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ON THE APPROXIMATION OF A RANDOM VARIABLE BY A CONDITIONAL EXPECTATION OF ANOTHER RANDOM VARIABLE

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Abstract. Let X and Y be $\boldsymbol{R}$-valued random variables on a non-atomic probability space $(\Omega, \mathfrak{F}, P)$. We give conditions under which Y can be approximated by a conditional expectation of X. In particular, we prove the following theorem:

Let X be an $\boldsymbol{R}$-valued random variable such that $EX^+ = EX^- = \infty$. Then for each random variable Y and arbitrary $\varepsilon > 0$ there exist $B \in \mathfrak{F}$ and a sub-σ-field $\mathfrak{A}$ of $\mathfrak{F}$ such that $P(B) \leqslant \varepsilon$ and $E(X \mid \mathfrak{A}) = Y$ a.s. on B^c.

We also review some facts on the conditional expectation of unintegrable random variables.

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0. INTRODUCTION

Let X be an $\boldsymbol{R}$-valued random variable on a non-atomic probability space $(\Omega, \mathfrak{F}, P)$. The paper is devoted to the question: which random variables can be obtained by conditioning of X with respect to sub-σ-fields of $\mathfrak{F}$. In general, it seems to be impossible to give any detailed characterization of the family of such random variables. Nevertheless we can give some sufficient conditions under which there exists a sub-σ-field $\mathfrak{A}$ of $\mathfrak{F}$ such that $Y = E(X \mid \mathfrak{A})$ outside a fixed set $B \in \mathfrak{F}$. The following lemma has been proved and used in [3] and [4].

0.1. LEMMA. *Let X be an integrable random variable and Y a random variable of the form*

$$Y = \sum_{i=1}^{k} \alpha_i \mathbf{1}_{A_i} + \beta \mathbf{1}_B,$$

such that $B, A_1, A_2, \ldots, A_k$ are pairwise disjoint,

$$B \cup \bigcup_{i=1}^{k} A_i = \Omega,$$

and

$$\sum_{i=1}^{k} |\alpha_i| P(A_i) + \max_{i=1,\dots,k} |\alpha_i| P(B) \leqslant \min\{EX^+ 1_B - EX^- 1_{B^c},\ EX^- 1_B - EX^+ 1_{B^c}\}.$$

Then there exists a sub-σ-field $\mathfrak{A}$ *of* $\mathfrak{F}$ *such that*

$$E(X \mid \mathfrak{A}) = Y \ a.s. \quad on\ B^c.$$

A generalization of this theorem is the main result of the paper. It is proved in Section 2. Corollaries to this theorem concern unintegrable random variables. The concept of the conditional expectation of unintegrable random variable is not new (see e.g. [1]). It seems, however, that the idea of the domain of conditional expectation lacks precise treatment in the literature. This topic is covered in Section 1.

1. CONDITIONAL EXPECTATION OF AN ARBITRARY RANDOM VARIABLE

Here and subsequently $(\Omega, \mathfrak{F}, P)$ denotes a non-atomic probability space. Let us recall now the definition of the conditional distribution.

1.1. DEFINITION. Let X be an $\boldsymbol{R}^n$-valued random variable on $(\Omega, \mathfrak{F}, P)$, and $\mathfrak{A}$ be a sub-σ-field of $\mathfrak{F}$. A function

$$P(X \in \cdot \mid \mathfrak{A})(\cdot)\colon \mathfrak{B}(\boldsymbol{R}^n) \times \Omega \to \boldsymbol{R}$$

is called a *version of the conditional distribution* of X given $\mathfrak{A}$ if

(i) for each $B \in \mathfrak{B}(\boldsymbol{R}^n)$ the map $\Omega \ni \omega \mapsto P(X \in B \mid \mathfrak{A})(\omega)$ is $\mathfrak{A}$-measurable,

(ii) for each $\omega \in \Omega$ the map $\mathfrak{B}(\boldsymbol{R}^n) \ni B \mapsto P(X \in B \mid \mathfrak{A})(\omega)$ is a probability distribution on $(\boldsymbol{R}^n, \mathfrak{B}(\boldsymbol{R}^n))$,

(iii) for each $A \in \mathfrak{A}$ and $B \in \mathfrak{B}(\boldsymbol{R}^n)$ we have

$$P(A \cap \{\omega\colon X(\omega) \in B\}) = \int_A P(X \in B \mid \mathfrak{A})(\omega)\, dP(\omega).$$

The existence of $P(X \in \cdot \mid \mathfrak{A})(\cdot)$ is guaranteed by the following

1.2. THEOREM. *Let X be an $\boldsymbol{R}^n$-valued random variable on $(\Omega, \mathfrak{F}, P)$, and $\mathfrak{A}$ be a sub-σ-field of $\mathfrak{F}$. Then there exists a version $P(X \in \cdot \mid \mathfrak{A})(\cdot)$ of the conditional distribution of X given $\mathfrak{A}$. Moreover, if $P_1(X \in \cdot \mid \mathfrak{A})(\cdot)$ is any other version of the conditional distribution of X given $\mathfrak{A}$, then*

$$P(X \in \cdot \mid \mathfrak{A})(\omega) = P_1(X \in \cdot \mid \mathfrak{A})(\omega) \ \ \omega\text{-}a.s.$$

Proof. See for instance [2]. ∎

In the expressions involving integrals with respect to $P(X \in \cdot \mid \mathfrak{A})(\omega)$ we shall often use the following, more convenient notation: $dp_{X|\mathfrak{A}}(\omega)$ or $dp_{X|\mathfrak{A}}$.

1.3. THEOREM. *Let X be an $\boldsymbol{R}^n$-valued random variable on $(\Omega, \mathfrak{F}, P)$, $\mathfrak{A}$ a sub-σ-field of $\mathfrak{F}$, $f\colon \boldsymbol{R}^n \to \boldsymbol{R}$ a Borel function, and $A \in \mathfrak{A}$. If the integral $\int_A f(X)\,dP$ exists (and is finite), then for almost every $\omega \in A$ the following integral exists (and is finite):*

$$\int_{\boldsymbol{R}^n} f(x)\,dp_{X|\mathfrak{A}}(\omega)(x);$$

moreover,

$$\int_A f(X)\,dP = \int_A \int_{\boldsymbol{R}^n} f(x)\,dp_{X|\mathfrak{A}}(\omega)(x)\,dP(\omega).$$

Proof. The theorem can be obtained by the standard "Lebesgue procedure". ∎

1.4. DEFINITION. Let X be an $\boldsymbol{R}$-valued random variable on $(\Omega, \mathfrak{F}, P)$, $\mathfrak{A}$ a sub-σ-field of $\mathfrak{F}$, and $p_{X|\mathfrak{A}}$ a version of the conditional distribution of X given $\mathfrak{A}$. Let us put

$$E(X \mid \mathfrak{A})(\omega) = \int_{\boldsymbol{R}} x\,dp_{X|\mathfrak{A}}(\omega)(x),$$

whenever the right-hand integral exists. By $D\big(E(X \mid \mathfrak{A})\big)$ we shall denote the set of points for which this integral exists.

By the equality $Y = E(X \mid \mathfrak{A})$ we mean that the random variable Y is a *version of the conditional expectation* of X given $\mathfrak{A}$. Usually, when no confusion can arise, we shall simply write $E(X \mid \mathfrak{A})$ to denote a version of the conditional expectation of X given $\mathfrak{A}$.

Definition 1.1 implies easily that $D\big(E(X \mid \mathfrak{A})\big) \in \mathfrak{A}$. By $\mathfrak{A}_{|D(E(X|\mathfrak{A}))}$ we shall denote the restriction of the σ-field $\mathfrak{A}$ to $D\big(E(X \mid \mathfrak{A})\big)$, that is

$$\mathfrak{A}_{|D(E(X|\mathfrak{A}))} = \big\{A \cap D\big(E(X \mid \mathfrak{A})\big)\colon\ A \in \mathfrak{A}\big\}.$$

Obviously, $\mathfrak{A}_{|D(E(X|\mathfrak{A}))} \subset \mathfrak{A}$. From Definition 1.1 we also conclude that any version of $E(X \mid \mathfrak{A})$ is $\mathfrak{A}_{|D(E(X|\mathfrak{A}))}$-measurable. Let $p^1_{X|\mathfrak{A}}$ and $p^2_{X|\mathfrak{A}}$ be two versions of the conditional distribution of X given $\mathfrak{A}$. Denote by $D_i\big(E(X \mid \mathfrak{A})\big)$ the domain of the version of $E(X \mid \mathfrak{A})$ derived from $p^i_{X|\mathfrak{A}}$. The second part of Theorem 1.3 implies

$$P\big(D_1\big(E(X \mid \mathfrak{A})\big) \triangle D_2\big(E(X \mid \mathfrak{A})\big)\big) = 0$$

and

$$\int_{\boldsymbol{R}} x\,dp^1_{X|\mathfrak{A}}(x) = \int_{\boldsymbol{R}} x\,dp^2_{X|\mathfrak{A}}(x) \text{ a.s.} \quad \text{on } D_1\big(E(X \mid \mathfrak{A})\big) \cap D_2\big(E(X \mid \mathfrak{A})\big).$$

These equalities imply that any two versions of $E(X \mid \mathfrak{A})$ are actually indistinguishable. This is why we do not have to indicate from which version of $p_{X|\mathfrak{A}}$ a version of the conditional expectation is derived.

1.5. PROPOSITION. *Let $E(X\mid\mathfrak{A})$ be a version of the conditional expectation of X given $\mathfrak{A}$. Suppose that $Y\colon D(E(X\mid\mathfrak{A}))\to \boldsymbol{R}$ is $\mathfrak{A}_{|D(E(X|\mathfrak{A}))}$-measurable and is almost surely equal to $E(X\mid\mathfrak{A})$. Then Y is another version of the conditional expectation of X given $\mathfrak{A}$.*

Proof. Suppose that both $E(X\mid\mathfrak{A})$ and $D(E(X\mid\mathfrak{A}))$ are derived from $p_{X|\mathfrak{A}}$. Let us put

$$A_0=\{\omega\in D(E(X\mid\mathfrak{A}))\colon\ Y(\omega)\neq E(X\mid\mathfrak{A})(\omega)\}$$

and define

$$\hat{p}_{X|\mathfrak{A}}(B)(\omega)=\begin{cases}p_{X|\mathfrak{A}}(B)(\omega) & \text{for } \omega\notin A_0,\\ \delta_{\{Y(\omega)\}}(B) & \text{for } \omega\in A_0.\end{cases}$$

It is easily seen that $\hat{p}_{X|\mathfrak{A}}$ is a version of the conditional distribution of X given $\mathfrak{A}$ and that

$$Y(\omega)=\int_{\boldsymbol{R}} x d\hat{p}_{X|\mathfrak{A}}(\omega)(x)\qquad \text{for each } \omega\in D(E(X\mid\mathfrak{A})).$$

This completes the proof. ∎

1.6. LEMMA. *If $A\in\mathfrak{A}$ and the integral $\int_A XdP$ exists, then $A\subset D(E(X\mid\mathfrak{A}))$.*

Proof. The lemma follows from Theorem 1.3. ∎

1.7. LEMMA. *Let X be an $\boldsymbol{R}$-valued random variable on $(\Omega,\mathfrak{F},P)$, and $\mathfrak{A}$ be a sub-σ-field of $\mathfrak{F}$. Then there exist pairwise disjoint sets A_i, B_i, $C_i\in\mathfrak{A}$, $i\geqslant 1$, such that*

(i) $\bigcup_{i\geqslant 1}A_i=\{\omega\in\Omega\colon\ |E(X\mid\mathfrak{A})|<\infty\}$,

(ii) $\bigcup_{i\geqslant 1}B_i=\{\omega\in\Omega\colon\ E(X\mid\mathfrak{A})=\infty\}$,

(iii) $\bigcup_{i\geqslant 1}C_i=\{\omega\in\Omega\colon\ E(X\mid\mathfrak{A})=-\infty\}$,

(iv) *integrals $\int_{B_i}XdP$ and $\int_{C_i}XdP$ exist and $\int_{A_i}|X|\,dP<\infty$.*

Proof. For $i\geqslant 1$ we set

$$A_i=\{\omega\in\Omega\colon\ i-1\leqslant \int_{\boldsymbol{R}}|x|\,dp_{X|\mathfrak{A}}(\omega)(x)<i\},$$

$$B_i=\{\omega\in\Omega\colon\ \int_{\boldsymbol{R}}xdp_{X|\mathfrak{A}}(\omega)(x)=\infty,\ i-1\leqslant\int_{\boldsymbol{R}}x^-\,dp_{X|\mathfrak{A}}(\omega)(x)<i\},$$

$$C_i=\{\omega\in\Omega\colon\ \int_{\boldsymbol{R}}xdp_{X|\mathfrak{A}}(\omega)(x)=-\infty,\ i-1\leqslant\int_{\boldsymbol{R}}x^+\,dp_{X|\mathfrak{A}}(\omega)(x)<i\}.$$

Now it is enough to use Theorem 1.3 to see that the above sets have the desired properties. ∎

From Lemma 1.7 we deduce

1.8. LEMMA. *Let X be an $\boldsymbol{R}$-valued random variable on $(\Omega,\mathfrak{F},P)$, and $\mathfrak{A}$ be a sub-σ-field of $\mathfrak{F}$. Then there exist pairwise disjoint sets $D_i\in\mathfrak{A}$, $i\geqslant 1$,* *such that*

(i) $\bigcup_{i\geqslant 1} D_i = D(E(X\,|\,\mathfrak{A}))$,

(ii) *integrals* $\int_{D_i} XdP$ *exist.*

The following lemma is elementary and we shall leave it without proof.

1.9. Lemma. *Let X and Y be* $\bar{\boldsymbol{R}}$*-valued random variables on* $(\Omega, \mathfrak{F}, P)$. *If the equality*

$$\int_A XdP = \int_A YdP$$

holds for each $A\in\mathfrak{F}$ *for which both integrals exist, then* $X = Y$ *a.s.*

The following characterization of the conditional expectation is analogous to the one for integrable random variables.

1.10. Theorem. *Let X be an* $\boldsymbol{R}$*-valued random variable on* $(\Omega, \mathfrak{F}, P)$, $\mathfrak{A}$ *a sub-σ-field of* $\mathfrak{F}$, *and Y an* $\boldsymbol{R}$*-valued random variable on* $D(E(X\,|\,\mathfrak{A}))$. *If Y is a version of* $E(X\,|\,\mathfrak{A})$, *then*

(i) *Y is* $\mathfrak{A}_{|D(E(X|\mathfrak{A}))}$*-measurable,*

(ii) *for each* $A\in\mathfrak{A}$, *if the integral* $\int_A XdP$ *exists, then* $\int_A YdP$ *also exists and*

$$\int_A XdP = \int_A YdP.$$

On the other hand, if Y is $\mathfrak{A}_{|D(E(X|\mathfrak{A}))}$*-measurable and the above equality holds for each* $A\in\mathfrak{A}_{|D(E(X|\mathfrak{A}))}$ *for which the above integrals exist, then Y is a version of* $E(X\,|\,\mathfrak{A})$.

Proof. Let Y be a version of $E(X\,|\,\mathfrak{A})$. We have already observed that Y is $\mathfrak{A}_{|D(E(X|\mathfrak{A}))}$-measurable. Part (ii) follows from Theorem 1.3.

Take an $\bar{\boldsymbol{R}}$-valued random variable Y satisfying the assumptions of the converse implication. Theorem 1.3 again implies

$$\int_A XdP = \int_A E(X\,|\,\mathfrak{A})\,dP \tag{1}$$

for each $A\in\mathfrak{A}_{|D(E(X|\mathfrak{A}))}$ such that both integrals exist. Take $A\in\mathfrak{A}_{|D(E(X|\mathfrak{A}))}$ such that the integrals $\int_A YdP$ and $\int_A E(X\,|\,\mathfrak{A})\,dP$ exist. Let D_i, $i\geqslant 1$, be a sequence of sets whose existence has been proved in Lemma 1.8. Now we have

$$\int_A YdP = \sum_{i\geqslant 1}\int_{A\cap D_i} YdP = \sum_{i\geqslant 1}\int_{A\cap D_i} XdP = \sum_{i\geqslant 1}\int_{A\cap D_i} E(X\,|\,\mathfrak{A})\,dP$$
$$= \int_A E(X\,|\,\mathfrak{A})\,dP.$$

The second equality is a consequence of our assumptions on Y and the third one follows from (1). It is worth mentioning at the moment that the third equality is the place where we have taken advantage of the existence of sets D_i on which integrals $\int_{D_i} XdP$ exist. Now Lemma 1.9 gives $Y = E(X\,|\,\mathfrak{A})$ a.s.

Since Y is $\mathfrak{A}_{|D(E(X|\mathfrak{A}))}$-measurable, it follows from Proposition 1.5 that Y is a version of $E(X|\mathfrak{A})$. ∎

Basic properties of the conditional expectation of integrable and unintegrable random variables are similar and we shall not discuss them here. However, some properties of the conditional expectation of X in the case when both EX^+ and EX^- are infinite seem to be pathological. The following theorem has been proved in [5].

1.11. THEOREM. *Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables with non-atomic distributions and such that*

$$EX_n^+ = EX_n^- = \infty \quad \textit{for } n \geq 1.$$

Then there exists an increasing sequence $(\mathfrak{A}_n)_{n\geq 1}$ of sub-σ-fields of $\mathfrak{F}$ such that

$$D\big(E(X_{n+1}|\mathfrak{A}_n)\big) = \Omega \ \textit{ for } n \geq 1, \qquad E(X_{n+1}|\mathfrak{A}_n) = X_n \ \textit{ for } n \geq 1,$$

but

$$D\big(E(X_{n+k}|\mathfrak{A}_n)\big) = \emptyset \quad \textit{for } n \geq 1 \textit{ and } k \geq 2.$$

2. THE MAIN RESULTS

2.1. THEOREM. *Let X be an integrable random variable on $(\Omega, \mathfrak{F}, P)$, and $B\in\mathfrak{F}$. For any random variable Y satisfying*

$$(2) \qquad E|Y|\mathbf{1}_{B^c} + \operatorname*{sup\,ess}_{\omega\in B^c} |Y(\omega)|\cdot P(B) < \min\{EX^+\mathbf{1}_B - EX^-\mathbf{1}_{B^c},\ EX^-\mathbf{1}_B - EX^+\mathbf{1}_{B^c}\}$$

there exists a sub-σ-field $\mathfrak{A}$ of $\mathfrak{F}$ such that

$$E(X|\mathfrak{A}) = Y \ \textit{a.s.} \quad \textit{on } B^c.$$

Proof. Let Y be a random variable satisfying (2). Let us suppose that (Y_n) is a sequence of simple random variables such that

$$\lim_{n\to\infty} Y_n = Y \text{ a.s.} \quad \text{on } B^c$$

and that $(\mathfrak{A}_n)$ is an increasing sequence of sub-σ-fields of $\mathfrak{F}$ satisfying

$$(3) \qquad E(X|\mathfrak{A}_n) = Y_n \text{ a.s.} \quad \text{on } B^c.$$

Putting

$$\mathfrak{A} = \sigma(\mathfrak{A}_1, \mathfrak{A}_2, \ldots)$$

we get

$$E(X|\mathfrak{A}) = \lim_{n\to\infty} E(X|\mathfrak{A}_n) = \lim_{n\to\infty} Y_n = Y \text{ a.s.} \quad \text{on } B^c.$$

To complete the proof we shall construct sequences (Y_n) and $(\mathfrak{A}_n)$.

Let N be an integer such that

$$|Y(\omega)| < N \text{ a.s.} \quad \text{on } B^c.$$

For $n \geqslant 1$ and $i \in \mathbf{Z}$ we write

$$\alpha_i^{(n)} = i2^{-n}, \tag{4}$$

$$\begin{aligned} E_i^{(n)} &= Y^{-1}(\alpha_{i-1}^{(n)}, \alpha_i^{(n)}] \cap \{X \leqslant Y\} \cap B^c, \\ F_i^{(n)} &= Y^{-1}(\alpha_{i-1}^{(n)}, \alpha_i^{(n)}] \cap \{X > Y\} \cap B^c. \end{aligned} \tag{5}$$

We can easily notice that

$$E_i^{(n)} = E_{2i-1}^{(n+1)} \cup E_{2i}^{(n+1)}, \quad F_i^{(n)} = F_{2i-1}^{(n+1)} \cup F_{2i}^{(n+1)}.$$

Let (Y_n) be a sequence of random variables given by

$$Y_n = \sum_{i=-N2^n}^{N2^n} \alpha_i^{(n)} \mathbf{1}_{E_i^{(n)}} + \sum_{i=-N2^n}^{N2^n} \alpha_{i-1}^{(n)} \mathbf{1}_{F_i^{(n)}}.$$

It can be easily seen that

$$\lim_{n\to\infty} Y_n = Y \text{ a.s.} \quad \text{on } B^c,$$

and

$$EX\mathbf{1}_{E_i^{(n)}} \leqslant \alpha_i^{(n)} P(E_i^{(n)}), \quad EX\mathbf{1}_{F_i^{(n)}} \geqslant \alpha_{i-1}^{(n)} P(F_i^{(n)}). \tag{6}$$

Let us also observe that

$$\lim_{n\to\infty} \sup_{\omega\in B^c} \operatorname{ess} |Y_n(\omega)| = \sup_{\omega\in B^c} \operatorname{ess} |Y(\omega)| \tag{7}$$

and

$$\lim_{n\to\infty} EY_n \mathbf{1}_{B^c} = EY\mathbf{1}_{B^c}. \tag{8}$$

From (2), (7) and (8) it follows that there exists an integer m such that for $n \geqslant m$ we have

$$E|Y_n|\mathbf{1}_{B^c} + \sup_{\omega\in B^c} \operatorname{ess} |Y_n(\omega)| \cdot P(B) < \min\{EX^+\mathbf{1}_B - EX^-\mathbf{1}_{B^c}, EX^-\mathbf{1}_B - EX^+\mathbf{1}_{B^c}\}.$$

Take an arbitrary random variable Z with a non-atomic distribution on $[0, 1]$, defined on $(\Omega, \mathfrak{F}, P)$, and put

$$B^+ = B \cap \{Z > 0\}, \quad B^- = B \cap \{Z < 0\}.$$

Now, using the same arguments as in the proof of Lemma 2.1 in [3] we find real numbers

$$0 \leqslant t_{-N2^m}^{(m)} \leqslant \ldots \leqslant t_0^{(m)} \leqslant \ldots \leqslant t_{N2^m}^{(m)} \leqslant 1$$

and

$$0 \leqslant s_{-N2^m}^{(m)} \leqslant \ldots \leqslant s_0^{(m)} \leqslant \ldots \leqslant s_{N2^m}^{(m)} \leqslant 1$$

satisfying

$$EX\mathbf{1}_{E_i^{(m)}\cup[B^+\cap Z^{-1}[t_{i-1}^{(m)},t_i^{(m)})]} = \alpha_i^{(m)} P\big(E_i^{(m)}\cup\big[B^+\cap Z^{-1}[t_{i-1}^{(m)}, t_i^{(m)})\big]\big) \tag{9}$$

and

$$EX\mathbf{1}_{F_i^{(m)}\cup[B^-\cap Z^{-1}[s_{i-1}^{(m)},s_i^{(m)})]} = \alpha_{i-1}^{(m)} P\big(F_i^{(m)}\cup\big[B^-\cap Z^{-1}[s_{i-1}^{(m)}, s_i^{(m)})\big]\big). \tag{10}$$

We set

$$G_i^{(m)} = E_i^{(m)}\cup\big[B^+\cap Z^{-1}[t_{i-1}^{(m)}, t_i^{(m)})\big], \quad H_i^{(m)} = F_i^{(m)}\cup\big[B^-\cap Z^{-1}[s_{i-1}^{(m)}, s_i^{(m)})\big].$$

Putting

$$\mathfrak{A}_m = \sigma(G_i^{(m)}, H_i^{(m)}\colon i = -N2^m, \ldots, N2^m)$$

from (9) and (10) we obtain

$$E(X\,|\,\mathfrak{A}_m) = Y_m \text{ a.s.} \quad \text{on } B^c.$$

Now we shall construct a σ-field $\mathfrak{A}_{m+1}$. It is easy to see that

$$Y_{m+1} = \sum_{i=-N2^m}^{N2^m} (\alpha_{2i-1}^{(m+1)}\mathbf{1}_{E_{2i-1}^{(m+1)}} + \alpha_{2i}^{(m+1)}\mathbf{1}_{E_{2i}^{(m+1)}}) + \sum_{i=-N2^m}^{N2^m} (\alpha_{2i-2}^{(m+1)}\mathbf{1}_{F_{2i-1}^{(m+1)}} + \alpha_{2i-1}^{(m+1)}\mathbf{1}_{F_{2i}^{(m+1)}}).$$

For $i\in\{-N2^m, \ldots, N2^m\}$ and $t\in[t_{i-1}^{(m)}, t_i^{(m)}]$ we put

$$T_{2i}(t) = EX\mathbf{1}_{E_{2i}^{(m+1)}\cup[B^+\cap Z^{-1}[t_{i-1}^{(m)},t)]} - \alpha_{2i}^{(m+1)} P\big(E_{2i}^{(m+1)}\cup\big[B^+\cap Z^{-1}[t_{i-1}^{(m)}, t)\big]\big).$$

Then by (6) we have

$$T_{2i}(t_{i-1}^{(m)}) \leqslant 0.$$

From (3)–(5) and (9) we obtain

$$\begin{aligned}
T_{2i}(t_i^{(m)}) &\geqslant EX\mathbf{1}_{E_{2i}^{(m+1)}\cup[B^+\cap Z^{-1}[t_{i-1}^{(m)},t_i^{(m)})]} \\
&\qquad - \alpha_{2i}^{(m+1)} P\big(E_{2i}^{(m+1)}\cup\big[B^+\cap Z^{-1}[t_{i-1}^{(m)}, t_i^{(m)})\big]\big) \\
&\qquad + EX\mathbf{1}_{E_{2i-1}^{(m+1)}} - \alpha_{2i}^{(m+1)} P(E_{2i-1}^{(m+1)}) \\
&= EX\mathbf{1}_{E_i^{(m)}\cup[B^+\cap Z^{-1}[t_{i-1}^{(m)},t_i^{(m)})]} - \alpha_i^{(m)} P\big(E_i^{(m)}\cup\big[B^+\cap Z^{-1}[t_{i-1}^{(m)}, t_i^{(m)})\big]\big) \\
&= 0.
\end{aligned}$$

Therefore there exists $t_{2i}^{(m+1)} \in [t_{i-1}^{(m)}, t_i^{(m)}]$ such that

(11) $$EX1_{E_{2i}^{(m+1)} \cup [B^+ \cap Z^{-1}[t_{i-1}^{(m)}, t_{2i}^{(m+1)})]}$$
$$= \alpha_{2i}^{(m+1)} P\left(E_{2i}^{(m+1)} \cup \left[B^+ \cap Z^{-1}[t_{i-1}^{(m)}, t_{2i}^{(m+1)})\right]\right).$$

Now for $t \in [t_{2i}^{(m+1)}, t_i^{(m)}]$ we put

$$T_{2i-1}(t) = EX1_{E_{2i-1}^{(m+1)} \cup [B^+ \cap Z^{-1}[t_{2i}^{(m+1)}, t)]} - \alpha_{2i-1}^{(m+1)} P\left(E_{2i-1}^{(m+1)} \cup \left[B^+ \cap Z^{-1}[t_{2i}^{(m+1)}, t)\right]\right).$$

By (6) we get

$$T_{2i-1}(t_{2i}^{(m+1)}) \leqslant 0.$$

From (5), (9), (11) and the inequality $\alpha_{2i-1}^{(m+1)} < \alpha_{2i}^{(m+1)}$ we have

$$\begin{aligned} T_{2i-1}(t_i^{(m)}) &= EX1_{E_i^{(m)} \cup [B^+ \cap Z^{-1}[t_{i-1}^{(m)}, t_i^{(m)})]} - EX1_{E_{2i}^{(m+1)} \cup [B^+ \cap Z^{-1}[t_{i-1}^{(m)}, t_{2i}^{(m+1)})]} \\ &\quad - \alpha_{2i-1}^{(m+1)} P\left(E_{2i-1}^{(m+1)} \cup \left[B^+ \cap Z^{-1}[t_{2i}^{(m+1)}, t_i^{(m)})\right]\right) \\ &= \alpha_i^{(m)} P\left(E_i^{(m)} \cup \left[B^+ \cap Z^{-1}[t_{i-1}^{(m)}, t_i^{(m)})\right]\right) \\ &\quad - \alpha_{2i}^{(m+1)} P\left(E_{2i}^{(m+1)} \cup \left[B^+ \cap Z^{-1}[t_{i-1}^{(m)}, t_{2i}^{(m+1)})\right]\right) \\ &\quad - \alpha_{2i-1}^{(m+1)} P\left(E_{2i-1}^{(m+1)} \cup \left[B^+ \cap Z^{-1}[t_{2i}^{(m+1)}, t_i^{(m)})\right]\right) \\ &\geqslant \alpha_i^{(m)} P\left(E_i^{(m)} \cup \left[B^+ \cap Z^{-1}[t_{i-1}^{(m)}, t_i^{(m)})\right]\right) \\ &\quad - \alpha_{2i}^{(m+1)} P\left(E_i^{(m)} \cup \left[B^+ \cap Z^{-1}[t_{i-1}^{(m)}, t_i^{(m)})\right]\right) \\ &= 0. \end{aligned}$$

So we conclude that there exists $t_{2i-1}^{(m+1)} \in [t_{2i}^{(m+1)}, t_i^{(m)}]$ such that

$$EX1_{E_{2i-1}^{(m+1)} \cup [B^+ \cap Z^{-1}[t_{2i}^{(m+1)}, t_{2i-1}^{(m+1)})]}$$
$$- \alpha_{2i-1}^{(m+1)} P\left(E_{2i-1}^{(m+1)} \cup \left[B^+ \cap Z^{-1}[t_{2i}^{(m+1)}, t_{2i-1}^{(m+1)})\right]\right) = 0.$$

Now we write

$$\begin{aligned} G_{2i-1}^{(m+1)} &= E_{2i-1}^{(m+1)} \cup \left[B^+ \cap Z^{-1}[t_{2i}^{(m+1)}, t_{2i-1}^{(m+1)})\right], \\ G_{2i}^{(m+1)} &= E_{2i}^{(m+1)} \cup \left[B^+ \cap Z^{-1}[t_{i-1}^{(m)}, t_{2i}^{(m+1)})\right]. \end{aligned}$$

It is easily seen that

(12) $$G_{2i-1}^{(m+1)} \cap G_{2i}^{(m+1)} = \varnothing$$

and

(13) $$G_{2i-1}^{(m+1)} \cap G_{2i}^{(m+1)} \subset G_i^{(m)}.$$

Similarly we find sets $H_{2i-1}^{(m+1)}$ and $H_{2i}^{(m+1)}$ satisfying

$$EX1_{H_{2i-1}^{(m+1)}} = \alpha_{2i-2}^{(m+1)} P(H_{2i-1}^{(m+1)}), \qquad EX1_{H_{2i}^{(m+1)}} = \alpha_{2i-1}^{(m+1)} P(H_{2i}^{(m+1)}),$$

(14) $$H_{2i-1}^{(m+1)} \cap H_{2i}^{(m+1)} = \emptyset,$$

(15) $$H_{2i-1}^{(m+1)} \cup H_{2i}^{(m+1)} \subset H_i^{(m)}.$$

Finally, we put

$$\mathfrak{A}_{m+1} = \sigma\big(G_{2i-1}^{(m+1)},\ G_{2i}^{(m+1)},\ G_i^{(m)} \setminus (G_{2i-1}^{(m+1)} \cup G_{2i}^{(m+1)}),$$
$$H_{2i-1}^{(m+1)},\ H_{2i}^{(m+1)},\ H_i^{(m)} \setminus (H_{2i-1}^{(m+1)} \cup H_{2i}^{(m+1)}),\ i \in \{-N2^m, \ldots, N2^m\}\big).$$

From (12)–(15) we conclude that

$$\mathfrak{A}_m \subset \mathfrak{A}_{m+1} \quad \text{and} \quad E(X \mid \mathfrak{A}_{m+1}) = Y_{m+1} \text{ a.s. } B^c.$$

Continuing inductively the above construction we find an increasing sequence $(\mathfrak{A}_n)$ of sub-σ-fields of $\mathfrak{F}$ satisfying (3). This completes the proof. ∎

2.2. Corollary. *Let X be an $\boldsymbol{R}$-valued random variable on $(\Omega, \mathfrak{F}, P)$ and $B \in \mathfrak{F}$. If*

(16) $$EX^+ \mathbf{1}_B = EX^- \mathbf{1}_B = \infty,$$

then for each random variable Y there exists a sub-σ-field $\mathfrak{A}$ of $\mathfrak{F}$ such that

$$E(X \mid \mathfrak{A}) = Y \ \text{a.s.} \quad \text{on } B^c.$$

Proof. Let $\varphi = (\varphi_1, \varphi_2)$ be a one-to-one mapping from $\boldsymbol{N}$ onto $\boldsymbol{N} \times \boldsymbol{N}$. For $n \geqslant 1$ we put

$$C_n = \{\omega \in \Omega\colon\ \varphi_1(n) - 1 \leqslant |X| < \varphi_1(n),\ \varphi_2(n) - 1 \leqslant |Y| < \varphi_2(n)\} \cap B^c.$$

It is easily observed that

$$\bigcup_{n=1}^{\infty} C_n = B^c.$$

By (16) there exists a positive number k_1 such that

(17) $$E|Y| \mathbf{1}_{C_1} + \operatorname*{sup\,ess}_{\omega \in C_1} |Y(\omega)| \cdot P(D_1) < \min\{EX^+ \mathbf{1}_{D_1} - EX^- \mathbf{1}_{C_1},\ EX^- \mathbf{1}_{D_1} - EX^+ \mathbf{1}_{C_1}\},$$

where $D_1 = B \cap \{|X| < k_1\}$. Similarly we can inductively define an increasing sequence (k_n) of real numbers such that (k_n) goes to infinity and

$$E|Y| \mathbf{1}_{C_n} + \operatorname*{sup\,ess}_{\omega \in C_n} |Y(\omega)| \cdot P(D_n) < \min\{EX^+ \mathbf{1}_{D_n} - EX^- \mathbf{1}_{C_n},\ EX^- \mathbf{1}_{D_n} - EX^+ \mathbf{1}_{C_n}\},$$

where $D_n = B \cap \{k_{n-1} \leqslant |X| < k_n\}$.

Now let us consider probability spaces $(\Omega_n, \mathfrak{F}_n, P_n)$ defined in the following way:

$$\Omega_n = D_n \cup C_n, \quad \mathfrak{F}_n = \{\Omega_n \cap F\colon\ F \in \mathfrak{F}\}, \quad P_n = P/P_n(\Omega_n).$$

It can be easily seen that

$$\Omega_n \cap \Omega_m = \emptyset \quad \text{for } n \neq m \tag{18}$$

and

$$\bigcup_{n=1}^{\infty} \Omega_n = \Omega. \tag{19}$$

From (17) and Theorem 2.1 we conclude that for each $n \geqslant 1$ there exists a σ-field $\mathfrak{A}_n \subset \mathfrak{F}_n$ such that

$$E_{(\Omega_n, \mathfrak{F}_n, P_n)}(X \mid \mathfrak{A}_n) = Y \text{ a.s.} \quad \text{on } C_n. \tag{20}$$

Finally, we set

$$\mathfrak{A} = \Big\{ \bigcup_{n=1}^{\infty} A_n \colon A_n \in \mathfrak{A}_n,\ n \geqslant 1 \Big\}.$$

By (19) and (20) we obtain

$$E(X \mid \mathfrak{A})(\omega) = Y(\omega) \text{ a.s.} \quad \text{on } B^c,$$

which completes the proof. ■

2.3. COROLLARY. *Let X be an $\mathbf{R}$-valued random variable on $(\Omega, \mathfrak{F}, P)$ such that*

$$EX^+ = EX^- = \infty. \tag{21}$$

Then for each random variable Y and arbitrary $\varepsilon > 0$ there exist $B \in \mathfrak{F}$ and a σ-field $\mathfrak{A} \subset \mathfrak{F}$ such that

$$P(B) \leqslant \varepsilon \quad \text{and} \quad E(X \mid \mathfrak{A}) = Y \text{ a.s.} \quad \text{on } B^c.$$

Proof. Fix $\varepsilon > 0$. It follows from (21) that there exists $B \in \mathfrak{F}$ such that $P(B) \leqslant \varepsilon$ and $EX^+ \mathbf{1}_B = EX^- \mathbf{1}_B = \infty$. Now we apply Corollary 2.2. ■

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