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CRITERIONS OF THE SIMILARITY FOR RANDOM WALKS AND BIRTH-AND-DEATH PROCESSES*

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Abstract. This paper is devoted to study the similarity of birth-and-death processes with a discrete and continuous time. We discuss some relations between the measures of orthogonality of the associated polynomials and the first return probabilities of two α-similar random walks and two ν-similar birth-and-death processes. We give the necessary and sufficient conditions for α-similarity of two random walks both in terms of the corresponding spectral measures. We consider analogous conditions for ν-similarity of two birth-and-death processes.

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1. INTRODUCTION

This work was intended as an attempt to study the similarity of birth-and-death processes with a discrete and continuous time. In Section 1 we are interested in a random walk with similar transition probabilities. We introduce a brief summary of such a process and its well-known properties. We recall the definition of α-similarity. Moreover, we give necessary and sufficient conditions for measures of orthogonality of the associated polynomials of the corresponding random walks $\mathscr{X}$ and $\tilde{\mathscr{X}}$ such that $\tilde{\mathscr{X}}$ is α-similar to $\mathscr{X}$. For such random walks we establish the relations between their first return probabilities. Section 2 contains a discussion of a birth-and-death process with a continuous time. We introduce the notion of ν-similarity and we obtain the analogous theorems but for the birth-and-death processes $\mathscr{Y}$ and $\tilde{\mathscr{Y}}$, where $\tilde{\mathscr{Y}}$ is ν-similar to $\mathscr{Y}$.

This work was inspired by the results of the papers by Schiefermayr (2003), Dette (2000) and Lenin et al. (2000).

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2. RANDOM WALKS WITH SIMILAR TRANSITION PROBABILITIES

Let $\mathscr{X} = \{X(n), n = 0, 1, \ldots\}$ denote a random walk on the nonnegative integers $\{0, 1, 2, \ldots\}$ and let

$$P_{ij}(n) = \Pr(\{X(m+n) = j\} \mid \{X(m) = i\}), \quad i, j \geqslant 0,$$

be the n-step transition probabilities. We will use the notation $p_j = P_{j,j+1}(1)$, $q_{j+1} = P_{j+1,j}(1)$, $r_j = P_{jj}(1)$, $j \geqslant 0$, and $P_{ij}(1) = 0$ for $|i-j| > 1$, $i, j \geqslant 0$. We assume that $p_j > 0$, $q_{j+1} > 0$, $r_j \geqslant 0$, $j \geqslant 0$, and $p_j + q_j + r_j \leqslant 1$, $j \geqslant 1$. The inequality $p_j + q_j + r_j < 1, j \geqslant 0$, corresponds to a permanent absorbing state j^* which can only be reached from state j with probability $1-(p_j+q_j+r_j)$.

Karlin and McGregor (1959) have shown that the n-step transition probability can be represented in the form

$$P_{ij}(n) = \pi_j \int_{-1}^{1} x^n Q_i(x) Q_j(x) d\psi(x), \quad i, j \geqslant 0, \; n \geqslant 0,$$

where

$$\pi_0 = 1, \quad \pi_j = \frac{p_0 p_1 \ldots p_{j-1}}{q_1 q_2 \ldots q_j}, \; j \geqslant 1.$$

ψ is a unique Borel measure with total mass 1 and infinite support in $[-1, 1]$, called the *random walk measure* of $\mathscr{X}$, and $Q_j(x)$ is a *random walk polynomial* of degree j defined recursively as follows:

$$\begin{aligned} &Q_{-1}(x) = 0, \quad Q_0(x) = 1, \\ &xQ_j(x) = q_j Q_{j-1}(x) + r_j Q_j(x) + p_j Q_{j+1}(x), \quad j \geqslant 0. \end{aligned} \tag{1}$$

The polynomials Q_j are orthogonal with respect to the random walk measure, i.e.

$$\pi_j \int_{-1}^{1} Q_i(x) Q_j(x) d\psi(x) = \delta_{ij},$$

where δ_{ij} denotes Kronecker's symbol.

Given the random walk, polynomials Q_j define the corresponding sequence of *first associated polynomials* $Q_j^{(1)}$ by replacing p_j, q_j and r_j by p_{j+1}, q_{j+1} and r_{j+1}, respectively, in the recurrence relation (1). Therefore the first associated polynomials satisfy the recurrence relation

$$Q_{-1}^{(1)}(x) = 0, \quad Q_0^{(1)}(x) = 1/p_0,$$

$$xQ_j^{(1)}(x) = q_{j+1} Q_{j-1}^{(1)}(x) + r_{j+1} Q_j^{(1)}(x) + p_{j+1} Q_{j+1}^{(1)}(x), \quad j \geqslant 0.$$

It follows from the arguments of Karlin and McGregor (1959) that there exists a random walk measure $\psi^{(1)}$ on the interval $[-1, 1]$ such that the first associated polynomials are orthogonal with respect to this one, i.e.

$$\pi_{j+1} p_0 q_1 \int_{-1}^{1} Q_i^{(1)}(x) Q_j^{(1)}(x)\, d\psi^{(1)}(x) = \delta_{ij}.$$

In the proofs we will use the monic associated polynomials

$$R_j^{(1)} = p_0 p_1 p_2 \ldots p_j Q_j^{(1)}(x), \quad j \geqslant 0,$$

which satisfy the recurrence relation

$$\begin{aligned} &R_{-1}^{(1)}(x) = 0, \quad R_0^{(1)}(x) = 1, \\ &R_{j+1}^{(1)}(x) = (x - r_{j+1}) R_j^{(1)}(x) - p_j q_{j+1} R_{j-1}^{(1)}(x), \quad j \geqslant 0. \end{aligned} \tag{2}$$

DEFINITION 1. For $\alpha > 0$, we call a random walk $\tilde{\mathscr{X}}$ *α-similar* to $\mathscr{X}$ if there exist constants $C_{ij} > 0$, $i, j \geqslant 0$, such that

$$\tilde{P}_{ij}(n) = \alpha^{-n} C_{ij} P_{ij}(n), \quad i, j \geqslant 0, \; n \geqslant 1.$$

In the following we will consider the random walk $\tilde{\mathscr{X}}$, α-similar to $\mathscr{X}$ ($\alpha > 0$), with parameters $\tilde{p}_j$, $\tilde{q}_j$, $\tilde{r}_j$, $j \geqslant 0$, its first associated polynomials $\tilde{Q}_j^{(1)}$ orthogonal with respect to the measure $\tilde{\psi}^{(1)}$ and the n-step transition probability $\tilde{P}_{ij}(n)$. We will use the same letter to denote the measure and its distribution function.

THEOREM 1. *The random walk $\tilde{\mathscr{X}}$ is α-similar to $\mathscr{X}$ if and only if the distribution functions of the random walk measures satisfy*

$$\tilde{\psi}^{(1)}(x) = \psi^{(1)}(\alpha x), \quad x \in \boldsymbol{R},$$

and $\alpha \geqslant \sup(\operatorname{supp}(\psi^{(1)}))$. In the case where $\tilde{\mathscr{X}}$ is α-similar to $\mathscr{X}$, we have the equalities for the first return probabilities to the origin:

$$\begin{aligned} &\tilde{P}_{i0}(n) = \alpha^{-n-1} \sqrt{\pi_{i-1}/\tilde{\pi}_{i-1}}\, P_{i0}(n), \quad i \geqslant 1, \; n \geqslant 1, \\ &\tilde{P}_{00}(n) = \alpha^{-n} P_{00}(n), \quad n \geqslant 2. \end{aligned}$$

Proof. Schiefermayr (2003) showed that the necessary and sufficient condition of α-similarity of $\tilde{\mathscr{X}}$ is the connection between parameters

$$\tilde{r}_j = \alpha^{-1} r_j, \quad \tilde{p}_j \tilde{q}_{j+1} = \alpha^{-2} p_j q_{j+1}, \quad j \geqslant 0.$$

Necessity. From the above remark and (2) we conclude that $\tilde{R}_j^{(1)}(x) = \alpha^{-j} R_j^{(1)}(\alpha x)$, which gives the equality

$$\tilde{Q}_j^{(1)}(x) = \sqrt{\pi_j/\tilde{\pi}_j}\, Q_j^{(1)}(\alpha x).$$

We proceed to show that $\tilde{\psi}^{(1)}(x) = \psi^{(1)}(\alpha x)$. We have

$$\delta_{ij} = \pi_{j+1} p_0 q_1 \int_{-1}^{1} Q_i^{(1)}(x) Q_j^{(1)}(x)\, d\psi^{(1)}(x) =$$

$$= \pi_{j+1} p_0 q_1 \int_{-1}^{1} Q_i^{(1)}(\alpha x) Q_j^{(1)}(\alpha x)\, d\psi^{(1)}(\alpha x)$$

$$= \pi_{j+1} \tilde{p}_0 \tilde{q}_1 \sqrt{\frac{\tilde{\pi}_{i+1}}{\pi_{i+1}}} \sqrt{\frac{\tilde{\pi}_{j+1}}{\pi_{j+1}}} \int_{-1}^{1} \tilde{Q}_i^{(1)}(x) \tilde{Q}_j^{(1)}(x)\, d\tilde{\psi}^{(1)}(x)$$

$$= \tilde{\pi}_{j+1} \tilde{p}_0 \tilde{q}_1 \int_{-1}^{1} \tilde{Q}_i^{(1)}(x) \tilde{Q}_j^{(1)}(x)\, d\tilde{\psi}^{(1)}(x).$$

Since $\operatorname{supp}(\tilde{\psi}^{(1)}) \subset [-1, 1]$, the parameter α has to satisfy $\alpha \geqslant \sup(\operatorname{supp}(\psi^{(1)}))$.

Sufficiency. Let $\tilde{\psi}_j^{(1)}(x) = \psi_j^{(1)}(\alpha x)$, $\alpha \geqslant \sup(\operatorname{supp}(\psi^{(1)}))$ and $R_j^{(1)}$ be the corresponding system of monic orthogonal polynomials of $\mathscr{X}$ satisfying the recurrence relation (2). Define

$$\tilde{R}_j^{(1)}(x) = \alpha^{-j} R_j^{(1)}(\alpha x). \tag{3}$$

Hence, for $i \neq j$,

$$0 = \int_{-1}^{1} R_i^{(1)}(x) R_j^{(1)}(x)\, d\psi^{(1)}(x) = \int_{-1}^{1} R_i^{(1)}(\alpha x) R_j^{(1)}(\alpha x)\, d\psi^{(1)}(\alpha x)$$

$$= \alpha^{i+j} \int_{-1}^{1} \tilde{R}_i^{(1)}(x) \tilde{R}_j^{(1)}(x)\, d\tilde{\psi}^{(1)}(x).$$

Thus $\tilde{R}_j^{(1)}$ is the corresponding system of monic orthogonal polynomials of $\tilde{\mathscr{X}}$. Using (3) we obtain the equivalent recurrence relation of $\tilde{R}_j^{(1)}(x)$, i.e.

$$R_{j+1}^{(1)}(x) = (\alpha x - \alpha \tilde{r}_{j+1}) R_j^{(1)}(x) - \alpha^2 \tilde{p}_j \tilde{q}_{j+1} R_{j-1}^{(1)}(x).$$

Consequently, it is obvious that the parameters of $\mathscr{X}$ and $\tilde{\mathscr{X}}$ satisfy the conditions $\tilde{r}_j = \alpha^{-1} r_j$ and $\tilde{p}_j \tilde{q}_{j+1} = \alpha^{-2} p_j q_{j+1}$. This completes the proof of α-similarity.

Using the results of Dette's (2000) work we can show the connections of the first return probabilities to the origin of $\mathscr{X}$ and $\tilde{\mathscr{X}}$. We have

$$\tilde{P}_{i0}(n) = \tilde{p}_0 \tilde{q}_1 \int_{-1}^{1} x^{n-1} \tilde{Q}_{i-1}^{(1)}(x)\, d\tilde{\psi}^{(1)}(x)$$

$$= \alpha^{-2} p_0 q_1 \sqrt{\frac{\pi_{i-1}}{\tilde{\pi}_{i-1}}} \int_{-1}^{1} x^{n-1} Q_{i-1}^{(1)}(\alpha x)\, d\psi^{(1)}(\alpha x)$$

$$= \alpha^{-n-1} p_0 q_1 \sqrt{\frac{\pi_{i-1}}{\tilde{\pi}_{i-1}}} \int_{-1}^{1} (\alpha x)^{n-1} Q_{i-1}^{(1)}(\alpha x)\, d\psi^{(1)}(\alpha x)$$

$$= \alpha^{-n-1} \sqrt{\frac{\pi_{i-1}}{\tilde{\pi}_{i-1}}} P_{i0}(n)$$

and

$$\tilde{P}_{00}(n) = \tilde{p}_0 \tilde{q}_1 \int_{-1}^{1} x^{n-2}\, d\tilde{\psi}^{(1)}(x) = \alpha^{-2} p_0 q_1 \int_{-1}^{1} x^{n-2}\, d\psi^{(1)}(\alpha x)$$

$$= \alpha^{-n} p_0 q_1 \int_{-1}^{1} (\alpha x)^{n-2}\, d\psi^{(1)}(\alpha x) = \alpha^{-n} P_{00}(n).$$

This is our claim. ■

The criterion of α-similarity does not depend on the initial transition probability for a sufficiently small population.

The kth associated orthogonal polynomials fulfil the recurrence relation

$$Q_{-1}^{(k)}(x) = 0, \qquad Q_0^{(k)}(x) = 1/p_{k-1},$$

$$xQ_j^{(k)}(x) = q_{j+k} Q_{j-1}^{(k)}(x) + r_{j+k} Q_j^{(k)}(x) + p_{j+k} Q_{j+1}^{(k)}(x), \qquad j \geqslant 0,\ k \geqslant 0,$$

and the corresponding measure of orthogonality $\psi^{(k)}$ plays a similar role in the consideration of connection between the first return probabilities $P_{ij}(n)$ and $\tilde{P}_{ij}(n)$, $i > j$, of $\mathscr{X}$ and $\tilde{\mathscr{X}}$, respectively.

COROLLARY 1. *The random walk $\tilde{\mathscr{X}}$ is α-similar to $\mathscr{X}$ if and only if the measures satisfy*

$$\tilde{\psi}^{(k)}(x) = \psi^{(k)}(\alpha x), \qquad x \in \mathbf{R},\ k \geqslant 0,$$

and $\alpha \geqslant \sup(\operatorname{supp}(\psi^{(k)}))$. In this case the relation between the first return probabilities to the state k of the systems $\mathscr{X}$ and $\tilde{\mathscr{X}}$ is the following:

$$\tilde{P}_{ik}(n) = \alpha^{-n-1} \sqrt{\frac{\pi_{i-k-1}}{\tilde{\pi}_{i-k-1}}}\, P_{ik}(n), \qquad i > k,\ i \geqslant 0.$$

Proof. Consider the random walk $\mathscr{X}^k$ with one-step probabilities

$$u_j^k = u_{j+k}, \qquad r_j^k = r_{j+k}, \qquad q_j^k = q_{j+k}$$

and the first associated orthogonal polynomials

$$\varphi_{-1}^{(1)}(x) = 0, \qquad \varphi_0^{(1)}(x) = 1/p_0^k,$$

$$x\varphi_j^{(1)}(x) = q_{j+1}^k \varphi_{j-1}^{(1)}(x) + r_{j+1}^k \varphi_j^{(1)}(x) + p_{j+1}^k \varphi_{j+1}^{(1)}(x), \qquad j \geqslant 0,\ k \geqslant 0.$$

We can build the monic associated polynomials for the above ones, and proceed analogously to the proof of Theorem 1 to give the conclusion for the systems $\mathscr{X}^k$ and α-similar $\tilde{\mathscr{X}}^k$ and the measures $\psi_k^{(1)}$ and $\tilde{\psi}_k^{(1)}$. The assertion of Corollary 1 follows from the recursive relation for the $(k+1)$st associated orthogonal polynomials. Using again results of Dette's (2000) work we can obtain the relation between the first return probabilities to the state k of $\mathscr{X}$ and $\tilde{\mathscr{X}}$:

$$\tilde{P}_{ik}(n) = \tilde{p}_k \tilde{q}_{k+1} \int_{-1}^{1} x^{n-1} \tilde{Q}_{i-k-1}^{(k+1)}(x)\, d\tilde{\psi}^{(k+1)}(x)$$

$$= \alpha^{-2} p_k q_{k+1} \sqrt{\frac{\pi_{i-k-1}}{\tilde{\pi}_{i-k-1}}} \int_{-1}^{1} x^{n-1} Q_{i-k-1}^{(k+1)}(\alpha x)\, d\psi^{(k+1)}(\alpha x)$$

$$= \alpha^{-n-1} p_k q_{k+1} \sqrt{\frac{\pi_{i-k-1}}{\tilde{\pi}_{i-k-1}}} \int_{-1}^{1} (\alpha x)^{n-1} Q_{i-k-1}^{(k+1)}(\alpha x)\, d\psi^{(k+1)}(\alpha x)$$

$$= \alpha^{-n-1} \sqrt{\frac{\pi_{i-k-1}}{\tilde{\pi}_{i-k-1}}} P_{ik}(n).$$

This completes our proof. ∎

By proving Theorem 1 and Corollary 1 we have also shown that $\psi^{(k)}(\alpha x)$ is a measure of orthogonality if and only if $\alpha \geqslant \sup(\operatorname{supp}(\psi^{(k)}))$, $k \geqslant 1$.

EXAMPLE 1. Let us consider a random walk $\mathscr{X}$ with constant parameters $p_j = p$, $q_j = q$, $r_j = 0$, $j \geqslant 0$, and $p+q = 1$. In this case the first associated polynomials are of the form

$$Q_j^{(1)}(x) = \left(\sqrt{\frac{q}{p}}\right)^j U_j\left(\frac{x}{2\sqrt{pq}}\right), \quad j \geqslant 0,$$

where $U_j(x)$ denotes the Chebyshev polynomials of the second kind. In such a situation $\sup(\operatorname{supp}(\psi^{(1)})) = 2\sqrt{pq}$. Since $\alpha \geqslant 2\sqrt{pq}$, let $b \geqslant 1$ such that $\alpha = 2b\sqrt{pq}$.

Schiefermayr (2003) showed that for $\mathscr{X}$ as in this example there exists a unique α-similar random walk $\tilde{\mathscr{X}}$ with parameters $\tilde{p}_j$, $\tilde{q}_j$, $\tilde{r}_j$ given by

$$\tilde{p}_j = \alpha^{-1} \frac{Q_{j+1}(\alpha)}{Q_j(\alpha)} p_j, \quad \tilde{q}_{j+1} = \alpha^{-1} \frac{Q_j(\alpha)}{Q_{j+1}(\alpha)} q_{j+1}, \quad \tilde{r}_j = \alpha^{-1} r_j, \quad j \geqslant 0,$$

where $\tilde{q}_0 = 0$. In our example $\tilde{\pi}_j = (Q_j(\alpha))^2 \cdot \pi_j$ and

$$P_{i0}(n) = p_0 q_1 \int_{-1}^{1} x^{n-1} Q_{i-1}^{(1)}(x)\, d\psi^{(1)}(x)$$

$$= \frac{2}{\pi} pq \int_{-1}^{1} x^{n-1} \left(\sqrt{\frac{p}{q}}\right)^{i-1} U_{i-1}\left(\frac{x}{2\sqrt{pq}}\right) \sqrt{1 - \frac{x^2}{4pq}}\, dx$$

$$= \frac{2}{\pi} pq \left(\sqrt{\frac{p}{q}}\right)^{i-1} \left(\frac{1}{2\sqrt{pq}}\right)^{1-n} \int_{-1}^{1} \left(\frac{x}{2\sqrt{pq}}\right)^{n-1} U_{i-1}\left(\frac{x}{2\sqrt{pq}}\right) \sqrt{1-\frac{x^2}{4pq}}\, dx$$

$$= \frac{2}{\pi} pq \left(\sqrt{\frac{p}{q}}\right)^{i-1} \left(\frac{1}{2\sqrt{pq}}\right)^{1-n} \int_{-1}^{1} x^{n-1} U_{i-1}(x) \sqrt{1-x^2}\, dx$$

$$= 2pq \left(\sqrt{\frac{p}{q}}\right)^{i-1} \left(\frac{1}{2\sqrt{pq}}\right)^{1-n} \frac{i}{n \cdot 2^n} \binom{n}{(n-i)/2}$$

if $i+n$ is even. For odd $i+n$, $P_{i0} = 0$. See Dette (2000) for more details.

Using the results of Theorem 1 we can calculate the first return probability to the origin for $\tilde{\mathscr{X}}$:

$$\tilde{P}_{i0}(n) = \frac{1}{U_{i-1}(b)} \cdot \frac{i}{n(2b)^{n+1}} \binom{n}{(n-i)/2}$$
$$= \frac{\sin(\arccos b)}{\sin(i \arccos b)} \cdot \frac{i}{n(2b)^{n+1}} \binom{n}{(n-i)/2}.$$

3. BIRTH-AND-DEATH PROCESSES WITH SIMILAR TRANSITION PROBABILITIES

We will deduce analogous criterions of the similarity for the birth-and-death processes, relations between their measures and first return probabilities.

Let $\mathscr{Y} = \{Y(t), t \geqslant 0\}$ denote a birth-and-death process, i.e. a stationary Markov process whose transition probability function

$$P_{ij}(t) = \Pr(\{Y(t) = j\} \mid \{Y(0) = i\})$$

satisfies the conditions

$$P_{j,j+1}(t) = \lambda_j t + o(t),$$
$$P_{j,j}(t) = 1 - (\lambda_j + \mu_j) t + o(t),$$
$$P_{j,j-1}(t) = \mu_j t + o(t)$$

as $t \to 0$. Constants λ_j (birth rates) and μ_j (death rates) may be thought of as the rates of absorption from state j into states $j+1$ and $j-1$, respectively ($\lambda_j > 0$, $\mu_j > 0, j = 0, 1, \ldots, \mu_0 \geqslant 0$). Karlin and McGregor (1957) have shown that the transition probabilities P_{ij} can be represented as

$$P_{ij}(t) = \kappa_j \int_0^{\infty} e^{-xt} G_i(x) G_j(x)\, d\varrho(x),$$

$$\kappa_0 = 1, \quad \kappa_j = \frac{\lambda_0 \lambda_1 \ldots \lambda_{j-1}}{\mu_1 \mu_2 \ldots \mu_j}, \quad j > 0.$$

$\{G_j(x)\}$ is a sequence of birth-and-death polynomials defined recursively:

$$G_{-1}(x) = 0, \quad G_0(x) = 1, \tag{4}$$
$$-xG_j(x) = \mu_j G_{j-1}(x) - (\lambda_j + \mu_j) G_j(x) + \lambda_j G_{j+1}(x), \quad j \geqslant 1,$$

and orthogonal with respect to the *spectral measure* ϱ, i.e.

$$\kappa_j \int_0^\infty G_i(x) G_j(x) d\varrho(x) = \delta_{ij}.$$

It is shown in the paper of Karlin and McGregor (1957) that there is at least one such measure with total mass 1 on $[0, \infty)$.

In the proofs we will use the monic polynomials

$$W_j(x) = (-1)^j \lambda_0 \lambda_1 \ldots \lambda_{j-1} G_j(x), \quad j \geqslant 1,$$

which satisfy the recurrence relation

$$W_{-1}(x) = 0, \quad W_0(x) = 1, \tag{5}$$
$$W_{j+1}(x) = (x - \lambda_j - \mu_j) W_j(x) - \lambda_{j-1} \mu_j W_{j-1}(x), \quad j \geqslant 0.$$

DEFINITION 2. The birth-and-death process $\tilde{\mathcal{Y}}$ is said to be *v-similar* to the birth-and-death process $\mathcal{Y}$ for some real number v if there are constants c_{ij}, $i, j \geqslant 0$, such that

$$\tilde{P}_{ij}(t) = c_{ij} e^{vt} P_{ij}(t), \quad i, j \geqslant 0, \ t \geqslant 0.$$

See Lenin et al. (2000) for more details.

$\tilde{\mathcal{Y}}$ is the process with parameters $\tilde{\lambda}_j$, $\tilde{\mu}_j$, $j \geqslant 0$, and polynomials $\tilde{G}_j$ orthogonal with respect to the measure $\tilde{\varrho}$.

THEOREM 2. *The birth-and-death process $\tilde{\mathcal{Y}}$ is v-similar to $\mathcal{Y}$ if and only if the distribution functions of the spectral measures satisfy*

$$\tilde{\varrho}(x) = \varrho(x - v), \quad x \in \mathbf{R},$$

and $v \leqslant \inf(\operatorname{supp}(\varrho))$.

Proof. Necessity. We claim that

$$\tilde{W}_j(x) = W_j(x - v). \tag{6}$$

This is implied by the fact that for the birth-and-death processes $\mathcal{Y}$ and $\tilde{\mathcal{Y}}$, where $\tilde{\mathcal{Y}}$ is v-similar to $\mathcal{Y}$, their rates are related as follows:

$$\tilde{\lambda}_j + \tilde{\mu}_j = \lambda_j + \mu_j - v, \quad \tilde{\lambda}_j \tilde{\mu}_{j+1} = \lambda_j \mu_{j+1}, \quad j \geqslant 0. \tag{7}$$

We conclude from (6) that $\tilde{G}_j(x) = \sqrt{\kappa_j / \tilde{\kappa}_j}\, G_j(x - v)$.

Next we claim that $\tilde{\varrho}(x)=\varrho(x-\nu)$ since

$$\delta_{ij}=\kappa_j\int\limits_0^\infty G_i(x)\,G_j(x)\,d\varrho(x)=\kappa_j\int\limits_0^\infty G_i(x-\nu)\,G_j(x-\nu)\,d\varrho(x-\nu)$$

$$=\kappa_j\sqrt{\frac{\tilde{\kappa}_j\tilde{\kappa}_i}{\kappa_j\kappa_i}}\int\limits_0^\infty \tilde{G}_i(x)\,\tilde{G}_j(x)\,d\tilde{\varrho}(x)=\tilde{\kappa}_j\int\limits_0^\infty \tilde{G}_i(x)\,\tilde{G}_j(x)\,d\tilde{\varrho}(x).$$

Since $\operatorname{supp}(\tilde{\varrho})\subset[0,\infty]$, the parameter ν has to satisfy $\nu\leqslant\inf(\operatorname{supp}(\varrho))$.

Sufficiency. Let ϱ and $\tilde{\varrho}$ be the spectral measures of $\mathscr{Y}$ and $\tilde{\mathscr{Y}}$, respectively. Let $\tilde{\varrho}(x)\doteq\varrho(x-\nu)$, $\nu\leqslant\inf(\operatorname{supp}(\varrho))$, and W_j be the corresponding system of monic orthogonal polynomials of $\mathscr{Y}$ satisfying the recurrence relation (5).

Let $\tilde{W}_j$ be defined by (6). Hence, for $i\neq j$,

$$0=\int\limits_0^\infty W_i(x)\,W_j(x)\,d\varrho(x)=\int\limits_0^\infty W_i(x-\nu)\,W_j(x-\nu)\,d\varrho(x-\nu)$$

$$=\int\limits_0^\infty \tilde{W}_i(x)\,\tilde{W}_j(x)\,d\tilde{\varrho}(x).$$

It follows that $\tilde{W}_j$ is the system of orthogonal polynomials of $\tilde{\mathscr{Y}}$. From the equation (6) we obtain the recurrence relation of $\tilde{W}_j$, i.e.

$$W_{j+1}(x)=\big(x-(\tilde{\lambda}_j+\tilde{\mu}_j-\nu)\big)\,W_j(x)-\tilde{\lambda}_{j-1}\,\tilde{\mu}_j\,W_{j-1}(x). \tag{8}$$

Comparing (5) and (8) we obtain the equalities for rates of $\mathscr{Y}$ and $\tilde{\mathscr{Y}}$ as in (7). Such connections of rates prove the ν-similarity of $\tilde{\mathscr{Y}}$, as shown by Lenin et al. (2000). ∎

We can formulate the analogous theorem for measures of the orthogonality $\varrho^{(1)}$ and $\tilde{\varrho}^{(1)}$ of the first associated polynomials $G_j^{(1)}$ and $\tilde{G}_j^{(1)}$, where

$$G_{-1}^{(1)}(x)=0,\qquad G_0^{(1)}(x)=-1/\lambda_0,$$

$$-xG_j^{(1)}(x)=\mu_{j+1}\,G_{j-1}^{(1)}(x)-(\lambda_{j+1}+\mu_{j+1})\,G_j^{(1)}(x)+\lambda_{j+1}\,G_{j+1}^{(1)}(x),\quad j\geqslant 0.$$

The monic form of these polynomials is

$$W_j^{(1)}(x)=(-1)^{j+1}\lambda_0\lambda_1\ldots\lambda_j\,G_j^{(1)}(x),\qquad j\geqslant 0,$$

and satisfies the recurrence relation

$$W_{-1}^{(1)}(x)=0,\qquad W_0^{(1)}(x)=1,$$
$$W_{j+1}^{(1)}(x)=(x-\lambda_{j+1}-\mu_{j+1})\,W_j^{(1)}(x)-\lambda_j\mu_{j+1}\,W_{j-1}^{(1)}(x),\qquad j\geqslant 0. \tag{9}$$

THEOREM 3. *The necessary and sufficient condition of the ν-similarity of $\tilde{\mathscr{Y}}$ when considering $\mathscr{Y}$ is the equality of measures*

$$\tilde{\varrho}^{(1)}(x)=\varrho^{(1)}(x-\nu),\qquad x\in\boldsymbol{R},$$

and $v \leqslant \inf(\mathrm{supp}(\varrho^{(1)}))$. *If* $\tilde{\mathscr{Y}}$ *is* v*-similar to* $\mathscr{Y}$, *we have the following relation for the first return probabilities to the origin:*

$$\tilde{P}_{i0}(t) = e^{-vt}\sqrt{\frac{\kappa_{i-1}}{\tilde{\kappa}_{i-1}}}\,P_{i0}, \qquad i \geqslant 1,\ t \geqslant 0.$$

Proof. The proof of the equivalence is analogous to the one of Theorem 2, but with using the polynomials $G_j^{(1)}(x)$ and $\tilde{G}_j^{(1)}(x)$ and their monic forms $W_j^{(1)}(x)$ and $\tilde{W}_j^{(1)}(x)$. Next we use the well-known formula for the probability of the first return to the origin (see van Doorn (2003)):

$$\begin{aligned}\tilde{P}_{i0}(t) &= \tilde{\lambda}_0\tilde{v}_1\int_0^\infty e^{-xt}\tilde{Q}_{i-1}^{(1)}(x)\,d\tilde{\varrho}^{(1)}(x)\\ &= \lambda_0 v_1\sqrt{\frac{\kappa_{i-1}}{\tilde{\kappa}_{i-1}}}\int_0^\infty e^{-xt}Q_{i-1}^{(1)}(x-v)\,d\varrho^{(1)}(x-v)\\ &= e^{-vt}\sqrt{\frac{\kappa_{i-1}}{\tilde{\kappa}_{i-1}}}\,\lambda_0 v_1\int_0^\infty e^{-(x-v)t}Q_{i-1}^{(1)}(x-v)\,d\varrho^{(1)}(x-v) = e^{-vt}\sqrt{\frac{\kappa_{i-1}}{\tilde{\kappa}_{i-1}}}\,P_{i0}(t).\end{aligned}$$

This is the desired conclusion. ∎

This result can be generalized. Let us consider the kth associated polynomials

$$G_{-1}^{(k)}(x) = 0, \qquad G_0^{(k)}(x) = -1/\lambda_{k-1},$$

$$-xG_j^{(k)}(x) = \mu_{j+k}G_{j-1}^{(k)}(x) - (\lambda_{j+k}+\mu_{j+k})G_j^{(k)}(x) + \lambda_{j+k}G_{j+1}^{(k)}(x), \qquad j \geqslant 0,\ k \geqslant 0,$$

orthogonal to the measure $\varrho^{(k)}(x)$. Our extension deals with the v-similarity and relation between such measures.

COROLLARY 2. *The birth-and-death process* $\tilde{\mathscr{Y}}$ *is* v*-similar to* $\mathscr{Y}$ *if and only if the measures satisfy*

$$\tilde{\varrho}^{(k)} = \varrho^{(k)}(x-v), \qquad x \in \mathbf{R},\ k \geqslant 0,$$

and $v \leqslant \inf(\mathrm{supp}(\varrho^{(k)}))$.

Proof. We can proceed analogously to the proof of Corollary 1 from the previous section. We can build the birth-and-death process $\mathscr{Y}^k$ with parameters

$$\mu_j^k = \mu_{j+k}, \qquad \lambda_j^k = \lambda_{j+k}$$

and with the corresponding monic associated polynomials orthogonal to the measure $\varrho_k^{(1)}(x)$. The assertion is obtained by Theorem 3. ∎

Remark. The proofs of Corollary 2 and Theorem 3 yield an additional information. It follows that $\varrho^{(k)}(x-v)$ is also a measure of the orthogonality of the kth associated polynomials if and only if $v \leqslant \inf(\mathrm{supp}(\varrho^{(k)}))$.

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