsilon$ and

$$(32) \qquad -\frac{\partial \phi}{\partial t}(t, x) - L_t \phi(t, x) - f\big(t, x, \phi(t, x), (\nabla\phi\sigma)(t, x)\big) \geqslant \varepsilon.$$

Let τ be a stopping time defined by

$$(33) \qquad \tau := (t_0 + \eta_\varepsilon) \wedge \inf\{s \geqslant t_0;\ |X_s^{t_0,x_0} - x_0| > \eta_\varepsilon\}.$$

Note that, for all $s \in [t_0, \tau]$,

$$u(s, X_s^{t_0,x_0}) \geqslant h(s, X_s^{t_0,x_0}) + \varepsilon.$$

Consequently, the process $(K_s^{t_0,x_0})$ is constant on $[t_0, \tau]$, and hence $(Y_s^{t_0,x_0}, s \in [t_0, \tau])$ is a solution of the classical BSDE associated with terminal value $Y_\tau^{t_0,x_0}$ and coefficient f.

Let us now compare the solution $Y_s^{t_0,x_0}$ to $\phi(s, X_s^{t_0,x_0})$ on the interval $[t_0, \tau]$. Now, Itô's lemma applied to $\phi(s, X_s^{t_0,x_0})$ gives

$$d\phi(s, X_s^{t_0,x_0}) = (\partial\phi/\partial t + L_t \phi)(s, X_s^{t_0,x_0})\,ds + (\nabla\phi\sigma)(s, X_s^{t_0,x_0})\,dW_s.$$

Consequently, the pair $\big(\phi(s, X_s^{t_0,x_0}), (\nabla\phi\sigma)(s, X_s^{t_0,x_0});\ s \in [t_0, \tau]\big)$ is a solution of the BSDE associated with terminal value $\phi(\tau, X_\tau^{t_0,x_0})$ and coefficient $-(\partial\phi/\partial t + L_t\phi)(s, X_s^{t_0,x_0})$. Now, by the assumption (32),

$$-(\partial\phi/\partial t + L_s \phi)(s, X_s^{t_0,x_0}) - f\big(s, X_s^{t_0,x_0}, \phi(s, X_s^{t_0,x_0}), (\nabla\phi\sigma)(s, X_s^{t_0,x_0})\big) \geqslant \varepsilon.$$

Also, $\phi(\tau, X_\tau^{t_0,x_0}) \geqslant u(\tau, X_\tau^{t_0,x_0}) = Y_\tau^{t_0,x_0}$. Consequently, the comparison theorem for classical quadratic BSDE's implies

$$\phi(t_0, x_0) > \phi(t_0, X_{t_0}^{t_0,x_0}) - \varepsilon(\tau - t_0) \geqslant Y_{t_0}^{t_0,x_0} = u(t_0, x_0),$$

which leads to a contradiction.

Let us show that u is a viscosity supersolution of (30). Let $(t_0, x_0) \in (0, T) \times \boldsymbol{R}^d$ and let $\phi \in C^{1,2}([0, T] \times \boldsymbol{R}^d)$ be such that $\phi(t_0, x_0) = u(t_0, x_0)$ and $\phi(t, x) \leqslant u(t, x)$ for all t, x.

First, note that since the solution $(Y_s^{t_0,x_0})$ of the RBSDE stays above the obstacle, we have clearly

$$u(t_0, x_0) \geqslant h(t_0, x_0).$$

Suppose now that

$$-\frac{\partial \phi}{\partial t}(t_0, x_0)-L_t \phi(t_0, x_0)-f\big(t, x, \phi(t_0, x_0), (\nabla\phi\sigma)(t_0, x_0)\big) < 0.$$

Let us show that this leads to a contradiction. Again, by continuity, we can suppose that there exists $\varepsilon > 0$ and there exists $\eta_\varepsilon > 0$ (small enough) such that: for each (t, x) such that $t_0 \leqslant t \leqslant t_0+\eta_\varepsilon$ and $|x-x_0| \leqslant \eta_\varepsilon$, we have

$$-\frac{\partial \phi}{\partial t}(t, x)-L_t \phi(t, x)-f\big(t, x, \phi(t, x), (\nabla\phi\sigma)(t, x)\big) \leqslant -\varepsilon. \tag{34}$$

Let τ be the stopping time defined as above by (33) and let us now compare the solution $Y_s^{t_0,x_0}$ to $\phi(s, X_s^{t_0,x_0})$ on the interval $[t_0, \tau]$. Now, as we have seen above, the pair $\big(\phi(s, X_s^{t_0,x_0}), (\nabla\phi\sigma)(s, X_s^{t_0,x_0}); s\in[t_0, \tau]\big)$ is a solution of the BSDE associated with terminal value $\phi(\tau, X_\tau^{t_0,x_0})$ and coefficient $-(\partial\phi/\partial t+L_t\phi)(s, X_s^{t_0,x_0})$.

Note now that the process $(Y^{t_0,x_0}; s\in[t_0, \tau])$ is a solution of the classical quadratic BSDE associated with terminal condition $Y_\tau^{t_0,x_0} = u(\tau, X_\tau^{t_0,x_0})$ and with coefficient $f(s, X_s^{t_0,x_0}, y, z)+dK_s^{t_0,x_0}$. Now, by the assumption (34),

$$-(\partial\phi/\partial t+L_s\phi)(s, X_s^{t_0,x_0})-f\big(s, X_s^{t_0,x_0}, \phi(s, X_s^{t_0,x_0}), (\nabla\phi\sigma)(s, X_s^{t_0,x_0})\big)$$
$$-dK_s^{t_0,x_0} \leqslant -\varepsilon.$$

Also, $\phi(\tau, X_\tau^{t_0,x_0}) \leqslant u(\tau, X_\tau^{t_0,x_0}) = Y_\tau^{t_0,x_0}$. Consequently, this with the comparison theorem for classical quadratic BSDE's implies that

$$\phi(t_0, x_0) < \phi(t_0, x_0)+\varepsilon(\tau-t_0) \leqslant Y_{t_0}^{t_0,x_0} = u(t_0, x_0),$$

which leads to a contradiction and the proof is completed. ∎

For the strong comparison principle between viscosity subsolution and supersolution we need in addition the following hypothesis of f:

$$\left|\frac{\partial f}{\partial x}(t, x, y, z)\right| \leqslant C(1+|z|^2).$$

Theorem 6 (Strong comparison principle for the obstacle problem). *Under the above hypothesis, if U is a viscosity subsolution and V is a viscosity supersolution of the obstacle problem* (30), *then* $U(t, x) \leqslant V(t, x)$ *for all* $t, x\in[0, T]\times \boldsymbol{R}^d$.

From this theorem we derive clearly the following corollary:

Corollary 2 (Uniqueness). *Under the above hypothesis, there exists a unique viscosity solution of the obstacle problem* (30).

Proof of Theorem 6. Let U and V be such as indicated in the theorem. The proof consists in showing that

$$M = \sup_{\substack{x\in\mathbf{R}^d\\ t\in[0,T]}} \{U(t, x) - V(t, x)\},$$

the supremum of the bounded function $U-V$, is negative.

Let us consider, for $\varepsilon, \eta > 0$,

$$\psi^{\varepsilon,\eta}(t, s, x, y) = U(t, x) - V(s, y) - \frac{|x-y|^2}{\varepsilon^2} - \frac{|t-s|^2}{\varepsilon^2} - \eta^2(|x|^2 + |y|^2).$$

Let $M^{\varepsilon,\eta}$ be a maximum of $\psi^{\varepsilon,\eta}$ (over t, s, x, y). As U and V are bounded, this maximum is reached at some point $(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta})$. We define

$$\Phi_1(t, x) = V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) + \frac{|x-y^{\varepsilon,\eta}|^2}{\varepsilon^2} + \frac{|t-s^{\varepsilon,\eta}|^2}{\varepsilon^2} + \eta^2(|x|^2 + |y^{\varepsilon,\eta}|^2),$$

$$\Phi_2(s, y) = U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - \frac{|x^{\varepsilon,\eta}-y|^2}{\varepsilon^2} - \frac{|t^{\varepsilon,\eta}-s|^2}{\varepsilon^2} - \eta^2(|x^{\varepsilon,\eta}|^2 + |y|^2).$$

As $(t, x) \mapsto (U-\Phi_1)(t, x)$ reaches its maximum at $(t^{\varepsilon,\eta}, x^{\varepsilon,\eta})$ and U is a subsolution, we have

- either $t^{\varepsilon,\eta} = T$, and then $U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) \leqslant g(x^{\varepsilon,\eta})$,
- or $t^{\varepsilon,\eta} \neq T$, and then

$$(35)\quad \min\Big(U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - h(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}),$$
$$\frac{\partial\Phi_1}{\partial t}(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - L_t\Phi_1(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - f\big(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}), (\nabla\Phi_1\sigma)(t^{\varepsilon,\eta}, x^{\varepsilon,\eta})\big)\Big) \leqslant 0.$$

As $(s, y) \mapsto (\Phi_2 - V)(s, y)$ reaches its maximum at $(s^{\varepsilon,\eta}, y^{\varepsilon,\eta})$ and V is a supersolution, we have

- either $s^{\varepsilon,\eta} = T$, and then $V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \leqslant g(y^{\varepsilon,\eta})$,
- or $s^{\varepsilon,\eta} \neq T$, and then

$$(36)\quad \min\Big(V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) - h(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}),$$
$$\frac{\partial\Phi_2}{\partial t}(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) - L_t\Phi_2(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) - f\big(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), (\nabla\Phi_2\sigma)(s^{\varepsilon,\eta}, y^{\varepsilon,\eta})\big)\Big) \geqslant 0.$$

We recall some notation and results from [6].

If a sequence $(a^{\varepsilon,\eta})$ satisfies

$$\limsup_{\eta\to 0}\,[\limsup_{\varepsilon\to 0} a^{\varepsilon,\eta}] = \liminf_{\eta\to 0}\,[\liminf_{\varepsilon\to 0} a^{\varepsilon,\eta}]$$

we denote this common limit by $\lim\limits_{\varepsilon\ll\eta\to 0} a^{\varepsilon,\eta}$.

It is shown in [6] that

(37) $$\lim_{\varepsilon \ll \eta \to 0} M^{\varepsilon,\eta} = M.$$

Furthermore, using the fact that $\psi^{\varepsilon,\eta}(t^{\varepsilon,\eta}, s^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \geqslant \psi^{\varepsilon,\eta}(0, 0, 0, 0)$ and that U and V are bounded, one can easily show that there exists a constant C such that

(38) $$|x^{\varepsilon,\eta}-y^{\varepsilon,\eta}|+|t^{\varepsilon,\eta}-s^{\varepsilon,\eta}| \leqslant C\varepsilon, \qquad |x^{\varepsilon,\eta}|, |y^{\varepsilon,\eta}| \leqslant C/\eta.$$

As $[0, T]$ is bounded and, by (38), extracting a subsequence, if necessary, we may suppose that for each η the sequences $(t^{\varepsilon,\eta})_\varepsilon$ and $(s^{\varepsilon,\eta})_\varepsilon$ converge to a common limit t^η. By (38) we may also suppose, extracting again, that for each η the sequences $(x^{\varepsilon,\eta})_\varepsilon$ and $(y^{\varepsilon,\eta})_\varepsilon$ converge to a common limit x^η.

1st case. There exists a subsequence of (t^η) such that $t^\eta = T$ for all η (of this subsequence).

As U is lower semicontinuous, for all η and for ε small enough we have

$$U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) \leqslant U(t^\eta, x^\eta)+\eta \leqslant g(x^\eta)+\eta,$$

and as V is upper semicontinuous, for all η and for ε small enough we obtain

$$V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \geqslant V(t^\eta, x^\eta)-\eta \geqslant g(x^\eta)-\eta.$$

Hence

$$U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta})-V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \leqslant 2\eta,$$

and consequently

$$\begin{aligned} M^{\varepsilon,\eta} &= U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta})-V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta})-\frac{|x^{\varepsilon,\eta}-y^{\varepsilon,\eta}|^2}{\varepsilon^2}-\frac{|t^{\varepsilon,\eta}-s^{\varepsilon,\eta}|^2}{\varepsilon^2}-\eta^2(|x^{\varepsilon,\eta}|^2+|y^{\varepsilon,\eta}|^2) \\ &\leqslant U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta})-V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \leqslant 2\eta. \end{aligned}$$

Letting $\varepsilon \to 0$, and then $\eta \to 0$, we see, using (37), that $M \leqslant 0$.

2nd case. There exists a subsequence such that $t^\eta \neq T$, and for all η belonging to this subsequence there exists a subsequence of $(x^{\varepsilon,\eta})_\varepsilon$ such that

$$U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta})-h(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) \leqslant 0.$$

As by (36) we have

$$V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta})-h(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \geqslant 0,$$

it follows that

$$M^{\varepsilon,\eta} \leqslant U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta})-V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) \leqslant h(t^{\varepsilon,\eta}, x^{\varepsilon,\eta})-h(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}).$$

Letting $\varepsilon \to 0$, and then $\eta \to 0$, we see, using (37) again, that $M \leqslant 0$.

Last case. We are left with the case when, for a subsequence of η, we have $t^{\eta} \neq T$ and for all η belonging to this subsequence there exists a subsequence of $(x^{\varepsilon,\eta})_{\varepsilon}$ such that

$$U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - h(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) > 0.$$

Then from (35) we obtain

$$\frac{\partial \Phi_1}{\partial t}(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - L_t \Phi_1(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - f\left(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}), (\nabla\Phi_1 \sigma)(t^{\varepsilon,\eta}, x^{\varepsilon,\eta})\right) \leqslant 0.$$

As the inequality

$$\frac{\partial \Phi_2}{\partial t}(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) - L_t \Phi_2(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}) - f\left(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), (\nabla\Phi_2 \sigma)(s^{\varepsilon,\eta}, y^{\varepsilon,\eta})\right) \geqslant 0$$

is always satisfied in view of (36), this case is the one treated in [6]. Hence, using the arguments of [6], one can show that $M \leqslant 0$, which completes the proof. ∎

6. APPLICATION TO THE PRICING OF AMERICAN OPTIONS IN AN INCOMPLETE MARKET

In the section, we present an application of quadratic RBSDE's to the pricing of American options in an incomplete market. Recall that in [4] El Karoui and Rouge proposed to price a European option in an incomplete market via exponential utility maximization and stated that such a price is a solution of a quadratic BSDE.

More precisely, let us consider a complete market (as in [4]) which contains d securities, whose (invertible) volatility matrix is denoted by σ_t. Suppose that only the first j ones are available for hedging and their volatility matrix is denoted by σ_t^1. The utility function is given by $u(x) = e^{-\gamma x}$, where γ ($\geqslant 0$) corresponds to the risk-aversion coefficient. Let C be a given contingent claim corresponding to an exercise time T; in other terms, C is a bounded $\mathscr{F}_T$-measurable variable. Let $\left(X_t(T, C)\right)$ (denoted also by (X_t)) be the forward price process defined via the exponential utility function as in [4]. By Theorem 5.1 in [4], there exists $z \in H^2(\boldsymbol{R}^d)$ such that the pair (X, z) is a solution of the quadratic BSDE:

$$-dX_t = \left\{-(\eta_t + \sigma_t^{-1} v_t^0) \cdot z_t + \frac{\gamma}{2}|\Pi(z_t)|^2\right\} dt - z_t\, dW_t, \qquad X_T = C,$$

where η is the classical relative risk process, v^0 is a given process (see [4]) and where $\Pi(z)$ denotes the orthogonal projection of z onto the kernel of σ_t^1.

Let us consider now the valuation of an American option $\{\xi_t, 0 \leqslant t \leqslant T\}$ which satisfies the assumptions of an obstacle. Recall that in this case the holder has the right to exercise the option at any stopping time τ between 0 and T.

If he exercises at time τ, then he receives the payoff ξ_τ. The forward price process (Y_t) of such an option is naturally defined by the right-continuous process (which exists) such that, for each t,

$$Y_t = \operatorname{ess\,sup}_{\tau \in T_t} X_t(\tau, \xi_\tau) \text{ a.s.} \tag{39}$$

Applying the previous results on RBSDE's, we derive the following characterization of Y:

PROPOSITION 6.1. *The forward price process Y is the unique solution of the RBSDE associated with obstacle ξ and coefficient f, where*

$$f(t, y, z) = -(\eta_t + \sigma_t^{-1} v_t^0) \cdot z + \frac{\gamma}{2} |\Pi(z)|^2.$$

Moreover, the stopping time $D_t = \inf\{u \geqslant t;\ Y_u = \xi_u\}$ is optimal, that is, $Y_t = X_t(D_t, \xi_{D_t})$ a.s.

Proof. Note first that if the coefficient v^0 is bounded, then the coefficient f satisfies the assumptions H3 and H4. Consequently, the result follows directly from our previous results. Suppose now that the coefficient v^0 is not bounded. Then the coefficient f does not satisfy the assumptions H3 and H4. However, the problem can be solved under another probability. Indeed, let us consider the probability Q^{v^0} (considered in [4]) which minimizes the entropy $h(Q^{v^0} \mid P)$. Recall that Q^{v^0} admits

$$H_T^{v^0} := \exp\left\{-\int_0^T (\eta_s + \sigma_s^{-1} v_s^0)^* \, dW_s - \frac{1}{2}\int_0^T |\eta_s + \sigma_s^{-1} v_s^0|^2 \, ds\right\}$$

as density with respect to P on $\mathscr{F}_T$. Note that there exists a representation theorem for $(\mathscr{F}, Q^{v^0})$-martingales with respect to the $(\mathscr{F}, Q^{v^0})$-Brownian motion

$$W_t^{v^0} := W_t + \int_0^t (\eta_s + \sigma_s^{-1} v_s^0) \, ds.$$

Thus, under Q^{v^0}, the forward price $X(T, C)$ of the European contingent claim C is characterized as the unique solution of the quadratic BSDE associated with Brownian W^{v^0}, terminal condition C and coefficient

$$F(t, z) = \frac{\gamma}{2} |\Pi(z)|^2,$$

which clearly satisfies H3 and H4. Hence, under Q^{v^0}, the forward price Y of the American contingent claim ξ is also clearly characterized as the unique solution of the quadratic RBSDE associated with Brownian W^{v^0}, obstacle ξ and coefficient F. ∎

Remark 5. Note that, as in the case of a European option, the price is increasing with respect to the risk-aversion coefficient γ. Furthermore, as γ goes to infinity, the price tends to the superhedging price (as defined in [2] for a European option and in [7] for an American option).

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