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ON THE STRONG LAWS OF LARGE NUMBERS FOR TWO-DIMENSIONAL ARRAYS OF BLOCKWISE INDEPENDENT AND BLOCKWISE ORTHOGONAL RANDOM VARIABLES*

BY

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Abstract. In this paper we obtain the conditions of the strong law of large numbers for two-dimensional arrays of random variables which are blockwise independent and blockwise orthogonal. Some well-known results on the strong laws of large numbers for two-dimensional arrays of random variables are extended.

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1. INTRODUCTION

Móricz [8] introduced the concepts of *blockwise independence* and *blockwise orthogonality* for a sequence of random variables. Móricz [8] and Gaposhkin [3] showed that some properties of independent sequences of random variables can be applied to sequences consisting of independent blocks. In particular, it was proved in Móricz [8] that if $\{X_i, i \geqslant 1\}$ is a sequence of random variables of mean 0 such that for each $k \geqslant 1$ the random variables $\{X_i, 2^k \leqslant i < 2^{k+1}\}$ are independent, then it satisfies the Kolmogorov theorem (see, e.g., Chow and Teicher [2], p. 124): the condition $\sum_{i=1}^{\infty} EX_i^2/i^2 < \infty$ implies the strong law of large numbers, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = 0 \text{ almost surely (a.s.).}$$

In [4] Gaposhkin obtained the sufficient conditions under which the strong law of large numbers is fulfilled for *blockwise independent* sequences and *blockwise orthogonal* sequences. However, the same problems for multidimensional arrays have not been studied yet.

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The aim of this paper is to establish the strong law of large numbers for two-dimensional arrays of *blockwise independent* and *blockwise orthogonal* random variables with arbitrary blocks. In the present work, we obtain, as corollaries, the Kolmogorov strong law of large numbers and the Rademacher–Mensov strong law of large numbers for two-dimensional arrays of random variables.

For $a, b \in \mathbf{R}$, $\min\{a, b\}$ and $\max\{a, b\}$ will be denoted by $a \wedge b$ and $a \vee b$, respectively. In this paper, the logarithms are to base 2.

2. PRELIMINARIES

In the sequel we will need the following lemmas.

LEMMA 2.1. *If $\{x_{mn}, m \geqslant 1, n \geqslant 1\}$ is an array of real numbers such that*

$$\lim_{m \vee n \to \infty} x_{mn} = 0,$$

then

$$\lim_{m \vee n \to \infty} 2^{-m-n} \sum_{i=1}^{m} \sum_{j=1}^{n} 2^{i+j} x_{ij} = 0.$$

Proof. Set $s = \sum_{n=1}^{\infty} n/2^{n-1}$. For all $\varepsilon > 0$, there exists n_0 such that $|x_{ij}| < \varepsilon/2s$ for $i \vee j \geqslant n_0$. On the other hand, since $\lim_{m \vee n \to \infty} 2^{-m-n} = 0$, there exists $m_0 > n_0$ such that

$$2^{-m-n} \sum_{i \wedge j < n_0} 2^{i+j} x_{ij} < \varepsilon/2 \quad \text{for } m \vee n \geqslant m_0.$$

Hence, for $m \vee n \geqslant m_0$,

$$\Big|2^{-m-n} \sum_{i=1}^{m} \sum_{j=1}^{n} 2^{i+j} x_{ij}\Big| \leqslant 2^{-m-n}\Big| \sum_{i \wedge j < n_0} 2^{i+j} x_{ij}\Big| + 2^{-m-n}\Big| \sum_{i \vee j \geqslant n_0, i \leqslant m, j \leqslant n} 2^{i+j} x_{ij}\Big|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2s} \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2s} s = \varepsilon.$$

The proof is completed. ■

The next lemma is the two-parameter version of the Kolmogorov inequality. It was obtained by Wichura [9].

LEMMA 2.2. *Let $\{X_{ij}, 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ be a collection of mn independent random variables. If $EX_{ij} = 0$ for all $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, then*

$$E\Big(\max_{1 \leqslant k \leqslant m, 1 \leqslant l \leqslant n} |S_{kl}|^2\Big) \leqslant 16 \sum_{i=1}^{m} \sum_{j=1}^{n} EX_{ij}^2,$$

where $S_{kl} = \sum_{i=1}^{k} \sum_{j=1}^{l} X_{ij}$, $1 \leqslant k \leqslant m$, $1 \leqslant l \leqslant n$.

The following lemma is the two-parameter version of the Rademacher–Mensov inequality. It was firstly achieved by Agnew [1]. It may also be found in the papers by Móricz [7] (Corollary 2) and Hong and Hwang [5] (Lemma 2.2).

LEMMA 2.3. *If* $\{X_{ij},\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$ *is an array of mutually orthogonal random variables,* $E|X_{ij}|^2 < \infty$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, *then*

$$E(\max_{1 \leqslant k \leqslant m, 1 \leqslant l \leqslant n} |S_{kl}|)^2 \leqslant (\log 2m)^2 (\log 2n)^2 \sum_{i=1}^{m} \sum_{j=1}^{n} EX_{ij}^2,$$

where $S_{kl} = \sum_{i=1}^{k} \sum_{j=1}^{l} X_{ij}$, $1 \leqslant k \leqslant m$, $1 \leqslant l \leqslant n$.

3. MAIN RESULTS

Let $\{\omega(k),\ k \geqslant 1\}$ and $\{\nu(k),\ k \geqslant 1\}$ be strictly increasing sequences of positive integers with $\omega(1) = \nu(1) = 1$ and set

$$\Delta_{kl} = [\omega(k),\ \omega(k+1)) \times [\nu(l),\ \nu(l+1)).$$

We say that an array $\{X_{ij},\ i \geqslant 1, j \geqslant 1\}$ of random variables is *blockwise independent* (resp., *blockwise orthogonal*) with respect to the blocks $\{\Delta_{kl},\ k \geqslant 1,\ l \geqslant 1\}$ if for each k and l the array $\{X_{ij},\ (i, j) \in \Delta_{kl}\}$ is independent (resp., orthogonal). For $\{\omega(k),\ k \geqslant 1\}$, $\{\nu(k),\ k \geqslant 1\}$ and $\{\Delta_{kl},\ k \geqslant 1,\ l \geqslant 1\}$ as above, and for $m \geqslant 0$, $n \geqslant 0$, $k \geqslant 1$, $l \geqslant 1$, we introduce the following notation:

$$\Delta^{(mn)} = \{(i, j):\ 2^m \leqslant i < 2^{m+1},\ 2^n \leqslant j < 2^{n+1}\},$$

$$\Delta_{kl}^{(mn)} = \Delta_{kl} \cap \Delta^{(mn)},$$

$$I_{mn} = \{(k, l):\ \Delta_{kl}^{(mn)} \neq \emptyset\},$$

$$r_k^{(m)} = \min\{m:\ m \in [\omega(k),\ \omega(k+1)) \cap [2^m,\ 2^{m+1})\},$$

$$s_l^{(n)} = \min\{n:\ n \in [\nu(l),\ \nu(l+1)) \cap [2^n,\ 2^{n+1})\},$$

$$r_k'^{(m)} = \max\{m:\ m \in [\omega(k),\ \omega(k+1)) \cap [2^m,\ 2^{m+1})\},$$

$$s_l'^{(n)} = \max\{n:\ n \in [\nu(l),\ \omega(l+1)) \cap [2^n,\ 2^{n+1})\},$$

$$|r_k^{(m)}| = r_k'^{(m)} - r_k^{(m)} + 1,$$

$$|s_l^{(n)}| = s_l'^{(n)} - s_l^{(n)} + 1,$$

$$s_{mn} = \operatorname{card} I_{mn},$$

$$\varphi(i, j) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} s_{ij} I_{\Delta^{(ij)}},$$

$$\varphi^*(i) = \log^2[\omega(k+1) - \omega(k) + 1] \text{ if } \omega(k) \leqslant i < \omega(k+1),$$

$$\psi^*(i) = \log^2[\nu(k+1)-\nu(k)+1] \text{ if } \nu(k) \leqslant i < \nu(k+1),$$
$$\phi(i,j) = \varphi^*(i)\psi^*(i),$$

where $I_{\Delta^{(ij)}}$ denotes the indicator function of the set $\Delta^{(ij)}$, $i \geqslant 0$, $j \geqslant 0$.

THEOREM 3.1. *If $\{X_{ij}, i \geqslant 1, j \geqslant 1\}$ is an array of blockwise independent random variables with respect to the blocks $\{\Delta_{kl}, k \geqslant 1, l \geqslant 1\}$, $EX_{ij} = 0$, $i \geqslant 1$, $j \geqslant 1$, then the condition*

(3.1)
$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{EX_{ij}^2}{i^2 j^2}\varphi(i,j) < \infty$$

implies

(3.2)
$$\lim_{m\vee n\to\infty} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij} = 0 \text{ a.s.}$$

Proof. Set

$$\gamma_{kl}^{(mn)} = \max_{(p,q)\in\Delta_{kl}^{(mn)}} \Big|\sum_{i=r_k^{(m)}}^{p}\sum_{j=s_l^{(n)}}^{q} X_{ij}\Big|, \quad (k,l)\in I_{mn},\ m\geqslant 0,\ n\geqslant 0,$$

and

$$\gamma_{mn} = 2^{-m-1}2^{-n-1}\sum_{(k,l)\in I_{mn}} \gamma_{kl}^{(mn)}, \quad m\geqslant 0,\ n\geqslant 0.$$

By Lemma 2.2, we have

$$E(\gamma_{kl}^{(mn)})^2 \leqslant 16E\Big(\sum_{(i,j)\in\Delta_{kl}^{(mn)}} X_{ij}\Big)^2 = 16\sum_{(i,j)\in\Delta_{kl}^{(mn)}} EX_{ij}^2.$$

Consequently,

$$E\gamma_{mn}^2 \leqslant 2^{-2m-2}2^{-2n-2}s_{mn}\sum_{(k,l)\in I_{mn}} E(\gamma_{kl}^{(mn)})^2 \leqslant 16\sum_{i=2^m}^{2^{m+1}-1}\sum_{j=2^n}^{2^{n+1}-1}\frac{EX_{ij}^2}{i^2 j^2}\varphi(i,j).$$

It thus follows from (3.1) that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E\gamma_{mn}^2 < \infty.$$

By Markov's inequality and the Borel–Cantelli lemma,

$$\gamma_{mn}\to 0 \text{ a.s.} \quad \text{as } m\vee n\to\infty.$$

On the other hand,

$$2^{-m}2^{-n}\sum_{k=1}^{m}\sum_{l=1}^{n}\sum_{(i,j)\in I_{kl}}\gamma_{ij}^{(kl)} = 2^{-m}2^{-n}\sum_{k=1}^{m}\sum_{l=1}^{n}2^{k+1}2^{l+1}\gamma_{kl}.$$

By Lemma 2.1,

$$\lim_{m\vee n\to\infty} 2^{-m}2^{-n}\sum_{k=1}^{m}\sum_{l=1}^{n}\sum_{(i,j)\in I_{kl}}\gamma_{ij}^{(kl)} = 0 \text{ a.s.}$$

Assume $(m, n) \in \Delta_{ij}^{(kl)}$. Then we have

$$0 \leqslant \left|m^{-1} n^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}\right| \leqslant 2^{-k} 2^{-l} \sum_{i=0}^{k} \sum_{j=0}^{l} \sum_{(\lambda,\mu)\in I_{ij}} \gamma_{\lambda\mu}^{(ij)},$$

which completes the proof. ∎

The following corollary extends Kolmogorov's strong law of large numbers for arrays.

COROLLARY 3.2. *If* $\omega(k) = [q_1^k]$, $\nu(l) = [q_2^l]$ $(q_1 > 1, q_2 > 1)$, *then the condition*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{EX_{ij}^2}{i^2 j^2} < \infty \tag{3.3}$$

implies (3.2) *for all* Δ_{kl}-*independent arrays* $\{X_{ij}, i \geqslant 1, j \geqslant 1\}$, $EX_{ij} = 0$, $i, j \geqslant 1$.

Proof. Indeed, in that case $\varphi(i, j) = O(1)$. From (3.3) we obtain (3.1). ∎

It is clear that the same statement is true for the case when $\omega(k)$ grows faster than 2^k, and $\nu(l)$ grows faster than 2^l. That is why the smaller blocks considered in other statements concerned are more interesting. This remark was made by Gaposhkin [4].

COROLLARY 3.3. *If* $\omega(k) = [2^{k^\alpha}]$, $\nu(l) = [2^{l^\beta}]$ $(0 < \alpha < 1, 0 < \beta < 1)$, *then the condition*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{EX_{ij}^2}{i^2 j^2} \log^{(1-\alpha)/\alpha} i \log^{(1-\beta)/\beta} j < \infty \tag{3.4}$$

implies (3.2) *for all* Δ_{kl}-*independent arrays* $\{X_{ij}, i \geqslant 1, j \geqslant 1\}$, $EX_{ij} = 0$, $i, j \geqslant 1$.

Proof. In that case, we have $\varphi(i, j) = O(\log^{(1-\alpha)/\alpha} i \log^{(1-\beta)/\beta} j)$. From (3.4) we get (3.1). ∎

COROLLARY 3.4. *If* $\omega(k) = [k^\alpha]$, $\nu(l) = [l^\beta]$ $(\alpha > 1, \beta > 1)$, *then the condition*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{EX_{ij}^2}{i^{2-1/\alpha} j^{2-1/\beta}} < \infty \tag{3.5}$$

implies (3.2) *for all* Δ_{kl}-*independent arrays* $\{X_{ij}, i \geqslant 1, j \geqslant 1\}$, $EX_{ij} = 0$, $i, j \geqslant 1$.

Proof. In that case, we have $\varphi(i, j) = O(i^{1/\alpha} j^{1/\beta})$. From (3.5) we infer that (3.1) is satisfied. ∎

COROLLARY 3.5. *If* $\{X_{ij}, i \geqslant 1, j \geqslant 1\}$ *is an array of arbitrary random variables*, $EX_{ij} = 0$, $i, j \geqslant 1$, *then the condition*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{EX_{ij}^2}{ij} < \infty \tag{3.6}$$

implies (3.2).

Proof. Indeed, for $\Delta_{kl} = [k, k+1) \times [l, l+1)$, any array of random variables is Δ_{kl}-independent and $\varphi(i, j) = O(ij)$. From (3.6) we get (3.1). ∎

In the following theorem, we obtain the condition of the strong law of large numbers for two-dimensional arrays of blockwise orthogonal random variables.

THEOREM 3.6. *If* $\{X_{ij}, i \geqslant 1, j \geqslant 1\}$ *is an array of blockwise orthogonal random variables with respect to the blocks* $\{\Delta_{kl}, k \geqslant 1, l \geqslant 1\}$, *then the condition*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{EX_{ij}^2}{i^2 j^2} \varphi(i, j) \phi(i, j) < \infty \tag{3.7}$$

implies (3.2).

Proof. Define $\gamma_{kl}^{(mn)}$, $(k, l) \in I_{mn}$, $m \geqslant 0$, $n \geqslant 0$, and γ_{mn}, $m \geqslant 0$, $n \geqslant 0$, as in the proof of Theorem 3.1. By Lemma 2.3 we have

$$\begin{aligned} E(\gamma_{kl}^{(mn)})^2 &\leqslant (\log 2|r_k^{(m)}|)^2 (\log 2|s_l^{(n)}|)^2 E\Big(\sum_{(i,j) \in \Delta_{kl}^{(mn)}} X_{ij}\Big)^2 \\ &= (\log 2|r_k^{(m)}|)^2 (\log 2|s_l^{(n)}|)^2 \sum_{(i,j) \in \Delta_{kl}^{(mn)}} EX_{ij}^2. \end{aligned}$$

Consequently,

$$\begin{aligned} E(\gamma_{mn})^2 &\leqslant 2^{-2m-2} 2^{-2n-2} s_{mn} \sum_{k=p_m}^{q_m} \sum_{l=u_n}^{v_n} E|\gamma_{kl}^{(mn)}|^2 \\ &\leqslant 2^{-2m-2} 2^{-2n-2} (\log 2|r_k^{(m)}|)^2 (\log 2|s_l^{(n)}|)^2 s_{mn} \sum_{i=2^m}^{2^{m+1}-1} \sum_{j=2^n}^{2^{n+1}-1} EX_{ij}^2 \\ &\leqslant C \sum_{i=2^m}^{2^{m+1}-1} \sum_{j=2^n}^{2^{n+1}-1} \frac{EX_{ij}^2}{i^2 j^2} \varphi(i, j) \phi(i, j), \end{aligned}$$

where C is a constant. It thus follows from (3.7) that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E(\gamma_{mn})^2 < \infty.$$

The rest of the argument is exactly the same as that at the end of the proof of Theorem 3.1. ∎

The following corollary extends the Rademacher–Mensov strong law of large numbers for arrays.

COROLLARY 3.7. *If* $\omega(k) = [q_1^k]$, $v(l) = [q_2^l]$ $(q_1 > 1, q_2 > 1)$, *then the condition*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{EX_{ij}^2}{i^2 j^2} \log^2 i \log^2 j < \infty \tag{3.8}$$

implies (3.2) *for all* Δ_{kl}*-orthogonal arrays* $\{X_{ij}, i \geqslant 1, j \geqslant 1\}$.

Proof. Indeed, in that case $\varphi(i, j) = O(1)$, $\phi(i, j) = O(\log^2 i \log^2 j)$. From (3.8) we obtain (3.7). ∎

Using the same techniques as in the case of the array of blockwise independent random variables, we get the following corollaries.

COROLLARY 3.8. *If* $\omega(k) = [2^{k^\alpha}]$, $\nu(l) = [2^{l^\beta}]$ $(0 < \alpha < \frac{1}{3}, 0 < \beta < \frac{1}{3})$, *then the condition*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{EX_{ij}^2}{i^2 j^2} \log^{2(1-\alpha)/\alpha} i \log^{2(1-\beta)/\beta} j < \infty \tag{3.9}$$

implies (3.2) *for all* Δ_{kl}*-orthogonal arrays* $\{X_{ij}, i \geqslant 1, j \geqslant 1\}$.

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