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A CALCULUS ON LÉVY EXPONENTS AND SELFDECOMPOSABILITY ON BANACH SPACES*

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Abstract. In infinite-dimensional Banach spaces there is no complete characterization of the Lévy exponents of infinitely divisible probability measures. Here we propose a *calculus on Lévy exponents* that is derived from some random integrals. As a consequence we prove that *each* selfdecomposable measure can by factorized as another selfdecomposable measure and its background driving measure that is s-selfdecomposable. This complements a result from the paper of Iksanov, Jurek and Schreiber in the Annals of Probability (2004).

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1. INTRODUCTION

Recall that a Borel probability measure μ, on a real separable Banach space E, is called *infinitely divisible* if for each natural number n there exists a probability measure μ_n such that $\mu_n^{*n} = \mu$; the class of all infinitely divisible measures will be denoted by ID. It is well known that their Fourier transforms (the *Lévy–Khintchine formulas*) can be written as follows:

(1.1) $\hat{\mu}(y) = e^{\Phi(y)},\ y \in E',$ and the exponents Φ are of the form

$$\Phi(y) = i\langle y, a\rangle - \frac{1}{2}\langle y, Ry\rangle + \int_{E\setminus\{0\}} [e^{i\langle y,x\rangle} - 1 - i\langle y, x\rangle 1_B(x)]M(dx),$$

where E' denotes the dual Banach space, $\langle \cdot, \cdot\rangle$ is an appropriate bilinear form between E' and E, a is a *shift vector*, R is a *covariance operator* corresponding to

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the Gaussian part of μ, and M is a *Lévy spectral measure*. There is a one-to-one correspondence between $\mu \in ID$ and the triples $[a, R, M]$ in its Lévy–Khintchine formula (1.1); cf. Araujo and Giné [1], Chapter 3, Section 6, p. 136. The function $\Phi(y)$ from (1.1) is called then the *Lévy exponent* of μ.

REMARK 1.1. (a) If E is a Hilbert space, then Lévy spectral measures M are completely characterized by the integrability condition $\int_E (1 \wedge \|x\|^2) M(dx) < \infty$, and Gaussian covariance operators R coincide with the positive trace-class operators; cf. Parthasarathy [12], Chapter VI, Theorem 4.10.

(b) When E is a Euclidean space, then Lévy exponents are completely characterized as continuous negative-definite functions; cf. Cuppens [4] and Schoenberg's theorem on p. 80.

Finally, a *Lévy process* $Y(t), t \geqslant 0$, means a continuous in probability process with stationary and independent increments and $Y(0) = 0$. Without loss of generality we may and do assume that it has paths in the Skorokhod space $D_E[0, \infty)$ of E-valued *càdlàg functions* (i.e., right continuous with left-hand limits). There is a one-to-one correspondence between the class ID and the class of Lévy processes.

The càdlàg paths of a process Y allow us to define *random integrals* of the form $\int_{(a,b]} h(s) Y\big(r(ds)\big)$ by the formal formula of integration by parts. Namely,

$$(1.2)\quad \int\limits_{(a,b]} h(s) Y\big(r(ds)\big) := h(b) Y\big(r(b)\big) - h(a) Y\big(r(a)\big) - \int\limits_{(a,b]} Y\big(r(s)\big) dh(s),$$

where h is a real-valued function of bounded variation and r is a monotone and right-continuous function. Furthermore, we have

$$(1.3)\quad \Big(\mathcal{L}\Big(\int\limits_{(a,b]} h(s) Y\big(r(ds)\big)\Big)\Big)\widehat{\ }(y) = \exp\Big[\int\limits_{(a,b]} \log\Big(\mathcal{L}\big(Y(1)\big)\Big)\widehat{\ }\big(h(s)y\big) r(ds)\Big],$$

where $\mathcal{L}(\cdot)$ denotes the probability distribution and $\hat{\mu}(\cdot)$ denotes the Fourier transform of a measure μ; cf. Jurek and Vervaat [10] or Jurek [7], or Jurek and Mason [9], Section 3.6, p. 116.

2. A CALCULUS ON LÉVY EXPONENTS

Let $\mathcal{E}$ denote the totality of all functions $\Phi : E' \to \mathbb{C}$ appearing as the exponent in the Lévy–Khintchine formula (1.1). Hence we have

$$(2.1)\quad \mathcal{E} + \mathcal{E} \subset \mathcal{E}, \quad \lambda \cdot \mathcal{E} \subset \mathcal{E} \text{ for all positive } \lambda,$$

which means that $\mathcal{E}$ forms a cone in the space of all complex-valued functions defined on E'. Furthermore, if $\Phi \in \mathcal{E}$, then all dilations $\Phi(a\cdot) \in \mathcal{E}$. These follow from the fact that infinite divisibility is preserved under convolution and under (convolution) powers to positive real numbers.

Here we consider two integral operators acting on $\mathcal{E}$ or its part. Namely,

$$
\begin{aligned}
&\mathcal{J}\colon \mathcal{E} \to \mathcal{E}, \quad (\mathcal{J}\Phi)(y) := \int_0^1 \Phi(sy)ds, \;\; y \in E';\\
(2.2)\qquad &\mathcal{I}\colon \mathcal{E}_{\log} \to \mathcal{E}, \quad (\mathcal{I}\Phi)(y) := \int_0^1 \Phi(sy)s^{-1}ds, \;\; y \in E'.
\end{aligned}
$$

Note that $\mathcal{J}$ is well defined on all of $\mathcal{E}$ since, by (1.3), $\mathcal{J}\Phi$ is the Lévy exponent of the well-defined integral $\int_{(0,1)} tdY(t)$, where $Y(1)$ has the Lévy exponent Φ; cf. Jurek [7] or [8]. On the other hand, $\mathcal{I}$ is only defined on $\mathcal{E}_{\log}$, which corresponds to infinitely divisible measures with finite logarithmic moments, since $\mathcal{I}\Phi$ is the Lévy exponent of the random integral $-\int_{(0,1)} tdY(-\ln t) = \int_{(0,\infty)} e^{-s}dY(s)$, where Φ is the Lévy exponent of $Y(1)$ that has finite logarithmic moment; cf. Jurek and Vervaat [10].

Here are the main algebraic properties of the mappings $\mathcal{J}$ and $\mathcal{I}$.

LEMMA 2.1. *The operators $\mathcal{I}$ and $\mathcal{J}$ acting on appropriate domains (Lévy exponents) have the following basic properties*:

(a) *$\mathcal{I}, \mathcal{J}$ are additive and positive homogeneous operators*;

(b) *$\mathcal{I}, \mathcal{J}$ commute under the composition and $\mathcal{J}\big(\mathcal{I}(\Phi)\big) = (\mathcal{I} - \mathcal{J})\Phi$.*

Other equivalent forms of the last property are:

$$\mathcal{J}(I + \mathcal{I}) = \mathcal{I}; \quad \mathcal{I}(I - \mathcal{J}) = \mathcal{J}; \quad (I - \mathcal{J})(I + \mathcal{I}) = I.$$

Proof. Part (a) follows from the fact that $\mathcal{E}$ forms a cone. For part (b) let us note that

$$
\begin{aligned}
\Big(\mathcal{J}\big(\mathcal{I}(\Phi)\big)\Big)(y) &= \int_0^1 \big(\mathcal{I}(\Phi)\big)(ty)\,dt = \int_0^1\int_0^1 \Phi(sty)s^{-1}dsdt\\
&= \int_0^1\int_0^t \Phi(ry)r^{-1}drdt = \int_0^1\int_r^1 \Phi(ry)dt\,r^{-1}dr\\
&= \int_0^1 \Phi(ry)r^{-1}dr - \int_0^1 \Phi(ry)dr = \mathcal{I}\Phi(y) - \mathcal{J}\Phi(y)\\
&= (\mathcal{I} - \mathcal{J})\Phi(y),
\end{aligned}
$$

which proves the equality in (b). Note that from the above (the first line of the above argument) we infer also that the operators $\mathcal{I}$ and $\mathcal{J}$ commute, which completes the argument. ■

LEMMA 2.2. *The operators $\mathcal{I}$ and $\mathcal{J}$, defined by* (2.2), *have the following additional properties*:

(a) *$\mathcal{J}: \mathcal{E}_{\log} \to \mathcal{E}_{\log}$ and $\mathcal{I}: \mathcal{E}_{(\log)^2} \to \mathcal{E}_{\log}$.*

(b) *If* $(I-\mathcal{J})\Phi \in \mathcal{E}$, *then the corresponding infinitely divisible measure* $\tilde{\mu}$ *with the Lévy exponent* $(I-\mathcal{J})\Phi(y)$, $y \in E'$, *has finite logarithmic moment.*

(c) $(I-\mathcal{J})\Phi + \mathcal{I}(I-\mathcal{J})\Phi = (I-\mathcal{J})\Phi + \mathcal{J}\Phi = \Phi$ *for all* $\Phi \in \mathcal{E}$.

Proof. (a) Since the function $E \ni x \to \log(1+\|x\|)$ is subadditive, for an infinitely divisible probability measure $\mu = [a, R, M]$ we have

$$\text{(2.3)} \qquad \int_E \log(1+\|x\|)\mu(dx) < \infty \ \text{ iff } \int_{\{\|x\|>1\}} \log(1+\|x\|)M(dx) < \infty$$
$$\text{iff } \int_{\{\|x\|>1\}} \log\|x\| M(dx) < \infty;$$

cf. Jurek and Mason [9], Proposition 1.8.13. Furthermore, if M is the spectral Lévy measure appearing in the Lévy exponent Φ, then $\mathcal{J}\Phi$ has a Lévy spectral measure $\mathcal{J}M$ (we keep that potentially conflicting notation), where

$$\text{(2.4)} \qquad (\mathcal{J}M)(A) := \int_{(0,1)} M(t^{-1}A)dt = \int_{(0,1)} \int_E 1_A(tx)M(dx)dt$$

for all Borel subsets A of $E \setminus \{0\}$. Hence

$$\int_{\{\|x\|>1\}} \log\|x\|(\mathcal{J}M)(dx) = \int_{(0,1)} \int_E 1_{\{\|x\|>1\}}(tx)\log(t\|x\|)M(dx)dt$$
$$= \int_{(0,1)} \int_{\{\|x\|>t^{-1}\}} \log(t\|x\|)M(dx)dt = \int_{\{\|x\|>1\}} \int_{\|x\|^{-1}}^{1} \log(t\|x\|)dt\, M(dx)$$
$$= \int_{\{\|x\|>1\}} \|x\|^{-1} \int_1^{\|x\|} \log w\, dw\, M(dx)$$
$$= \int_{\{\|x\|>1\}} \|x\|^{-1}[\|x\|\log\|x\| - \|x\| + 1]M(dx)$$
$$= \int_{\{\|x\|>1\}} \log\|x\| M(dx) - \int_{\{\|x\|>1\}} [1-\|x\|^{-1}]M(dx).$$

Since the last integral is always finite as we integrate a bounded function with respect to a finite measure, we get the first part of (a). For the second one, let us note that

$$\int_{\{\|x\|>1\}} \log\|x\|(\mathcal{I}M)(dx) = \int_0^\infty \int_{\{\|x\|>1\}} \log\|x\| M(e^t dx)dt$$
$$= \frac{1}{2} \int_{\{\|x\|>1\}} \log^2\|x\| M(dx),$$

where $\mathcal{I}M$ is the Lévy spectral measure corresponding to the Lévy exponent $\mathcal{I}\Phi$.

For the part (b), note that the assumption made there implies that the measure

$$\widetilde{M}(A) := M(A) - \int_{(0,1)} M(t^{-1}A)dt \geqslant 0 \text{ for all Borel sets } A \subset E \setminus \{0\} \tag{2.5}$$

is the Lévy spectral measure of some $\tilde{\mu}$. [Note that there is no restriction on the Gaussian part.] In fact, if $\widetilde{M}$ is a nonnegative measure, then it is necessarily a Lévy spectral measure because $0 \leqslant \widetilde{M} \leqslant M$ and M is a Lévy spectral measure; cf. Araujo and Giné [1], Chapter 3, Theorem 4.7, p. 119. To establish the logarithmic moment of $\tilde{\mu}$ we argue as follows. Observe that for any constant $k > 1$ we have

$$\begin{aligned}
0 &\leqslant \int_{\{1<\|x\|\leqslant k\}} \log\|x\| \widetilde{M}(dx) \\
&= \int_{\{1<\|x\|\leqslant k\}} \log\|x\| M(dx) - \int_{(0,1)} \int_{\{1<\|x\|\leqslant k\}} \log\|x\| M(t^{-1}dx)dt \\
&= \int_{\{1<\|x\|\leqslant k\}} \log\|x\| M(dx) - \int_{(0,1)} \int_{\{t^{-1}<\|x\|\leqslant kt^{-1}\}} \log(t\|x\|)\, M(dx)dt \\
&= \int_{\{1<\|x\|\leqslant k\}} \log\|x\| M(dx) - \int_{\{1<\|x\|\leqslant k\}} \int_{\|x\|^{-1}}^{1} \log(t\|x\|)dt\, M(dx) \\
&\quad - \int_{\{k<\|x\|\}} \int_{\|x\|^{-1}}^{k\|x\|^{-1}} \log(t\|x\|)dt\, M(dx) \\
&= \int_{\{1<\|x\|\leqslant k\}} \log\|x\| M(dx) - \int_{\{1<\|x\|\leqslant k\}} \|x\|^{-1} \int_{1}^{\|x\|} \log(w)\, dw\, M(dx) \\
&\quad - \int_{\{k<\|x\|\}} \|x\|^{-1} \int_{1}^{k} \log(w)\, dw\, M(dx) \\
&= \int_{\{1<\|x\|\leqslant k\}} \log\|x\| M(dx) - \int_{\{1<\|x\|\leqslant k\}} \|x\|^{-1}(\|x\|\log\|x\| - \|x\| + 1)M(dx) \\
&\quad - (k\log k - k + 1) \int_{\{\|x\|>k\}} \|x\|^{-1} M(dx) \\
&= \int_{\{1<\|x\|\leqslant k\}} (1 - \|x\|^{-1})M(dx) - (k\log k - k + 1) \int_{\{\|x\|>k\}} \|x\|^{-1} M(dx) \\
&\leqslant M(\|x\| > 1) < \infty,
\end{aligned}$$

and consequently $\int_{\{\|x\|>1\}} \log\|x\| \widetilde{M}(dx) < \infty$. This with property (2.3) completes the proof of the part (b).

Finally, since $(I - \mathcal{J})\Phi$ is in a domain of definition of the operator $\mathcal{I}$, so the part (c) is a consequence of Lemma 2.1 (b). Thus the proof is complete. ∎

3. FACTORIZATIONS OF SELFDECOMPOSABLE DISTRIBUTIONS

The classes of limit laws $\mathcal{U}$ and L are obtained by non-linear shrinking transformations and linear transformations (multiplications by scalars), respectively; cf. Jurek [7] and references therein. However, there are many (unexpected) relations between $\mathcal{U}$ and L as was already proved in Jurek [7] and more recently in Iksanov et al. [6]. Furthermore, more recently selfdecomposable distributions are used in modelling real phenomena, in particular in mathematical finance; for instance cf. Bingham [2], Carr et al. [3] or Eberlein and Keller [5]. This motivates further studies on factorizations and other relations between the classes $\mathcal{U}$ and L, like those in Theorems 3.1 and 3.2 below.

In this section we will apply the operators $\mathcal{I}$ and $\mathcal{J}$ to Lévy exponents of selfdecomposable (the class L) and s-selfdecomposable (the class $\mathcal{U}$) probability measures. For the convenience of the readers recall here that

$$\begin{aligned}(3.1)\qquad \mu \in L \ \text{ iff } \ &\forall (t>0) \exists \nu_t \ \ \mu = T_{e^{-t}}\,\mu * \nu_t \\ &\text{iff } \ \mu = \mathcal{L}\Big(\int\limits_{(0,\infty)} e^{-t} dY(t)\Big), \ \ \mathcal{L}\big(Y(1)\big) \in ID_{\log}; \\ \mu \in \mathcal{U} \ \text{ iff } \ &\mu = \mathcal{L}\Big(\int\limits_{(0,1]} t\, dY(t)\Big), \ \ \mathcal{L}\big(Y(1)\big) \in ID.\end{aligned}$$

Measures from the class $\mathcal{U}$ are called *s-selfdecomposable*; cf Jurek [7], [8]. The corresponding Fourier transforms of measures from L and $\mathcal{U}$ follow easily from (1.2) and (1.3); cf. Jurek and Vervaat [10] or the above references.

LEMMA 3.1. *If μ is a selfdecomposable probability measure on a Banach space E with characteristic function $\hat{\mu}(y) = \exp[\Phi(y)]$, $y \in E'$, then*

$$\widetilde{\Phi}(y) := \Phi(y) - \int\limits_{(0,1)} \Phi(sy) ds = (I - \mathcal{J})\Phi(y), \qquad y \in E',$$

is a Lévy exponent corresponding to an infinitely divisible probability measure with finite logarithmic moment.

Equivalently, if M is the Lévy spectral measure of a selfdecomposable μ, then the measure $\widetilde{M}$ given by

$$\widetilde{M}(A) := M(A) - \int\limits_0^1 M(t^{-1}A) dt, \qquad A \subset E \setminus \{0\},$$

is a Lévy spectral measure on E that additionally integrates the logarithmic function on the complement of any neighborhood of zero.

P r o o f. If $\mu = [a, R, M]$ is selfdecomposable (or, in other words, a class L distribution), then we infer that

$$M(A) - M(e^t A) \geqslant 0 \quad \text{ for all } \ t > 0 \text{ and Borel } \ A \subset E \setminus \{0\},$$

and that there is no restriction on the remaining two parameters (the shift vector and the Gaussian covariance operator) in the Lévy–Khintchine formula (1.1). Multiplying both sides by e^{-t} and then integrating over the positive half-line we conclude that $\widetilde{M}$, given by (2.5), is a non-negative measure. Since $\widetilde{M} \leqslant M$ and M is a Lévy spectral measure, so is $\widetilde{M}$; cf. Theorem 4.7 in Chapter 3 of Araujo and Giné [1]. Finally, our Lemma 2.2 (b) gives the finiteness of the logarithmic moment. Thus the proof is complete. ■

THEOREM 3.1. *For each selfdecomposable probability measure μ, on a Banach space E, there exists a unique s-selfdecomposable probability measure $\tilde{\mu}$ with finite logarithmic moment such that*

$$\mu = \tilde{\mu} * \mathcal{I}(\tilde{\mu}) \quad \textit{and} \quad \mathcal{J}(\mu) = \mathcal{I}(\tilde{\mu}). \tag{3.2}$$

In fact, if $\hat{\mu}(y) = \exp[\Phi(y)]$, then $(\tilde{\mu})\hat{}(y) = \exp\left[\Phi(y) - \int_{(0,1)} \Phi(ty)dt\right]$, $y \in E'$.

In other words, if Φ is the Lévy exponent of a selfdecomposable probability measure, then $(I - \mathcal{J})\Phi$ is the Lévy exponent of an s-selfdecomposable measure with the finite logarithmic moment and

$$\Phi = (I - \mathcal{J})\Phi + \mathcal{I}(I - \mathcal{J})\Phi = (I - \mathcal{J})\Phi + \mathcal{J}\Phi. \tag{3.3}$$

Proof. Let $\hat{\mu}(y) = \exp[\Phi(y)] \in L$. From the factorization in (3.1) (the first line) we infer that $\Phi_t(y) := \Phi(y) - \Phi(e^{-t}y)$ are Lévy exponents as well. Hence,

$$\widetilde{\Phi}(y) := \int_{(0,\infty)} \Phi_t(ty)e^{-t}dt = \Phi(y) - \int_{(0,\infty)} \Phi(e^{-t}y)e^{-t}dt = \big((I - \mathcal{J})\Phi\big)(y)$$

is a Lévy exponent as well, because of Lemma 3.1. Again by Lemma 3.1 (or Lemma 2.2 (b)), a probability measure $\tilde{\mu}$ defined by the Fourier transform $(\tilde{\mu})\hat{}(y) = \exp(I - \mathcal{J})\Phi(y)$ has logarithmic moment. Consequently, $\mathcal{I}(\tilde{\mu})$ is a well-defined probability measure whose Lévy exponent is equal to $\mathcal{I}(I - \mathcal{J})\Phi$. Finally, Lemmas 2.1 (b) and 2.2 (c) give the factorization (3.2).

Since $\mathcal{I}(\tilde{\mu}) \in L$ has the property that $\tilde{\mu} * \mathcal{I}(\tilde{\mu})$ is again in L, therefore Theorem 1 from Iksanov et al. [6] implies that $\tilde{\mu} \in \mathcal{U}$, i.e., it is an s-selfdecomposable probability distribution.

To see the second equality in (3.3) one should observe that it is equivalent to the equality $\mathcal{J}\Phi = \mathcal{I}(I - \mathcal{J})\Phi$ that indeed holds true in view of Lemma 2.1 (b).

Suppose there exists another factorization of the form $\mu = \rho * \mathcal{I}(\rho)$ and let $\Xi(y)$ be the Lévy exponent of ρ. Then we see that $\Phi(y) = \Xi(y) + (\mathcal{I}\,\Xi)(y) = (I + \mathcal{I})\,\Xi(y)$. Hence, applying to both sides $\mathcal{I} - \mathcal{J}$ we conclude that

$$(I - \mathcal{J})\Phi = \big((I - \mathcal{J})(I + \mathcal{I})\big)\,\Xi = \Xi,$$

where the last equality is from Lemma 2.1 (b). This proves the uniqueness of $\tilde{\mu}$ in the representation (3.2), and thus the proof of Theorem 3.1 is completed. ■

REMARK 3.1. The factorization (3.2) in Theorem 3.1 can be also derived from previous papers as follows:

For each selfdecomposable (or class L) μ there exists a unique $\rho \in ID_{\log}$ such that $\mu = \mathcal{I}(\rho)$; Jurek and Vervaat [10]. Since $\tilde{\mu} := \mathcal{J}(\rho)$ is an s-selfdecomposable (class $\mathcal{U}$) with logarithmic moment (cf. Jurek [7]), therefore $\mathcal{I}(\tilde{\mu}) * \tilde{\mu} \in L$ in view of Iksanov et al. [6]. Finally, again by Jurek [7], $\mathcal{I}(\tilde{\mu}) * \tilde{\mu} = \mathcal{J}\big(\mathcal{I}(\rho) * \rho\big) = \mathcal{I}(\rho) = \mu$, which gives the decomposition.

However, the present proof is less involved, more straightforward and, moreover, the result and the proof of finiteness of the logarithmic moment in Lemma 2.2 (b) are completely new. Last but not least, the "calculus" on Lévy exponents, introduced in this note, is of an interest in itself.

REMARK 3.2. In the case of Euclidean space $\mathbb{R}^d$, using Schoenberg's theorem, it follows immediately that $\widetilde{\Phi}$ is a Lévy exponent; cf. Cuppens [4], pp. 80–82.

Following Iksanov et al. [6], p. 1360, we will say that a selfdecomposable probability measure μ has the *factorization property* if $\mu * \mathcal{I}^{-1}(\mu)$ is selfdecomposable as well. In other words, a class L probability measure convolved with its background driving probability distribution is again class L distribution. As in Iksanov et al. [6], Proposition 1, if L^f denotes the set of all class L distributions with the factorization property, then

$$L^f = \mathcal{I}\big(\mathcal{J}(ID_{\log})\big) = \mathcal{J}\big(\mathcal{I}(ID_{\log})\big) = \mathcal{J}(L) \quad \text{and} \quad L^f \subset L \subset \mathcal{U}. \tag{3.4}$$

COROLLARY 3.1. *Each selfdecomposable measure μ admits a factorization $\mu = \nu_1 * \nu_2$, where ν_1 is an s-selfdecomposable measure* (*i.e.*, $\nu_1 \in \mathcal{U}$) *and ν_2 is a selfdecomposable one with the factorization property* (*i.e.*, $\nu_2 \in L^f$). *That is, besides the inclusion $L^f \subset L \subset \mathcal{U}$ we also have $L \subset L^f * \mathcal{U}$.*

Proof. Because of (3.2), $\nu_1 := \tilde{\mu}$ is an s-selfdecomposable measure. Furthermore, $\nu_2 := \mathcal{I}(\tilde{\mu}) \in L$ has the factorization property, i.e., $\nu_2 \in L^f$, which completes the proof. ■

EXAMPLE 3.1. Let Σ_p be a symmetric stable distribution on a Banach space E, with the exponent p. Then its Lévy exponent Φ_p is equal to

$$\Phi_p(y) = -\int_S |\langle y, x\rangle|^p \, m(dx),$$

where m is a finite Borel measure on the unit sphere S of E; cf. Samorodnitsky and Taqqu [13]. Hence $(I - \mathcal{J})\Phi_p(y) = p/(p+1)\Phi_p(y)$, which means that in Corollary 3.1, both ν_1 and ν_2 are stable with the exponent p and measures $m_1 := \big(p/(p+1)\big)m$ and $m_2 := \big(1/(p+1)\big)m$, respectively.

EXAMPLE 3.2. Let η denote the Laplace (double exponential) distribution on the real line $\mathbb{R}$; cf. Jurek and Yor [11]. Then its Lévy exponent Φ_η is equal to $\Phi_\eta(t) := -\log(1+t^2)$, $t \in \mathbb{R}$. Consequently, it follows that $(I-\mathcal{J})\Phi_\eta(t) = 2(\arctan t - t)t^{-1}$ is the Lévy exponent of the class $\mathcal{U}$ probability measure ν_1 from Corollary 3.1, and $\big(2t - \arctan t - t\log(1+t^2)\big)t^{-1}$ is the Lévy exponent of the class L^f measure ν_2 from Corollary 3.1.

Before we formulate the next result we need to recall that, by (3.1), the class $\mathcal{U}$ is defined here as $\mathcal{U} = \mathcal{J}(ID)$. Consequently, by iteration argument we can define

$$\mathcal{U}^{\langle 1\rangle} := \mathcal{U}, \quad \mathcal{U}^{\langle k+1\rangle} := \mathcal{J}(\mathcal{U}^{\langle k\rangle}) = \mathcal{J}^{k+1}(ID),\ k = 1, 2, \ldots; \tag{3.5}$$

cf. Jurek [8] for other characterization of classes $\mathcal{U}^{\langle k\rangle}$. Elements from the semigroups $\mathcal{U}^{\langle k\rangle}$ are called *k-times s-selfdecomposable probability measures*.

THEOREM 3.2. *Let n be any natural number and μ be a selfdecomposable probability measure. Then there exist k-times s-selfdecomposable probability measures $\tilde{\mu}_k$, for $k = 1, 2, \ldots, n$, such that*

$$\mu = \tilde{\mu}_1 * \tilde{\mu}_2 * \ldots * \tilde{\mu}_n * \mathcal{I}(\tilde{\mu}_n), \quad \mathcal{J}^k(\mu) = \mathcal{I}(\tilde{\mu}_k),\ k = 1, 2, \ldots, n. \tag{3.6}$$

In fact, if Φ is the exponent of μ, then $\tilde{\mu}_k$ has the exponent $\mathcal{I}^{k-1}(I-\mathcal{J})^k\Phi = (I-\mathcal{J})\mathcal{J}^{k-1}\Phi$ and

$$\begin{aligned}\Phi = (I-\mathcal{J})\Phi + (I-\mathcal{J})\mathcal{J}\Phi + \ldots + (I-\mathcal{J})\mathcal{J}^{k-1}\Phi + \\ \ldots + (I-\mathcal{J})\mathcal{J}^{n-1}\Phi + \mathcal{J}^n\Phi = (I-\mathcal{J}^n)\Phi + \mathcal{J}^n\Phi.\end{aligned} \tag{3.7}$$

Proof. For $n = 1$ the factorization (3.6) and the formula (3.7) are true by Theorem 3.1, with $\tilde{\mu}_1 := \tilde{\mu}$. Suppose our claim (3.6) is true for n. Since $\rho := \mathcal{I}(\tilde{\mu}_n)$ is selfdecomposable, applying to it Theorem 3.1, we have $\rho = \tilde{\rho} * \mathcal{I}(\tilde{\rho})$, where $\tilde{\rho}$ has the Lévy exponent $(I-\mathcal{J})\mathcal{J}^n\Phi = \mathcal{J}^n(I-\mathcal{J})\Phi$, and thus it corresponds to an $(n+1)$-times s-selfdecomposable probability because, by Theorem 3.1, $(I-\mathcal{J})\Phi$ is already s-selfdecomposable; then we apply n times the operator $\mathcal{J}$; cf. the definition (3.5). Thus the factorization (3.6) holds for $n+1$, which completes the proof of the first part of the theorem.

Similarly, applying inductively the decomposition (3.3) and using Lemma 2.1 (b), we get the formula (3.6). Thus the proof is complete. ■

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REFERENCES

[1] A. Araujo and E. Giné, *The Central Limit Theorem for Real and Banach Valued Random Variables*, Wiley, New York 1980.

[2] N. H. Bingham, *Lévy processes and self-decomposability in finance*, Probab. Math. Statist. 26 (2) (2006), pp. 367–378.

[3] P. Carr, H. Geman, D. Madan and M. Yor, *Pricing options on realized variance*, Finance Stoch. 9 (4) (2005), pp. 453–475.
[4] R. Cuppens, *Decomposition of Multivariate Probabilities*, Academic Press, New York 1975.
[5] E. Eberlein and U. Keller, *Hyperbolic distributions in finance*, Bernoulli 1 (1995), pp. 281–299.
[6] A. M. Iksanov, Z. J. Jurek and B. M. Schreiber, *A new factorization property of the selfdecomposable probability measures*, Ann. Probab. 32 (2) (2004), pp. 1356–1369.
[7] Z. J. Jurek, *Relations between the s-selfdecomposable and selfdecomposable measures*, Ann. Probab. 13 (2) (1985), pp. 592–608.
[8] Z. J. Jurek, *The random integral representation hypothesis revisited: new classes of s-selfdecomposable laws*, in: *Abstract and Applied Analysis*, Proc. International Conf. ICAAA, Hanoi, August 2002, World Scientific, Hongkong 2004, pp. 495–514.
[9] Z. J. Jurek and J. D. Mason, *Operator-limit Distributions in Probability Theory*, Wiley, New York 1993.
[10] Z. J. Jurek and W. Vervaat, *An integral representation for selfdecomposable Banach space valued random variables*, Z. Wahrscheinlichkeitstheorie verw. Gebiete 62 (1983), pp. 247–262.
[11] Z. J. Jurek and M. Yor, *Selfdecomposable laws associated with hyperbolic functions*, Probab. Math. Statist. 24 (1) (2004), pp. 180–190.
[12] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, New York–London 1967.
[13] G. Samorodnitsky and M. S. Taqqu, *Stable Non-gaussian Random Processes*, Chapman and Hall, New York 1994.

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