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CHARACTERIZATIONS OF POLYNOMIAL-GAUSSIAN PROCESSES THAT ARE MARKOVIAN

BY

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Abstract. We consider questions of characterizing a stochastic process $\mathscr{X} = (X_t, t \geqslant 0)$ by the properties of the first two conditional moments. Our first result is a new version of the classical P. Lévy characterization theorem for martingales. Next we deal with a characterization of processes without continuous trajectories. We consider a special form of the initial state. Namely, we suppose that the r.v. X_0 has a polynomial-normal distribution (*PND*), i.e. the density of X_0 is the product of a positive polynomial and a normal density.

1. INTRODUCTION

We consider questions of characterizing a stochastic process $\mathscr{X} = (X_t, t \geqslant 0)$ by the properties of the first two conditional moments.

Let $X_{d+1} = (X_0, X_{t_1}, \ldots, X_{t_d})$ denote a $(d+1)$-dimensional random vector for $0 < t_1 < \ldots < t_d$. We shall consider a special form of the initial state. Namely, we suppose that the r.v. X_0 has a *polynomial-normal distribution* (*PND*), i.e. the density of X_0 is the product of a positive polynomial and a normal density. Such a density is called by Evans and Swartz [4] a *polynomial-normal density*. For simplicity we use the symbol *PND* not only for densities but also for the class of r.v.'s with *PND* densities. We investigate the class *PND* using Hermite polynomials. It is known that every polynomial can be represented as a linear combination of Hermite polynomials.

Thus we will consider the r.v. X with density of the following form:

$$f(x) = \frac{1}{\sqrt{2\pi}\,\delta} p_{2l}(x, \delta^2) \exp\left(-\frac{x^2}{2\delta^2}\right), \tag{1.1}$$

where

$$p_{2l}(x, \delta^2) = \sum_{r=0}^{2l} c_r \delta^{-r} H_r(x/\delta)$$

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is a positive polynomial in x of degree $2l$ ($l \geqslant 0$), c_r are parameters, H_r is the Hermite polynomial of degree r, and $\delta > 0$.

The distribution given by (1.1) will be denoted by

$$PND(2l, \delta^2, \boldsymbol{C}_{2l}), \quad \text{where } \boldsymbol{C}_{2l} = (c_0, \ldots, c_{2l}).$$

Let $\tilde{\mathscr{X}} = (\tilde{X}_t, t \geqslant 0)$ be a zero mean Gaussian Markov process with non-degenerate distributions. It is known (see for example Timoszyk [9] and Adler et al. [1]) that its covariance function is the product $k(s, t) = E(\tilde{X}_s \tilde{X}_t) = \varphi(s)\psi(t)$ for $s \leqslant t$. We suppose that $\psi(t) \neq 0$.

We construct a stochastic process $\mathscr{X} = (X_t, t \geqslant 0)$ in such a way that various properties of Gaussian Markov processes are preserved but the one-dimensional distributions of $\mathscr{X}$ are *PND*.

Namely, we define a *polynomial-Gaussian Markov process* (*PGMP*) in the following way:

$$\mathscr{X} = (X_t, t \geqslant 0) = \left(\tilde{X}_t + \frac{\psi(t)}{\psi(0)}(X_0 - \tilde{X}_0),\ t \geqslant 0\right), \tag{1.2}$$

where $X_0 \sim PND(2l, k(0, 0), \boldsymbol{C}_{2l})$ and $X_0, \tilde{X}$ are independent.

Define

$$g(t) = \varphi(t)/\psi(t).$$

It is evident that g is an increasing function.

The density of the one-dimensional distribution of $\mathscr{X}$ has the following form (see Plucińska and Bisińska [7]):

$$f(x) = P_{2l}(x, k(t, t))\tilde{f}(x) = P_{2l}(x, k(t, t))\frac{1}{\sqrt{2\pi k(t, t)}}\exp\left\{-\frac{x^2}{2k(t, t)}\right\}, \tag{1.3}$$

where

$$P_{2l}(x, k(t, t)) = \sum_{r=0}^{2l} c_r \left[\frac{\psi(t)}{\psi^2(0)\varphi(t)}\right]^{r/2} H_r\left(\frac{x}{\sqrt{k(t, t)}}\right). \tag{1.4}$$

The aim of this paper is to give some characterizations of *PGMP* based on properties of conditional moments. For simplicity we use the symbol *PGMP* not only for processes but also for the class of suitable distributions (analogically to *PND*).

Our first result is a new version of the classical P. Lévy characterization theorem for martingales.

Next we consider a characterization of processes without continuous trajectories.

We use conditioning with respect to the past σ-field $\mathscr{F}_s = \sigma(X_w : w \leqslant s)$ of the stochastic process $\mathscr{X}$. We shall also use σ-fields that allow some insight into the future, namely, we put

$$\mathscr{F}_{s,u} = \sigma(X_w : w \leqslant s \text{ or } w = u), \quad s < u.$$

We are going to prove the following propositions.

PROPOSITION 1.1. *Let* $\mathscr{X} = (X_t, t \geqslant 0)$ *be a square-integrable stochastic process with continuous trajectories and* $EX_0^n < \infty$ *for* $n = 1, 2, \ldots$ *Suppose there exist a positive definite function* $k(s, t) = \varphi(s)\psi(t)$, $s \leqslant t$, *and a positive polynomial* P_{2l} *of the form* (1.4) *such that for all* $s \leqslant t$

$$E(X_t \mid \mathscr{F}_s) = \frac{\psi(t)}{\psi(s)} X_s, \tag{1.5}$$

$$\mathrm{Var}(X_t \mid \mathscr{F}_s) \text{ is non-random,} \tag{1.6}$$

and for some $t > 0$

$$E(X_0 \mid X_t = y) = \frac{\varphi(0)}{\varphi(t)} y + [\varphi(t)\psi(0) - \varphi(0)\psi(t)] \frac{(\partial/\partial y) P_{2l}(y, k(t, t))}{P_{2l}(y, k(t, t))}. \tag{1.7}$$

Then $\mathscr{X}$ *is a PGMP and its densities are given by* (2.2).

The next proposition concerns characterizations of processes without continuous trajectories.

PROPOSITION 1.2. *Let* $\mathscr{X} = (X_t, 0 \leqslant t \leqslant T)$ *be a square-integrable stochastic process such that* (2.3), (2.4), (2.6), (2.7) *hold, there exists a polynomial* P_{2l} *of the form* (1.4) *such that for some* $t > 0$ *formula* (1.7) *is satisfied, and* $EX_0^n < \infty$ *for* $n = 1, 2, \ldots$ *Then* $\mathscr{X}$ *is a PGMP and its densities are given by* (2.2).

For convenience of the reader we state here the classical P. Lévy characterization theorem.

THEOREM 1.3. *If a stochastic process* $\mathscr{X} = (X_t, t \in [0, 1])$ *has continuous trajectories, is square-integrable and*

$$E(X_t \mid \mathscr{F}_s) = X_s, \qquad \mathrm{Var}(X_t \mid \mathscr{F}_s) = t - s, \qquad X_0 = 0,$$

then $\mathscr{X}$ *is the Wiener process.*

If we put $\psi(t) = 1$, $\varphi(t) = t$, $P_{2l} = \text{const}$ in Proposition 1.1, we get Theorem 1.3. The condition $X_0 = 0$ in Theorem 1.3 is essential. In some papers (e.g. Bryc [2], Theorem 8.2.1) this condition is omitted; that is a mistake.

In our considerations condition (1.7) is essential. It concerns some conditioning of the initial state with respect to the future. It is similar to the conditioning of the parameters with respect to the future.

Characterizations of distributions by their posterior conditional expectations are considered by various authors. For example, mixtures of normal or gamma densities $f(x \mid Q)$ with respect to the parameter Q are considered by Cacoullos and Papageorgiou [3]. In the present paper instead of the parameter Q we have the initial state X_0.

2. PROPERTIES OF *PGMP*

We shall use Propositions 2.1–2.4 given in Plucińska and Bisińska [7].

PROPOSITION 2.1. *If $X \sim PND\,(2l, \delta^2, C_{2l})$, then the moments are given by the formulas*

$$EH_n\left(\frac{X}{\delta}\right) = \begin{cases} c_r\,\delta^{-n}\,n! & \text{for } n = 1, \ldots, 2l, \\ 0 & \text{for } n > 2l. \end{cases} \tag{2.1}$$

PROPOSITION 2.2. *The sum of two independent r.v.'s with polynomial-normal distribution has a polynomial-normal distribution.*

PROPOSITION 2.3. *Let $\mathscr{X}$ be a PGMP. Then for every $d \geqslant 1$, $0 \leqslant t_1 < \ldots < t_d$, the density function of the vector $\boldsymbol{X}_d = (X_{t_1}, \ldots, X_{t_d})$ has the following form:*

$$\begin{aligned} f_d(\boldsymbol{x}_d) &= \frac{\sqrt{\det \mathscr{A}}}{(2\pi)^{d/2}} P_{2l}\big(x_1, k(t_1, t_1)\big) \exp\left\{-\tfrac{1}{2}(\mathscr{A}\boldsymbol{x}_d, \boldsymbol{x}_d)\right\} \\ &= P_{2l}\big(x_1, k(t_1, t_1)\big)\, \tilde{f}(\boldsymbol{x}_d), \end{aligned} \tag{2.2}$$

where

$$P_{2l}\big(x_1, k(t_1, t_1)\big) = \sum_{r=0}^{2l} c_r \left[\frac{\psi(t_1)}{\psi^2(0)\,\varphi(t_1)}\right]^{r/2} H_r\left(\frac{x_1}{\sqrt{\varphi(t_1)\,\psi(t_1)}}\right)$$

$k(t, t) = \varphi(t)\psi(t)$, $\boldsymbol{x}_d = (x_1, \ldots, x_d) \in R^d$, $\tilde{f}$ is the density of the d-dimensional normal distribution $\mathscr{N}(0, \mathscr{K})$, $\mathscr{K} = [\varphi(t_i)\psi(t_j)]_{1 \leqslant i \leqslant j \leqslant d}$, $\mathscr{A} = \mathscr{K}^{-1}$. The distribution given by (2.2) will be denoted by

$$PND_d(2l, \mathscr{K}, C_{2l}).$$

PROPOSITION 2.4. *Let the density of $\boldsymbol{X}_d$ be given by (2.2). Then the characteristic function of $\boldsymbol{X}_d$ has the following form:*

$$\varphi(\zeta_d) = E\exp\left[i(\zeta_d, \boldsymbol{X}_d)\right] = \sum_{r=0}^{2l} c_r (i\eta)^r \exp\left[-\frac{1}{2}\sum_{r,s=1}^{d} k(t_r, t_s)\,\zeta_r\,\zeta_s\right],$$

where

$$\eta = \frac{1}{k(0, 0)}\left[\zeta_1\,k(0, t_1) + \ldots + \zeta_d\,k(0, t_d)\right].$$

It is evident that $c_0 = 1$.

We are going to prove the following proposition.

PROPOSITION 2.5. *The conditional moments of PGMP for $s < t < u$ are given by the formula*

$$E(X_t \mid \mathscr{F}_s) = \frac{\psi(t)}{\psi(s)} X_s, \tag{2.3}$$

(2.4) $$E(X_t \mid \mathscr{F}_{s,u}) = \frac{g(u)-g(t)}{g(u)-g(s)} \frac{\psi(t)}{\psi(s)} X_s + \frac{g(t)-g(s)}{g(u)-g(s)} \frac{\psi(t)}{\psi(u)} X_u,$$

(2.5) $$E(X_s \mid X_t = y) = \frac{\varphi(s)}{\varphi(t)} y + \frac{\psi(s)}{\psi(t)} [g(t)-g(s)] \frac{(\partial/\partial y) P_{2l}(y, k(t, t))}{P_{2l}(y, k(t, t))},$$

(2.6) $$\operatorname{Var}(X_t \mid \mathscr{F}_s) \text{ is non-random,}$$

(2.7) $$\operatorname{Var}(X_t \mid \mathscr{F}_{s,u}) \text{ is non-random.}$$

The proof of Proposition 2.5 will be based on Lemma 2.1 below. First we introduce some notation.

Let us put

$$Y_t = X_t/\psi(t), \qquad \tilde{Y}_t = \tilde{X}_t/\psi(t).$$

Then for $s \leqslant t$

$$E(\tilde{Y}_s \tilde{Y}_t) = g(s).$$

Next we change the time. Taking into account that g is an increasing function we introduce the new processes

$$\mathscr{Z} = (Z_t, t \geqslant 0), \qquad \tilde{\mathscr{Z}} = (\tilde{Z}_t, t \geqslant 0),$$

where

$$Z_t = Y_{g^{-1}(t+\alpha)}, \qquad \tilde{Z}_t = \tilde{Y}_{g^{-1}(t+\alpha)}, \qquad g(g^{-1}(x)) = x, \qquad g^{-1}(\alpha) = 0.$$

It is evident that for $s \leqslant t$

$$\tilde{k}(s, t) = E(\tilde{Z}_s \tilde{Z}_t) = \tilde{\varphi}(s) \tilde{\psi}(t) = s+\alpha,$$

i.e. $\tilde{\psi}(t) = 1$, $\tilde{\varphi}(s) = s+\alpha$. It is obvious that $(\tilde{Z}_t - \tilde{Z}_0, t \geqslant 0)$ is the Wiener process.

Proposition 2.5 for the process

(2.8) $$\mathscr{Z} = (Z_t, t \geqslant 0) = (\tilde{Z}_t - \tilde{Z}_0 + Z_0, \ t \geqslant 0),$$

where $(\tilde{Z}_t, t \geqslant 0)$ is a zero mean Gaussian Markov process with covariance function $\tilde{k}(s, t)$, takes a simpler form, which we name Lemma 2.1.

LEMMA 2.1. *Let $\mathscr{Z} = (Z_t, t \geqslant 0)$ be a PGMP given by* (2.8). *Then the conditional moments for $s < t < u$ are given by*

(2.9) $$E(Z_t \mid \tilde{\mathscr{F}}_s) = Z_s,$$

(2.10) $$E(Z_t \mid \tilde{\mathscr{F}}_{su}) = \frac{u-t}{u-s} Z_s + \frac{t-s}{u-s} Z_u,$$

(2.11) $$E(Z_s \mid Z_t = z) = \frac{s+\alpha}{t+\alpha} z + (t-s) \frac{(\partial/\partial z) \tilde{P}_{2l}(z, t+\alpha)}{P_{2l}(z, t+\alpha)},$$

$$\operatorname{Var}(Z_t \mid \tilde{\mathscr{F}}_s) = t-s, \tag{2.12}$$

$$\operatorname{Var}(Z_t \mid \tilde{\mathscr{F}}_{s,u}) = \frac{(u-t)(t-s)}{u-s}, \tag{2.13}$$

where $\tilde{\mathscr{F}}_s = \sigma(Z_w\colon\ w \leqslant s)$, $\tilde{\mathscr{F}}_{s,u} = \sigma(Z_w\colon\ w \leqslant s \text{ or } w = u)$, *and*

$$\tilde{P}_{2l}(z, t+\alpha) = \sum_{r=0}^{2l} c_r (t+\alpha)^{-r/2} H_r\left(\frac{z}{\sqrt{t+\alpha}}\right). \tag{2.14}$$

Proof of Lemma 2.1. First observe that by definition (2.8) we have

$$Z_t - Z_s = \tilde{Z}_t - \tilde{Z}_s, \tag{2.15}$$

$$Z_t - \frac{u-t}{u-s} Z_s - \frac{t-s}{u-s} Z_u = \tilde{Z}_t - \frac{u-t}{u-s}\tilde{Z}_s - \frac{t-s}{u-s}\tilde{Z}_u. \tag{2.16}$$

Taking into account that $(\tilde{Z}_t - \tilde{Z}_0,\ t \geqslant 0)$ is the Wiener process and formulas (2.15), (2.16) we get (2.9), (2.10), (2.12) and (2.13).

Now we are going to prove (2.11). Taking into account (2.2), (4.1)–(4.4), substituting

$$u = x\sqrt{\frac{t+\alpha}{(s+\alpha)(t-s)}}, \qquad w = z\sqrt{\frac{s+\alpha}{(t+\alpha)(t-s)}}, \qquad C = \sqrt{\frac{t-s}{t+\alpha}} \tag{2.17}$$

and integrating by parts, we get

$$E(Z_s \mid Z_t = z) = \sqrt{\frac{t+\alpha}{2\pi(s+\alpha)(t-s)}}\,\frac{1}{\tilde{P}_{2l}(z, t+\alpha)}$$
$$\times \int \sum_{r=0}^{2l} \frac{c_r x}{(s+\alpha)^{r/2}} H_r\left(\frac{x}{\sqrt{s+\alpha}}\right) \exp\left\{-\frac{x^2}{2(s+\alpha)} - \frac{(z-x)^2}{2(t-s)} + \frac{z^2}{2(t+\alpha)}\right\} dx$$
$$= \sqrt{\frac{(s+\alpha)(t-s)}{2\pi(t+\alpha)}}\,\frac{1}{\tilde{P}_{2l}(z, t+\alpha)} \sum_{r=0}^{2l} \frac{c_r}{(s+\alpha)^{r/2}} \int u H_r(cu) \exp\left\{-\frac{(u-w)^2}{2}\right\} du$$
$$= \sqrt{\frac{(s+\alpha)(t-s)}{2\pi(t+\alpha)}}\,\frac{1}{\tilde{P}_{2l}(z, t+\alpha)} \sum_{r=0}^{2l} \frac{c_r}{(s+\alpha)^{r/2}}$$
$$\times\left[\sqrt{\frac{(t+\alpha)(t-s)}{s+\alpha}}\,\frac{d}{dz} H_r\left(-\frac{z}{\sqrt{t+\alpha}}\right) + \sqrt{\frac{s+\alpha}{(t+\alpha)(t-s)}}\, z H_r\left(-\frac{z}{\sqrt{t+\alpha}}\right)\right]$$
$$= z\frac{s+\alpha}{t+\alpha} + (t-s)\frac{(d/dz)\tilde{P}_{2l}(z, t+\alpha)}{\tilde{P}_{2l}(z, t+\alpha)}.$$

Formula (2.11) is thus proved. This completes the proof of Lemma 2.1.

Proof of Proposition 2.5. First observe that by definition (1.3) we have

$$X_t - \frac{\psi(t)}{\psi(s)} X_s = \tilde{X}_t - \frac{\psi(t)}{\psi(s)} \tilde{X}_s, \tag{2.18}$$

$$\begin{aligned} X_t - \frac{g(u)-g(t)}{g(u)-g(s)} \frac{\psi(t)}{\psi(s)} X_s - \frac{g(t)-g(s)}{g(u)-g(s)} \frac{\psi(t)}{\psi(u)} X_u \\ = \tilde{X}_t - \frac{g(u)-g(t)}{g(u)-g(s)} \frac{\psi(t)}{\psi(s)} \tilde{X}_s - \frac{g(t)-g(s)}{g(u)-g(s)} \frac{\psi(t)}{\psi(u)} \tilde{X}_u. \end{aligned} \tag{2.19}$$

Taking into account that $\tilde{X}$ is a Gaussian Markov process and (2.18), (2.19) we get formulas (2.3), (2.4), (2.6) and (2.7). Formula (2.5) follows easily from (2.11). ∎

3. PROOFS OF PROPOSITIONS 1.1 AND 1.2

First we prove a simpler version of Proposition 1.1.

LEMMA 3.1. *Let $\mathscr{Z} = (Z_t, 0 \leqslant t \leqslant T)$ be a square-integrable stochastic process with continuous trajectories and $EZ_0^n < \infty$, $n = 1, 2, \ldots$ Suppose that there exists a polynomial $\tilde{P}_{2l}$ of the form* (2.14) *such that for $s < t$*

$$E(Z_t \mid Z_s) = Z_s, \tag{3.1}$$

$$\operatorname{Var}(Z_t \mid Z_s) = t - s, \tag{3.2}$$

and for some $t > 0$

$$E(Z_0 \mid Z_t = z) = \frac{\alpha}{t+\alpha} z + t \frac{(\partial/\partial z) \tilde{P}_{2l}(z, t+\alpha)}{\tilde{P}_{2l}(z, t+\alpha)}. \tag{3.3}$$

Then $\mathscr{Z}$ is a PGMP.

Proof of Lemma 3.1. The difference between Theorem 1.3 and Lemma 3.1 concerns the initial state Z_0. The assumptions connected with the conditional distributions of $Z_{t_n} \mid Z_{t_1}, \ldots, Z_{t_{n-1}}$ are identical. Thus in virtue of Theorem 1.3 the conditional distributions of $Z_{t_n} \mid Z_{t_1}, \ldots, Z_{t_{n-1}}$ for $t_1 < \ldots < t_n$ are Gaussian. We must only find the distribution of Z_0.

Since $Z_t - Z_0 \sim N(0, t)$, $EZ_0^n < \infty$, and Z_0, $Z_t - Z_0$ are independent, we have $EZ_t^n < \infty$.

Let us write

$$\frac{\partial}{\partial z} \tilde{P}_{2l}(z, t) = \tilde{P}'_{2l}(z, t).$$

Taking into account (4.1)–(4.4) for every natural n we have

$$(3.4)\quad E\left[Z_0 H_n\left(\frac{Z_t}{\sqrt{t+\alpha}}\right)\tilde{P}_{2l}(Z_t, t+\alpha)\right]$$

$$= E\left[H_n\left(\frac{Z_t}{\sqrt{t+\alpha}}\right)\tilde{P}_{2l}(Z_t, t+\alpha)\, E(Z_0 \mid Z_t)\right]$$

$$= EH_n\left(\frac{Z_t}{\sqrt{t+\alpha}}\right)\sum_{r=0}^{2l}\frac{c_r}{(t+\alpha)^{r/2}}H_r\left(\frac{Z_t}{\sqrt{t+\alpha}}\right)\left[\frac{\alpha}{t+\alpha}Z_t + t\frac{\tilde{P}'_{2l}(Z_t, t+\alpha)}{\tilde{P}_{2l}(Z_t, t+\alpha)}\right]$$

$$= EH_n\left(\frac{Z_t}{\sqrt{t+\alpha}}\right)\sum_{r=0}^{2l}\frac{c_r}{(t+\alpha)^{r/2}}$$

$$\times\left\{\frac{\alpha}{\sqrt{t+\alpha}}\left[H_{r+1}\left(\frac{Z_t}{\sqrt{t+\alpha}}\right) + rH_{r-1}\left(\frac{Z_t}{\sqrt{t+\alpha}}\right)\right]\frac{t}{\sqrt{t+\alpha}}rH_{r-1}\left(\frac{Z_t}{\sqrt{t+\alpha}}\right)\right\}$$

$$= EH_n\left(\frac{Z_t}{\sqrt{t+\alpha}}\right)\sum_{r=0}^{2l}\frac{c_r}{(t+\alpha)^{r/2}}$$

$$\times\left\{\frac{\alpha}{\sqrt{t+\alpha}}H_{r+1}\left(\frac{Z_t}{\sqrt{t+\alpha}}\right) + r\sqrt{t+\alpha}\,H_{r-1}\left(\frac{Z_t}{\sqrt{t+\alpha}}\right)\right\}$$

$$= \sum_{r=0}^{2l}\frac{c_r}{(t+\alpha)^{r/2}}\left[\frac{\alpha}{\sqrt{t+\alpha}}\sum_{k=0}^{\min(n,r+1)}k!\binom{n}{k}\binom{r+1}{k}\left(\frac{\alpha}{t+\alpha}\right)^{(n+r+1)/2-k}\right.$$

$$\times EH_{n+r+1-2k}\left(\frac{Z_0}{\sqrt{\alpha}}\right) + \sqrt{t+\alpha}\sum_{k=0}^{\min(n,r+1)}k!\binom{n}{k}\binom{r-1}{k}$$

$$\left.\times\left(\frac{\alpha}{t+\alpha}\right)^{(n+r+1)/2-k} EH_{n+r-1-2k}\left(\frac{Z_0}{\sqrt{\alpha}}\right)\right].$$

On the other hand, the left-hand side of (3.4) can be represented in the following form:

$$(3.5)\quad E\left[Z_0 H_n\left(\frac{Z_t}{\sqrt{t+\alpha}}\right)\tilde{P}_{2l}(Z_t, t+\alpha)\right]$$

$$= E\left\{Z_0 E\left[H_n\left(\frac{Z_t}{\sqrt{t+\alpha}}\right)\tilde{P}_{2l}(Z_t, t+\alpha) \mid Z_0\right]\right\}$$

$$= EZ_0\sum_{r=0}^{2l}\frac{c_r}{(t+\alpha)^{r/2}}\sum_{k=0}^{\min(n,r)}k!\binom{n}{k}\binom{r}{k}\left(\frac{\alpha}{t+\alpha}\right)^{(n+r)/2-k}H_{n+r-2k}\left(\frac{Z_0}{\sqrt{\alpha}}\right)$$

$$= \sum_{r=0}^{2l}\frac{c_r\sqrt{\alpha}}{(t+\alpha)^{r/2}}\sum_{k=0}^{\min(n,r)}k!\binom{n}{k}\binom{r}{k}\left(\frac{\alpha}{t+\alpha}\right)^{(n+r)/2-k}$$

$$\times\left[EH_{n+r+1-2k}\left(\frac{Z_0}{\sqrt{\alpha}}\right) + (n+r-2k)\,EH_{n+r-1-2k}\left(\frac{Z_0}{\sqrt{\alpha}}\right)\right].$$

In view of (3.4) and (3.5) we have

$$\sum_{r=0}^{2l}\sum_{k=0}^{\min(n,r)}\frac{c_r}{(t+\alpha)^{r-k}}\alpha^{(r+1)/2-k}k!\binom{n}{k}\binom{r}{k} \tag{3.6}$$

$$\times\left[EH_{n+r+1-2k}\left(\frac{Z_0}{\sqrt{\alpha}}\right)+(n+r-2k)\,EH_{n+r-1-2k}\left(\frac{Z_0}{\sqrt{\alpha}}\right)\right]$$

$$-\sum_{r=0}^{2l}\sum_{k=0}^{\min(n,r+1)}\frac{c_r}{(t+\alpha)^{r+1-k}}\alpha^{(r+3)/2-k}k!\binom{n}{k}\binom{r+1}{k}EH_{n+r+1-2k}\left(\frac{Z_0}{\sqrt{\alpha}}\right)$$

$$-\sum_{r=0}^{2l}\sum_{k=0}^{\min(n,r-1)}\frac{c_r}{(t+\alpha)^{r-1-k}}\alpha^{(r-1)/2-k}rk!\binom{n}{k}\binom{r-1}{k}EH_{n+r-1-2k}\left(\frac{Z_0}{\sqrt{\alpha}}\right)=0.$$

The left-hand side of (3.6) is a polynomial in $(t+\alpha)^{-1}$. The equation (3.6) must be satisfied for every t. In particular, the constant term (independent of t) must disappear. Thus in view of (3.6) for $n=0$ we get

$$EH_1\left(\frac{Z_0}{\sqrt{\alpha}}\right)=\frac{c_1}{\sqrt{\alpha}}.$$

From (3.6) for $n\geqslant 1$ we get

$$\sum_{r=0}^{n}c_r\alpha^{(1-r)/2}r!\binom{n}{r}\left[EH_{n-r+1}\left(\frac{Z_0}{\sqrt{\alpha}}\right)+(n-r)\,EH_{n-r-1}\left(\frac{Z_0}{\sqrt{\alpha}}\right)\right] \tag{3.7}$$

$$-\sum_{r=0}^{n}c_r\alpha^{(1-r)/2}(r+1)!\binom{n}{r+1}EH_{n-r-1}\left(\frac{Z_0}{\sqrt{\alpha}}\right)$$

$$-\sum_{r=0}^{n}c_r\alpha^{(1-r)/2}r!\binom{n}{r-1}EH_{n-r+1}\left(\frac{Z_0}{\sqrt{\alpha}}\right)$$

$$=\alpha^{1/2}EH_{n+1}\left(\frac{Z_0}{\sqrt{\alpha}}\right)+\sum_{r=0}^{n}c_r\alpha^{(1-r)/2}\frac{n!\,(n-2r+1)}{(n-r+1)!}EH_{n-r+1}\left(\frac{Z_0}{\sqrt{\alpha}}\right)$$

$$.-c_{n+1}(n+1)!\alpha^{-n/2}=0.$$

We now prove by induction on n that

$$EH_r\left(\frac{Z_0}{\sqrt{\alpha}}\right)=c_r\alpha^{-r/2}r!\qquad\text{for } r=1,\ldots,n,\ n<2l. \tag{3.8}$$

We have already settled the case of $n=1$. Suppose that (3.8) holds for some n with $n+1<2l$. Then taking into account (3.7) and substituting $m=n-2r+1$ we get

$$\sum_{r=1}^{n}c_r\alpha^{(1-r)/2}\frac{n-2r+1}{(n-1+r)!}EH_{n-r+1}\left(\frac{Z_0}{\sqrt{\alpha}}\right)=\alpha^{-n/2}\sum_{r=1}^{n}(n-2r+1)\,c_rc_{n-r+1} \tag{3.9}$$

$$=\alpha^{-n/2}\sum_{m=1-n}^{n-1}mc_{(n+1-m)/2}c_{(n+1+m)/2}=\alpha^{-n/2}S_n.$$

We define

$$h(n, m) = mc_{(n+1-m)/2}\, c_{(n+1+m)/2}.$$

Evidently, $h(n, m) = -h(n, -m)$. Thus

$$S_n = 0. \tag{3.10}$$

It follows from (3.7)–(3.10) that

$$EH_{n+1}\left(\frac{Z_0}{\sqrt{\alpha}}\right) = c_{n+1}\,\alpha^{-(n+1)/2}(n+1)!, \tag{3.11}$$

where $n < 2l$.

Now we take $n = 2l$. Then by (3.7) we have

$$0 = \sum_{r=0}^{n} c_r \alpha^{(1-r)/2} r! \binom{2l}{r} EH_{2l-r+1}\left(\frac{Z_0}{\sqrt{\alpha}}\right) - \sum_{r=1}^{n} c_r \alpha^{(1-r)/2} r! \binom{2l}{r-1} EH_{2l-r+1}\left(\frac{Z_0}{\sqrt{\alpha}}\right)$$

$$= \alpha^{1/2} EH_{2l+1}\left(\frac{Z_0}{\sqrt{\alpha}}\right) + (2l)! \sum_{r=1}^{n-2l} c_r \alpha^{(1-2r+1)/2} \frac{2l-2r+1}{(2l-r+1)!} EH_{2l+1-r}\left(\frac{Z_0}{\sqrt{\alpha}}\right)$$

$$= \alpha^{1/2} EH_{2l+1}\left(\frac{Z_0}{\sqrt{\alpha}}\right) + (2l)!\,\alpha^{-l} \sum_{m=1-2l}^{2l-1} mc_{(2l+1-m)/2}\, c_{(2l+1-m)/2}$$

$$= \alpha^{1/2} EH_{2l+1}\left(\frac{Z_0}{\sqrt{\alpha}}\right).$$

Thus

$$EH_{2l+1}\left(\frac{Z_0}{\sqrt{\alpha}}\right) = 0.$$

Now let $n = 2l+1, \ldots, 4l$. Then by (3.7) we have

$$c_0\,\alpha^{1/2} E\left(H_{n+1}\frac{Z_0}{\sqrt{\alpha}}\right) + n!\,\alpha^{-n/2} \times \left[\sum_{r=1}^{n-2l} c_r \alpha^{(1-2r+1)/2} \frac{n-2r+1}{(n-r+1)!} EH_{n-r+1}\left(\frac{Z_0}{\sqrt{\alpha}}\right) + \sum_{r=n-2l+1}^{2l} c_r \alpha^{(1-r+n)/2} \frac{n-2r+1}{(n-r+1)!} E\left(H_{n-r+1}\left(\frac{Z_0}{\sqrt{\alpha}}\right)\right)\right] = 0. \tag{3.12}$$

We now prove by induction on N that

$$E\left(H_r\left(\frac{Z_0}{\sqrt{\alpha}}\right)\right) = 0 \quad \text{for } r = 2l+1, \ldots, N,\ N < 4l. \tag{3.13}$$

The case $N = 2l+1$ is already settled. Suppose that (3.13) holds for some N with $N+1 < 4l$. We substitute $m = n-2+1$. Then by (3.11)–(3.13) we get

$$(3.14) \qquad c_0 \alpha^{1/2} E\left(H_{N+1}\left(\frac{Z_0}{\sqrt{\alpha}}\right)\right) + N!\,\alpha^{-N/2} \sum_{m=-(n-1)}^{n-1} m c_{(n+1+m)/2}\, c_{(n+1-m)/2} = 0.$$

We repeat the considerations given in (3.9) and (3.10). Then, by (3.14),

$$EH_{N+1}\left(\frac{Z_0}{\sqrt{\alpha}}\right) = 0.$$

Thus (3.13) holds.
Analogously we show that

$$EH_r\left(\frac{Z_0}{\sqrt{\alpha}}\right) = 0 \quad \text{for } r > 4l.$$

Finally, all the moments are as in (2.1). By the Carleman criterion the moment problem has a unique solution (see Plucińska [6]). Then

$$Z_0 \sim PND\ (2l, \alpha, C_{2l}),$$

and therefore $\mathscr{X}$ is a $PGMP$. ∎

Proof of Proposition 1.1. It is evident that

$$X_t = \psi(t)\, Z_{g(t+\alpha)}.$$

Therefore Proposition 1.1 is an immediate consequence of Lemma 3.1. ∎

Proof of Proposition 1.2. We use Theorem 1 of Plucińska [5] and Theorem 2.1 of Wesołowski [10]. By these theorems all the conditional distributions $X_{t_n} | X_{t_1}, \ldots, X_{t_{n-1}}$ are Gaussian. Using the methods of Lemma 3.1 we show that $X_0 \sim PND$. Proposition 1.2 is thus proved. ∎

4. APPENDIX

For convenience of the reader we give some formulas for Hermite polynomials taken (after some easy transformations) from the book of Prudnikov et al. [8]:

$$(4.1) \qquad H_{r+1}\left(\frac{x}{v}\right) = \frac{x}{v} H_r\left(\frac{x}{v}\right) - \sqrt{v}\frac{d}{dx} H_r\left(\frac{x}{v}\right), \quad v > 0,$$

$$(4.2) \qquad \frac{1}{\sqrt{2\pi}} \int \left[\frac{d^k}{du^k} H_r(cu)\right] \exp\left\{-\frac{(u-v)^2}{2}\right\} du$$

$$= (1-c^2)^{r/2} \frac{d^k}{dv^k} H_r\left(\frac{cv}{\sqrt{1-c^2}}\right), \quad c^2 < 1,\ k = 0, 1, 2.$$

Let $(X_t, t \geqslant 0)$ be a zero mean Gaussian Markov process, $E(X_s X_t) = EX_s^2 = v_1$, $EX_t^2 = v_2$ for $s < t$. Then

$$(4.3)\quad E\left(H_r\left(\frac{X_t}{\sqrt{v_2}} \mid X_s\right) = x_1\right)$$

$$= \frac{1}{\sqrt{2\pi(v_2 - v_1)}} \int H_r\left(\frac{x_2}{\sqrt{v_2}}\right) \exp\left\{-\frac{(x_2 - x_1)^2}{2(v_2 - v_1)}\right\} dx = \left(\frac{v_1}{v_2}\right)^{r/2} H_r\left(\frac{x_1}{v_1}\right),$$

$$(4.4)\quad E\left(H_m\left(\frac{X_t}{\sqrt{v_2}}\right) H_n\left(\frac{X_t}{\sqrt{v_2}}\right) \mid X_s\right)$$

$$= \sum_{k=0}^{\min(m,n)} k! \binom{m}{k} \binom{n}{k} \left(\frac{v_1}{v_2}\right)^{(m+n)/2-k} H_{m+n-2k}\left(\frac{X_s}{\sqrt{v_1}}\right).$$

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