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ASYMPTOTIC PROPERTIES OF PERIODOGRAM FOR ALMOST PERIODICALLY CORRELATED TIME SERIES

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Abstract. The main purpose of this paper is to establish the asymptotic properties of the expectation and variance of periodogram for nonstationary, almost periodically correlated time series. We expand our consideration to the whole bifrequency square $(0,2\pi]^2$. We show the exact form of asymptotic covariance between two values of periodogram which are calculated at different points. This result implies that periodogram is not consistent in mean square sense for any point from bifrequency square $(0,2\pi]^2$. Finally, under the moment and α-mixing condition, we prove the consistency of smoothed periodogram.

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1. INTRODUCTION

The concept of a class of periodically and almost periodically correlated (PC and APC) stochastic processes and time series was firstly introduced by Gladyshev [9], [10] and Hurd [12]–[14]. Such a class of stochastic processes and time series occurs in many fields including telecommunications (see [6], [7]), meteorology (see [1], [17]), finance (see [2], [4], [20]), econometrics (see [21], [22]) and many other fields (see [8]). Therefore, in the last years this field is a topic of the intensive research.

It can be stated that the spectral theory of continuous time APC processes was more broadly investigated than its discrete-time counterpart. Hurd [13] has shown that a smoothed periodogram is consistent (in mean square sense) for a PC Gaussian stochastic process. Leśkow [19] has shown that for ϕ-mixing APC stochastic processes observed in continuous time a smoothed periodogram is consistent and asymptotically normal. However, these results have not been applied yet to the time series context. It is well known that the theory of PC and APC stochastic processes observed in continuous time and the theory of PC and APC time series must be considered separately (for more details see [8]). This difference leads to other assumptions concerning theorems for these processes and other techniques in proofs. In the literature the PC and APC time series are often called *cyclostationary* (CS) and *almost cyclostationary* (ACS). The definition of PC and APC time series is the same as the definition of cyclostationary and almost cyclostationary ones in the wide sense, respectively. The review of the theory of cyclostationarity for time series and stochastic processes can be found in [8].

This paper is devoted to a study of properties of bias and variance of smoothed and non-smoothed periodogram for APC time series $\{X_t : t \in \mathbb{Z}\}$. In Section 3 we focus our attention on the bias of a periodogram for all points from bifrequency square $(0, 2\pi]^2$. We show the rate of convergence of bias of periodogram for points which belong to spectral mass location and beyond this set.

On the other hand, Section 4 is devoted to a study of second order properties of periodogram and consistency of smoothed periodogram. We show the exact form of asymptotic covariance between two values of periodogram which are calculated at different points. Moreover, this result holds for a more general class of time series than the APC case. The asymptotic form of variance of periodogram suggests inconsistency of the periodogram on the whole bifrequency square (in mean square sense). Finally, we show consistency (in mean square sense) of smoothed periodogram in the APC case. Recall that in the literature the asymptotic properties of the periodogram and its modification were considered only for points from spectral mass location. It was done under the ϕ-mixing condition in the PC and APC case (see [13], [16]) and under the α-mixing condition in the stationary case (see [25]). It is innovative in this work that all results in Sections 3 and 4 are presented for all points from bifrequency square $(0, 2\pi]^2$. Moreover, in our considerations we use the α-mixing condition instead of the ϕ-mixing condition. All proofs are contained in the Appendix.

2. BASIC DEFINITIONS AND ASSUMPTIONS

In this section we recall basic definitions and introduce the notation that is helpful for subsequent work. Let us start with the definition of an almost periodic function taken from [3]. It will be used to introduce the definitions of PC and APC time series (see [9], [14]).

DEFINITION 2.1. A function $f(t) : \mathbb{Z} \to \mathbb{R}$ is said to be *almost periodic* in $t \in \mathbb{Z}$ if for any $\epsilon > 0$ there exists an integer $L_\epsilon > 0$ such that among any $L_\epsilon > 0$ consecutive integers there is an integer $p_\epsilon > 0$ such that

$$\sup_{t \in \mathbb{Z}} |f(t + p_\epsilon) - f(t)| < \epsilon.$$

Notice that any periodic function is also almost periodic. A simple example of an almost periodic function is $f(t) = \sin(\omega t)$, where the argument $t \in \mathbb{Z}$ and ω is a frequency from the interval $(0, 2\pi]$. If we take an ω such that $\omega/2\pi$ is a rational number, then easy calculations show that $f(t)$ is a periodic function. If we take, for example, $\omega = \sqrt{2}$, then we get an almost periodic function which is not periodic.

DEFINITION 2.2. A second order real-valued time series $\{X_t : t \in \mathbb{Z}\}$ is called *almost periodically correlated* (APC) if both mean $\mu(t) = E(X_t)$ and autocovariance functions $B(t, \tau) = \operatorname{cov}(X_t, X_{t+\tau})$ are almost periodic functions at t for every $\tau \in \mathbb{Z}$. We say that the time series is *periodically correlated* (PC) if the mean and autocovariance functions are periodic in t for every $\tau \in \mathbb{Z}$.

In this paper we are interested in the second order structure of APC time series, so it is assumed that our time series is zero-mean, i.e. $\mu(t) \equiv 0$.

Take any $\tau \in \mathbb{Z}$. Then the autocovariance function $B(\cdot, \tau)$ has the Fourier representation (see [14])

$$B(t, \tau) \sim \sum_{\lambda \in \Lambda_\tau} a(\lambda, \tau) e^{i\lambda t}, \tag{2.1}$$

where $a(\lambda, \tau)$ are Fourier coefficients of the following form:

$$a(\lambda, \tau) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} B(j, \tau) e^{-i\lambda j}$$

and for fixed τ the set $\Lambda_\tau \subset [0, 2\pi)$ is the set of all frequencies λ for which $a(\lambda, \tau) \neq 0$. Let Λ denote the sum $\Lambda = \bigcup_{\tau\in\mathbb{Z}} \Lambda_\tau$. Below we formulate some assumptions regarding the set Λ and the summability of the Fourier coefficient $a(\lambda, \tau)$. These assumptions are used in the next sections.

ASSUMPTION 2.1. Assume that for the APC time series $\{X_t : t \in \mathbb{Z}\}$

(a) the set Λ is finite;

(b) there exists a real number $C_0 < \infty$ such that $\sum_{\tau=-\infty}^{\infty} |a(\lambda, \tau)| < C_0$ for any $\lambda \in \Lambda$;

(b′) there exists a real number $C_1 < \infty$ such that $\sum_{\tau=-\infty}^{\infty} |\tau||a(\lambda, \tau)| < C_1$ for any $\lambda \in \Lambda$.

Notice that under Assumption 2.1 (a) the Fourier representation (2.1) becomes equality and the set Λ_τ can be replaced by Λ. Moreover, it is easy to see that Assumption 2.1 (b′) implies (b).

Let us introduce the frequency domain theory for APC time series. We start from the assumption that our APC time series $\{X_t : t \in \mathbb{Z}\}$ is harmonizable, so it can be represented as a stochastic integral

$$X_t = \int_0^{2\pi} e^{i\xi t} Z(d\xi),$$

where $\{Z(\xi) : 0 < \xi \leqslant 2\pi\}$ is a zero-mean complex-valued random process. Then a signed measure defined on the bifrequency plane $(0, 2\pi]^2$ as $R\big((a, b] \times (c, d]\big) =$

$E\big[\big(Z(b)-Z(a)\big)\big(Z(d)-Z(c)\big)\big]$, where $0<a\leqslant b\leqslant 2\pi$, $0<c\leqslant d\leqslant 2\pi$, has a support contained in the set S of parallel lines:

$$S=\bigcup_{\lambda\in\Lambda}\{(\xi_1,\xi_2)\in(0,2\pi]^2:\xi_2=\xi_1-\lambda\}.$$

Gladyshev [9] has shown that all PC time series are harmonizable. Moreover, for the PC case the set Λ is finite and is contained in the set $\{\lambda=2k\pi/T, k=0,1,2,\ldots,T-1\}$. But this property does not hold for the APC case.

The Fourier coefficients $\{a(\lambda,\tau)\colon \lambda\in\Lambda\}$ are Fourier transforms of complex measures $r_\lambda(\cdot)$:

$$a(\lambda,\tau)=\int_0^{2\pi} e^{i\xi\tau}r_\lambda(d\xi)$$

(see [13]), where the measure r_λ can be identified with the restriction of the signed measure $R(\cdot,\cdot)$ to the line $\xi_2=\xi_1-\lambda$. Assume in addition that for any $\lambda\in\Lambda$ there exists a spectral density function $g_\lambda(\cdot)$ such that

$$\text{(2.2)}\qquad a(\lambda,\tau)=\int_0^{2\pi} e^{i\xi\tau}g_\lambda(\xi)d\xi \quad\text{and}\quad g_\lambda(\nu)=\frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty}a(\lambda,\tau)e^{-i\nu\tau}.$$

Notice that, under Assumption 2.1 (b) there exists a complex density function satisfying (2.2) for any $\lambda\in\Lambda$.

Let us recall definitions of Fourier transform and periodogram for APC time series. Assume that we have a sample $\{X_1,X_2,\ldots,X_n\}$ from APC time series $\{X_t:t\in\mathbb{Z}\}$. Denote by $I_n(\nu)=(2\pi n)^{-1/2}\sum_{t=1}^n X_te^{-i\nu t}$ the Fourier transform at point $\nu\in\mathbb{R}$. Then the bifrequency periodogram $\hat{P}_n(\nu,\omega)$ computed at point $(\nu,\omega)\in\mathbb{R}^2$ takes the form $\hat{P}_n(\nu,\omega)=I_n(\nu)\overline{I_n(\omega)}$, where $\overline{z}$ denotes the conjugation of the complex number z. Denote by $P_n(\nu,\omega)$ the expectation value of bifrequency periodogram. It will be shown later that for any point $(\nu,\omega)\in(0,2\pi]^2$ the limit $\lim_{n\to\infty}P_n(\nu,\omega)$ exists. Denote such a limit by $P(\nu,\omega)$.

Notice that the periodogram $\hat{P}_n(\nu,\omega)$ is a periodic function at ν and at ω with the same period equal to 2π. Therefore, we study the properties of periodogram for $(\nu,\omega)\in(0,2\pi]^2$.

Now we introduce the well-known definition of α-mixing sequence. The theoretical background for α-mixing time series can be found, for example, in [5].

DEFINITION 2.3. The time series $\{X(t)\colon t\in\mathbb{Z}\}$ is called *α-mixing* (or *strongly mixing*) if $\alpha(s)\to 0$ for $s\to\infty$, where

$$\alpha(s)=\sup_{t\in\mathbb{Z}}\sup_{\substack{A\in\mathcal{F}_X(-\infty,t)\\ B\in\mathcal{F}_X(t+s,\infty)}}|P(A\cap B)-P(A)P(B)|$$

and $\mathcal{F}_X(t_1,t_2)$ stands for the σ-algebra generated by $\{X(t)\colon t_1\leqslant t\leqslant t_2\}$.

In the remark below we show the connection between Assumption 2.1 and the α-mixing sequence, which corresponds to our time series $\{X_t : t \in \mathbb{Z}\}$. The appropriate summability of the α-mixing sequence is assumed in Section 4 instead of Assumption 2.1 (b) and (b′).

REMARK 2.1. *Let $\alpha(\cdot)$ be an α-mixing sequence for the time series $\{X_t : t \in \mathbb{Z}\}$. If we assume that there exists a real number $\delta > 0$ such that $\sup_{t\in\mathbb{Z}} \|X_t\|_{2+\delta} < \Delta < \infty$ and*

$$\sum_{k=1}^{\infty} \alpha^{\delta/(2+\delta)}(k) < \infty, \tag{2.3}$$

then the assumption (b) *holds. If instead of the condition* (2.3) *we assume that*

$$\sum_{k=1}^{\infty} k\alpha^{\delta/(2+\delta)}(k) < \infty,$$

then the assumption (b′) *holds.*

This follows from the inequality

$$|a(\lambda, \tau)| \leqslant 8\Delta^2 \alpha^{\delta/(2+\delta)}(|\tau|), \tag{2.4}$$

which is given by the estimation

$$\begin{aligned} |a(\lambda, \tau)| = \lim_{n\to\infty} \left| \frac{1}{n} \sum_{j=1}^{n} B(j, \tau) e^{-i\lambda j} \right| &\leqslant \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} |B(j, \tau)| \\ &\leqslant \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} |\operatorname{cov}(X_j, X_{j+\tau})|, \end{aligned}$$

and Lemma 6.1 in the Appendix.

3. ASYMPTOTIC PROPERTIES OF THE BIAS OF A PERIODOGRAM

In this section we study asymptotic properties of the bias of a periodogram for APC time series. We start with showing that for any point $(\nu, \omega) \notin S$ the absolute value of the expectation of periodogram is bounded by $O(n^{-1})$ times some function that depends on point (ν, ω).

THEOREM 3.1. *Let $\{X_1, X_2, \ldots, X_n\}$ be a sample from zero-mean APC time series $\{X_t : t \in \mathbb{Z}\}$. Suppose that Assumptions* 2.1 (a) *and* (b) *hold. Then for any $(\nu, \omega) \in (0, 2\pi]^2$ such that $(\nu, \omega) \notin S$ we have*

$$|P_n(\nu, \omega)| \leqslant \frac{D(\omega - \nu) C_0 |\Lambda|}{2\pi n},$$

where $|\Lambda|$ *is the power of the set* Λ *and* $D(\cdot)$ *is a function which is defined on the complement of the set* Λ *and has the following form*:

$$D(\alpha) = \max_{\lambda \in \Lambda} \{2/\sqrt{2 - 2\cos(\lambda - \alpha)}\}.$$

The result above shows the rate of convergence to zero for expectation of a periodogram on the complement of the support set S. For our subsequent considerations it will be also important to examine the convergence of $P_n(\nu, \omega)$ for points from the support set S. The result below provides the answer to such a question.

THEOREM 3.2. *Assume that we have a sample* $\{X_1, X_2, \ldots, X_n\}$ *from zero-mean APC time series* $\{X_t : t \in \mathbb{Z}\}$. *Then*, *under Assumptions* 2.1 (a) *and* (b), *for any* $|\lambda| \in \Lambda$ *and* $(\nu, \nu - \lambda) \in (0, 2\pi]^2$

$$P_n(\nu, \nu - \lambda) = g_\lambda(\nu) + o(1).$$

If Assumptions 2.1 (a) *and* (b′) *hold*, *then for any* $|\lambda| \in \Lambda$ *and* $(\nu, \nu - \lambda) \in (0, 2\pi]^2$

$$P_n(\nu, \nu - \lambda) = g_\lambda(\nu) + O(n^{-1})C_1.$$

4. ASYMPTOTIC VARIANCE OF THE PERIODOGRAM AND CONSISTENT ESTIMATES

The first theorem in this section concerns the exact form of asymptotic covariance between two values of periodogram. This form suggests inconsistency of periodogram in mean square sense for a wider class of time series than APC, that is, those mentioned in Remark 4.1. Consistency is proved for smoothed periodogram in Theorem 4.2. All results hold under the mixing and moment condition.

THEOREM 4.1. *Let* $\{X_t : t \in \mathbb{Z}\}$ *be a zero-mean*, *APC and* α*-mixing time series for which Assumption* 2.1 (a) *holds. Assume that there exists a real number* $\delta > 0$ *such that*

(i) $\sup_{t \in \mathbb{Z}} \|X_t\|_{6+3\delta} \leqslant \Delta < \infty$;

(ii) $\sum_{k=1}^{\infty} k^2 \alpha(k)^{\delta/(2+\delta)} \leqslant K < \infty$.

Then

$$\text{(4.1)} \quad \lim_{n \to \infty} \operatorname{cov}\big(\hat{P}_n(\nu_1, \omega_1), \hat{P}_n(\nu_2, \omega_2)\big) = P(\nu_1, \nu_2)\overline{P(\omega_1, \omega_2)} + P(\nu_1, 2\pi - \omega_2)\overline{P(\nu_2, 2\pi - \omega_1)}$$

for any $(\nu_1, \omega_1), (\nu_2, \omega_2) \in (0, 2\pi]^2$.

Now we formulate the corollary concerning the asymptotic variance of the periodogram.

COROLLARY 4.1. *Assume that all the assumptions of Theorem* 4.1 *hold. Then for any* $(\nu, \omega) \in (0, 2\pi]^2$ *we have*

$$\lim_{n\to\infty} \operatorname{var}\big(\hat{P}_n(\nu, \omega)\big) = g_0(\nu)g_0(\omega) + |P(\nu, 2\pi - \omega)|^2.$$

If we assume that

$$g_0(\nu)g_0(\omega) + |P(\nu, 2\pi - \omega)|^2 > 0 \quad \text{for some } (\nu, \omega) \in (0, 2\pi]^2,$$

then the above corollary implies inconsistency of the estimator $\hat{P}_n(\nu, \omega)$ in mean square sense. If we assume that

$$g_0(\xi) > 0 \quad \text{for any } \xi \in (0, 2\pi],$$

then the estimator $\hat{P}_n(\nu, \omega)$ is inconsistent for any point $(\nu, \omega) \in (0, 2\pi]^2$.

To obtain the consistent estimator of $P(\nu, \omega)$ we use the well-known technique based on smoothing operation. Let $\{X_1, X_2, \ldots, X_n\}$ be a sample from the APC time series $\{X_t : t \in \mathbb{Z}\}$. Consider the following class of smoothed estimators of $P(\nu, \omega)$:

$$\hat{G}_n(\nu, \omega) = \frac{1}{2\pi n} \sum_{t=1}^{n} \sum_{s=1}^{n} K_n(s - t) X_t X_s e^{-i\nu t} e^{i\omega s}, \tag{4.2}$$

where $K_n(\cdot)$ is a lag window function such that $K_n(\tau) = K_{L_n}(\tau) = 0$ holds for $|\tau| > L_n$, $\tau \in \mathbb{Z}$, and L_n is a sequence of positive integers tending to infinity with n. Moreover, we assume that $L_n/n \to 0$. This estimator was first introduced by Grenander and Rosenblatt in [11] for stationary time series in case $\nu = \omega$. The first and second order properties of this estimator in stationary case can be found in [24] and [25].

To prove consistency for the estimator (4.2) we take the following assumption in this work:

ASSUMPTION 4.1. Assume that

(A1) $|K_n(\tau)| \leqslant M < \infty$, where M is a constant which is independent of n and τ;

(A2) there exists a sequence $\{a_n\}$ of positive integers such that $a_n \leqslant L_n$, $a_n \to \infty$ and

$$\sup_{|\tau| \leqslant a_n} |K_n(\tau) - 1| \to 0.$$

The following theorem concerns consistency (in mean square sense) of the estimator (4.2) for APC case and for any $(\nu, \omega) \in (0, 2\pi]^2$.

THEOREM 4.2. *Let* $\{X_1, X_2, \ldots, X_n\}$ *be a sample from zero-mean, APC and* α*-mixing time series* $\{X_t : t \in \mathbb{Z}\}$*. Assume that there exists a real number* $\delta > 0$ *such that*:

(i) $\sup_{t\in\mathbb{Z}} \|X_t\|_{4+2\delta} \leqslant \Delta_1 < \infty$;
(ii) $\sum_{k=0}^{\infty} \alpha(k)^{\delta/(2+\delta)} \leqslant K_1 < \infty$.

Let $K_n(\cdot)$ be a lag window such that Assumption 4.1 holds. Assume that the sequence L_n, corresponding to the lag window $K_n(\cdot)$, has the property $L_n^3/n \to 0$, where $L_n \to \infty$ and $n \to \infty$. Then for any point $(\nu,\omega) \in (0,2\pi]^2$ the estimator (4.2) is consistent for $P(\nu,\omega)$ in mean square sense, which means that $E|\hat{G}_n(\nu,\omega) - P(\nu,\omega)|^2 \longrightarrow 0$ for $n \to \infty$.

REMARK 4.1. Notice that in Theorem 4.1 it is sufficient to assume that the limit $\lim_{n\to\infty} P_n(\nu,\omega)$ exists for any $(\nu,\omega) \in (0,2\pi]^2$, instead of the assumption that we have in the APC case (see the proof of this theorem). Corollary 4.1 is also true under this weaker assumption. We need only write $P(\nu,\nu)$ instead of $g_0(\nu)$. Therefore, these results can be used for the class of time series which are not APC. One of the examples of such time series is the class of time series which are harmonizable but not APC.

REMARK 4.2. Notice that Assumptions 4.1 (A1) and (A2) hold for the following windows:

– Bartlett's window

$$K_n(\tau) = (1 - |\tau|/L_n)I(|\tau| \leqslant L_n),$$

– "truncated periodogram" window

$$K_n(\tau) = I(|\tau| \leqslant L_n),$$

– rectangular window

$$K_n(\tau) = \begin{cases} \dfrac{\sin(\pi\tau/L_n)}{\pi\tau/L_n} I(|\tau| \leqslant L_n), & \tau \neq 0, \\ 1, & \tau = 0, \end{cases}$$

– general Tukey window

$$K_n(\tau) = \big(1 - 2a + 2a\cos(\pi\tau/L_n)\big)I(|\tau| \leqslant L_n), \quad a \in \mathbb{R},$$

and other windows which are presented in [24] for stationary time series. Easy calculations show that for the above lag window functions $K_n(\cdot)$ the condition (A1) holds for $M = 1$ and the condition (A2) holds for $a_n = [\sqrt{L_n}]$, where $[x]$ denotes the integer part of a real number x.

5. CONCLUSIONS

In this work we were focused on consistency of the periodogram and its modification in mean square sense. The theorems contained in Section 3 state about the rate of convergence of the bias of a periodogram in two cases. First we concentrate attention on convergence beyond the support S and then we supplement considerations to the case of points from the support S. Section 4 presents inconsistency of the periodogram in mean square sense for all points from bifrequency square $(0, 2\pi]^2$. We show that this result holds for a more general class of time series than APC case. Finally, the consistent form of the estimator is considered for an α-mixing APC time series. Notice that the problem of consistency of resampling methods is not applied yet for PC and APC time series in frequency domain. If we use bifrequency square for testing if time series is stationary, then we need to know the properties of periodogram on the whole square $(0, 2\pi]^2$. Therefore, the results from Sections 3 and 4 play an important role for considering theoretical background of these graphical tests.

6. APPENDIX

LEMMA 6.1 (Politis et al. [23]). *Let $\{X_t : t \in \mathbb{Z}\}$ be a random sequence with corresponding α-mixing sequence $\alpha(\cdot)$. Let the random variables ξ and ζ be $\mathcal{F}_{-\infty}^{n}$- and $\mathcal{F}_{n+k}^{\infty}$-measurable, respectively, with $\|\xi\|_p < \infty$ and $\|\zeta\|_q < \infty$ for some $p, q > 1$ such that $1/p + 1/q < 1$. Then*

$$|\operatorname{cov}(\xi, \zeta)| \leqslant 8\|\xi\|_p \|\zeta\|_q \alpha^{1-1/p-1/q}(k).$$

LEMMA 6.2. *Let $\{X_t : t \in \mathbb{Z}\}$ be a zero-mean, APC time series such that Assumptions 2.1 (a) and (b) hold. Then for any $m \in \mathbb{N}$ and $\tau, q \in \mathbb{Z}$ we have the following inequalities:*

$$\text{(i)} \qquad \Big|\sum_{j=q+1}^{q+m} B(j,\tau)e^{-i\alpha j}\Big| \leqslant D(\alpha)\sum_{\lambda\in\Lambda}|a(\lambda,\tau)| \leqslant D(\alpha)C_0|\Lambda| < \infty,$$

where $\alpha \notin \Lambda$ and $D(\alpha) = \max_{\lambda\in\Lambda}\{2/\sqrt{2-2\cos(\lambda-\alpha)}\}$;

$$\text{(ii)} \quad \left|a(\alpha,\tau) - \frac{1}{m}\sum_{j=q+1}^{q+m} B(j,\tau)e^{-i\alpha j}\right| \leqslant B\frac{1}{m}\sum_{\substack{\lambda\in\Lambda\\ \lambda\neq\alpha}}|a(\lambda,\tau)| \leqslant \frac{B}{m}C_0|\Lambda| < \infty,$$

where $\alpha \in \Lambda$ and

$$B = \max_{\substack{\lambda_1\neq\lambda_2\\ (\lambda_1,\lambda_2)\in\Lambda\times\Lambda}} \{2/\sqrt{2-2\cos(\lambda_1-\lambda_2)}\}.$$

P r o o f o f L e m m a 6.2. For $\alpha \notin \Lambda$ easy calculations give

$$\Big|\sum_{j=q+1}^{q+m} B(j,\tau)e^{-i\alpha j}\Big| = \Big|\sum_{j=q+1}^{q+m}\sum_{\lambda\in\Lambda} a(\lambda,\tau)e^{i\lambda j}e^{-i\alpha j}\Big| = \Big|\sum_{\lambda\in\Lambda} a(\lambda,\tau)\sum_{j=q+1}^{q+m} e^{i(\lambda-\alpha)j}\Big|$$
$$= \left|\sum_{\lambda\in\Lambda} a(\lambda,\tau)\frac{e^{i(\lambda-\alpha)(q+1)}(1-e^{i(\lambda-\alpha)m})}{1-e^{i(\lambda-\alpha)}}\right| \leqslant \sum_{\lambda\in\Lambda} |a(\lambda,\tau)|C_m(\lambda-\alpha),$$

where $C_m(\lambda-\alpha) = |(1-e^{i(\lambda-\alpha)m})/(1-e^{i(\lambda-\alpha)})|$. The sequence $C_m(\lambda-\alpha)$ is bounded by

$$C_m(\lambda-\alpha) = \left|\frac{1-e^{i(\lambda-\alpha)m}}{1-e^{i(\lambda-\alpha)}}\right| \leqslant \frac{2}{\sqrt{2-2\cos(\lambda-\alpha)}} \leqslant D(\alpha)$$

for any $\lambda \in \Lambda$. This completes the proof of the inequalities (i).

If we assume that $\alpha \in \Lambda$, then similar steps as before give

$$\left|a(\alpha,\tau)-\frac{1}{m}\sum_{j=q+1}^{m+q} B(j,\tau)e^{-i\alpha j}\right| = \left|\frac{1}{m}\sum_{j=q+1}^{m+q}\sum_{\substack{\lambda\in\Lambda\\ \lambda\neq\alpha}} a(\lambda,\tau)e^{i\lambda j}e^{-i\alpha j}\right|$$
$$= \left|\sum_{\substack{\lambda\in\Lambda\\ \lambda\neq\alpha}} a(\lambda,\tau)\frac{1}{m}\sum_{j=q+1}^{q+m} e^{i(\lambda-\alpha)j}\right| \leqslant \frac{1}{m}\sum_{\substack{\lambda\in\Lambda\\ \lambda\neq\alpha}}|a(\lambda,\tau)|C_m(\lambda-\alpha) \leqslant \frac{B}{m}\sum_{\substack{\lambda\in\Lambda\\ \lambda\neq\alpha}}|a(\lambda,\tau)|,$$

where

$$B = \max_{\substack{\lambda_1\neq\lambda_2\\ (\lambda_1,\lambda_2)\in\Lambda\times\Lambda}} \{2/\sqrt{2-2\cos(\lambda_1-\lambda_2)}\}.$$

This completes the proof. ∎

LEMMA 6.3 (presented without proof in Kunsch [18] for stationary case). *Let $\{X_t : t \in \mathbb{Z}\}$ be a zero-mean and α-mixing time series. Assume that there exists a real number $\delta > 0$ such that*

(i) $\sup_{t\in\mathbb{Z}} \|X_t\|_{6+3\delta} \leqslant \Delta < \infty$;

(ii) $\sum_{k=0}^{\infty} k^2\alpha^{\delta/(2+\delta)}(k) \leqslant K < \infty$.

Then for any $s \leqslant t \leqslant u \leqslant v$

$$|E(X_sX_tX_uX_v) - E(X_sX_t)E(X_uX_v) - E(X_sX_u)E(X_tX_v) - E(X_sX_v)E(X_tX_u)| \leqslant 32\Delta^4\alpha^{\delta/(2+\delta)}(k),$$

where $k = \max\{t-s, u-t, v-u\}$.

P r o o f o f L e m m a 6.3. Notice that using repeatedly the Hölder inequality we get

$$\begin{aligned}\|X_sX_tX_u\|_{2+\delta} &= \big(E(|X_s|^{2+\delta}|X_tX_u|^{2+\delta})\big)^{1/(2+\delta)} \\ &\leqslant \|X_s\|_{6+3\delta}\,\big(E|X_tX_u|^{3(2+\delta)/2}\big)^{2/\left(3(2+\delta)\right)} \\ &\leqslant \|X_s\|_{6+3\delta}\|X_t\|_{6+3\delta}\|X_u\|_{6+3\delta} \leqslant \Delta^3,\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}\|X_sX_t\|_{2+\delta} &= \big(E(|X_s|^{2+\delta}|X_t|^{2+\delta})\big)^{1/(2+\delta)} \\ &\leqslant \|X_s\|_{4+2\delta}\|X_t\|_{4+2\delta} \leqslant \|X_s\|_{6+3\delta}\|X_t\|_{6+3\delta} \leqslant \Delta^2\end{aligned} \tag{6.2}$$

for any $s, t, u \in \mathbb{Z}$. Moreover,

$$\begin{aligned}|\mathrm{cov}(X_s, X_t)| = |E(X_sX_t)| &\leqslant E|X_sX_t| \leqslant \|X_s\|_2\|X_t\|_2 \\ &\leqslant \|X_s\|_{6+3\delta}\|X_t\|_{6+3\delta} \leqslant \Delta^2\end{aligned} \tag{6.3}$$

for any $s, t \in \mathbb{Z}$, which follows from the Jensen and Hölder inequalities. From the definition of α-mixing sequence we get

$$\alpha(k_1) \leqslant \alpha(k_2) \leqslant 1 \tag{6.4}$$

for any integers k_1, k_2 such that $0 \leqslant k_2 \leqslant k_1$. Let us consider the following cases:

C a s e 1. Assume that $k = |t - s|$. Then using Lemma 6.1 and the inequalities (6.1), (6.3), (6.4) we get

$$\begin{aligned}&|E(X_sX_tX_uX_v) - E(X_sX_t)E(X_uX_v) - E(X_sX_u)E(X_tX_v) \\ &\quad - E(X_sX_v)E(X_tX_u)| \\ &= |\mathrm{cov}(X_s, X_tX_uX_v) - E(X_sX_t)E(X_uX_v) - E(X_sX_u)E(X_tX_v) \\ &\quad - E(X_sX_v)E(X_tX_u)| \\ &\leqslant |\mathrm{cov}(X_s, X_tX_uX_v)| + |\mathrm{cov}(X_s, X_t)|\,|\mathrm{cov}(X_u, X_v)| \\ &\quad + |\mathrm{cov}(X_s, X_u)|\,|\mathrm{cov}(X_t, X_v)| + |\mathrm{cov}(X_s, X_v)|\,|\mathrm{cov}(X_t, X_u)| \\ &\leqslant 8\,\|X_s\|_{2+\delta}\|X_tX_uX_v\|_{2+\delta}\;\alpha^{\delta/(2+\delta)}(k) + \big(8\,\|X_s\|_{2+\delta}\|X_t\|_{2+\delta}\;\alpha^{\delta/(2+\delta)}(k)\big)\Delta^2 \\ &\quad + \big(8\,\|X_s\|_{2+\delta}\|X_u\|_{2+\delta}\;\alpha^{\delta/(2+\delta)}(k)\big)\Delta^2 \\ &\quad + \big(8\,\|X_s\|_{2+\delta}\|X_v\|_{2+\delta}\;\alpha^{\delta/(2+\delta)}(k)\big)\Delta^2 \\ &\leqslant 32\Delta^4\alpha^{\delta/(2+\delta)}(k).\end{aligned}$$

Case 2. Assume that $k = |u - t|$. Then using Lemma 6.1 and inequalities (6.2)–(6.4) we get

$$\begin{aligned}
&|E(X_sX_tX_uX_v) - E(X_sX_t)E(X_uX_v) - E(X_sX_u)E(X_tX_v)\\
&\quad - E(X_sX_v)E(X_tX_u)|\\
&= |\mathrm{cov}(X_sX_t, X_uX_v) - E(X_sX_u)E(X_tX_v) - E(X_sX_v)E(X_tX_u)|\\
&\leqslant |\mathrm{cov}(X_sX_t, X_uX_v)| + |\mathrm{cov}(X_s, X_u)|\,|\mathrm{cov}(X_t, X_v)|\\
&\quad + |\mathrm{cov}(X_s, X_v)|\,|\mathrm{cov}(X_t, X_u)|\\
&\leqslant 8\,\|X_sX_t\|_{2+\delta}\|X_uX_v\|_{2+\delta}\ \alpha^{\delta/(2+\delta)}(k) + \big(8\,\|X_s\|_{2+\delta}\|X_u\|_{2+\delta}\ \alpha^{\delta/(2+\delta)}(k)\big)\Delta^2\\
&\quad + \big(8\,\|X_s\|_{2+\delta}\|X_v\|_{2+\delta}\ \alpha^{\delta/(2+\delta)}(k)\big)\Delta^2 \leqslant 24\Delta^4\alpha^{\delta/(2+\delta)}(k).
\end{aligned}$$

Case 3. Assume that $k = |v - u|$. Using analogous steps to those in Case 1 we obtain

$$\begin{aligned}
&|E(X_sX_tX_uX_v) - E(X_sX_t)E(X_uX_v) - E(X_sX_u)E(X_tX_v)\\
&\qquad\qquad - E(X_sX_v)E(X_tX_u)| \leqslant 32\Delta^4\alpha^{\delta/(2+\delta)}(k).
\end{aligned}$$

This completes the proof of lemma. ■

Proof of Theorem 3.1. Changing the variables and using a simple decomposition we get

$$\begin{aligned}
&|2\pi n P_n(\nu,\omega)| = \Big|\sum_{s=1}^{n}\sum_{t=1}^{n} E(X_sX_t)e^{-i\nu s}e^{i\omega t}\Big|\\
&= \Big|\sum_{u=-(n-1)}^{-1}\ \sum_{v=-(u-1)}^{n} B(v,u)e^{-i\nu u}e^{i(\omega-\nu)v} + \sum_{u=1}^{n-1}\sum_{v=1}^{n-u} B(v,u)e^{-i\nu u}e^{i(\omega-\nu)v}\\
&\qquad\qquad + \sum_{v=1}^{n} B(v,0)e^{i(\omega-\nu)v}\Big|\\
&\leqslant \Big|\sum_{u=-(n-1)}^{-1} e^{-i\nu u}\sum_{v=-(u-1)}^{n} B(v,u)e^{i(\omega-\nu)v}\Big| + \Big|\sum_{u=1}^{n-1} e^{-i\nu u}\sum_{v=1}^{n-u} B(v,u)e^{i(\omega-\nu)v}\Big|\\
&\quad + \Big|\sum_{v=1}^{n} B(v,0)e^{i(\omega-\nu)v}\Big|\\
&\leqslant \sum_{u=-(n-1)}^{-1}\Big|\sum_{v=-(u-1)}^{n} B(v,u)e^{i(\omega-\nu)v}\Big| + \sum_{u=1}^{n-1}\Big|\sum_{v=1}^{n-u} B(v,u)e^{i(\omega-\nu)v}\Big|\\
&\quad + \Big|\sum_{v=1}^{n} B(v,0)e^{i(\omega-\nu)v}\Big|.
\end{aligned}$$

Hence, using Assumptions 2.1 (a), (b) and Lemma 6.2 (i) we get

$$|2\pi n P_n(\nu,\omega)| \leqslant D(\omega-\nu) \sum_{u=-(n-1)}^{-1} \sum_{\lambda\in\Lambda} |a(\lambda,u)| + D(\omega-\nu) \sum_{u=1}^{n-1} \sum_{\lambda\in\Lambda} |a(\lambda,u)|$$

$$+ D(\omega-\nu) \sum_{\lambda\in\Lambda} |a(\lambda,0)| = D(\omega-\nu) \sum_{u=-(n-1)}^{n-1} \sum_{\lambda\in\Lambda} |a(\lambda,u)| < D(\omega-\nu) C_0 |\Lambda|,$$

which completes the proof. ■

P r o o f o f T h e o r e m 3.2. Notice that

$$P_n(\nu,\nu-\lambda) = \frac{1}{2\pi n} \sum_{s=1}^{n} \sum_{t=1}^{n} E(X_s X_t) e^{-i\lambda t} e^{-i\nu(s-t)}$$

$$= \frac{1}{2\pi n} \sum_{j=1}^{n} \sum_{\tau=1-j}^{n-j} B(j,\tau) e^{-i\lambda j} e^{-i\nu\tau}$$

$$= \frac{1}{2\pi n} \sum_{j=1}^{n} \sum_{\tau=1-j}^{n-j} \sum_{\gamma\in\Lambda} a(\gamma,\tau) e^{i(\gamma-\lambda)j} e^{-i\nu\tau}$$

$$= \frac{1}{2\pi n} \sum_{j=1}^{n} \sum_{\tau=1-j}^{n-j} a(\lambda,\tau) e^{-i\nu\tau} + \frac{1}{2\pi n} \sum_{\gamma\neq\lambda} \sum_{j=1}^{n} \sum_{\tau=1-j}^{n-j} a(\gamma,\tau) e^{i(\gamma-\lambda)j} e^{-i\nu\tau}$$

$$= \frac{1}{2\pi} \sum_{|\tau|<n} \left(1 - \frac{|\tau|}{n}\right) a(\lambda,\tau) e^{-i\nu\tau} + \frac{1}{2\pi n} \sum_{\gamma\neq\lambda} \sum_{j=1}^{n} \sum_{\tau=1-j}^{n-j} a(\gamma,\tau) e^{i(\gamma-\lambda)j} e^{-i\nu\tau}.$$

Denote the first and the second term of the last equality by c_n and ϵ_n, respectively. Notice that c_n is a Cesáro means for the patrial sum $s_n = \sum_{|\tau|<n} a(\lambda,\tau) e^{-i\nu\tau}$. Therefore, under Assumption 2.1 (b), c_n goes to $g_\lambda(\nu)$. If we assume in addition that Assumption 2.1 (b′) holds then the term c_n goes to $g_\lambda(\nu)$ with the rate $O(n^{-1})$, which follows immediately from the decomposition

$$\begin{aligned} c_n &= \sum_{|\tau|<n} a(\lambda,\tau) e^{-i\nu\tau} - \frac{1}{n} \sum_{|\tau|<n} |\tau| a(\lambda,\tau) e^{-i\nu\tau} \\ &= g_\lambda(\nu) - \sum_{|\tau|\geqslant n} a(\lambda,\tau) e^{-i\nu\tau} - \frac{1}{n} \sum_{|\tau|<n} |\tau| a(\lambda,\tau) e^{-i\nu\tau} \end{aligned}$$

and the inequalities

$$\Big| \sum_{|\tau|\geqslant n} a(\lambda,\tau) e^{-i\nu\tau} \Big| \leqslant \frac{1}{n} \sum_{|\tau|\geqslant n} n |a(\lambda,\tau)| \leqslant \frac{1}{n} \sum_{|\tau|\geqslant n} |\tau| |a(\lambda,\tau)| \leqslant \frac{C_1}{n},$$

$$\left|\frac{1}{n}\sum_{|\tau|<n}|\tau|a(\lambda,\tau)e^{-i\nu\tau}\right| \leqslant \frac{1}{n}\sum_{|\tau|<n}|\tau||a(\lambda,\tau)| \leqslant \frac{C_1}{n}.$$

The second term ϵ_n is equal to $O(n^{-1})$, which follows immediately from Assumptions 2.1 (a), (b) and the same steps as in the proof of Lemma 6.2 (i). This remark completes the proof. ■

Proof of Theorem 4.1. Notice that

$$\begin{aligned}
&(2\pi n)^2\big|\operatorname{cov}\big(\hat{P}_n(\nu_1,\omega_1),\hat{P}_n(\nu_2,\omega_2)\big) - P_n(\nu_1,\nu_2)\overline{P_n(\omega_1,\omega_2)} \\
&\qquad\qquad - P_n(\nu_1,2\pi-\omega_2)\overline{P_n(\nu_2,2\pi-\omega_1)}\big| \\
&= \big|\sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{u=1}^{n}\sum_{v=1}^{n}\operatorname{cov}(X_sX_t,X_uX_v)e^{-i(\nu_1 s-\omega_1 t-\nu_2 u+\omega_2 v)} \\
&\quad - \sum_{s=1}^{n}\sum_{u=1}^{n}E(X_sX_u)e^{-i(\nu_1 s-\nu_2 u)}\sum_{t=1}^{n}\sum_{v=1}^{n}E(X_tX_v)e^{-i(-\omega_1 t+\omega_2 v)} \\
&\quad - \sum_{s=1}^{n}\sum_{v=1}^{n}E(X_sX_v)e^{-i(\nu_1 s+\omega_2 v)}\sum_{t=1}^{n}\sum_{u=1}^{n}E(X_tX_u)e^{-i(-\omega_1 t-\nu_2 u)}\big| \\
&= \big|\sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{u=1}^{n}\sum_{v=1}^{n}\big(E(X_sX_tX_uX_v)-E(X_sX_t)E(X_uX_v)\big)e^{-i(\nu_1 s-\omega_1 t-\nu_2 u+\omega_2 v)} \\
&\quad - \sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{u=1}^{n}\sum_{v=1}^{n}E(X_sX_u)E(X_tX_v)e^{-i(\nu_1 s-\omega_1 t-\nu_2 u+\omega_2 v)} \\
&\quad - \sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{u=1}^{n}\sum_{v=1}^{n}E(X_sX_v)E(X_tX_u)e^{-i(\nu_1 s-\omega_1 t-\nu_2 u+\omega_2 v)}\big| \\
&\leqslant \sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{u=1}^{n}\sum_{v=1}^{n}|E(X_sX_tX_uX_v)-E(X_sX_t)E(X_uX_v)-E(X_sX_u)E(X_tX_v) \\
&\qquad\qquad -E(X_sX_v)E(X_tX_u)| \\
&\leqslant 4!\sum_{1\leqslant s\leqslant t\leqslant u\leqslant v\leqslant n}|E(X_sX_tX_uX_v)-E(X_sX_t)E(X_uX_v)-E(X_sX_u)E(X_tX_v) \\
&\qquad\qquad -E(X_sX_v)E(X_tX_u)|.
\end{aligned}$$

Using now Lemma 6.3 we get

$$\begin{aligned}
U_n(\nu,\omega) &= (2\pi n)^2\big|\operatorname{cov}\big(\hat{P}_n(\nu_1,\omega_1),\hat{P}_n(\nu_2,\omega_2)\big) - P_n(\nu_1,\nu_2)\overline{P_n(\omega_1,\omega_2)} \\
&\qquad - P_n(\nu_1,2\pi-\omega_2)\overline{P_n(\nu_2,2\pi-\omega_1)}\big| \\
&\leqslant 4!\,32\Delta^4\sum_{1\leqslant s\leqslant t\leqslant u\leqslant v\leqslant n}\alpha^{\delta/(2+\delta)}(\max\{t-s,u-t,v-u\}) \\
&\leqslant 4!\,32\Delta^4\sum_{k=0}^{n-1}|a_{n,k}|\,\alpha^{\delta/(2+\delta)}(k),
\end{aligned}$$

where $|a_{n,k}|$ is a power of the set $a_{n,k}$ and

$$a_{n,k} = \big\{(s,t,u,v) \in \mathbb{N}^4 \colon 1 \leqslant s \leqslant t \leqslant u \leqslant v \leqslant n,\ \max\{t-s, u-t, v-u\} = k\big\}.$$

It can be shown that $|a_{n,k}| \leqslant 3(n-k)4k4k$ for $k > 0$ and $|a_{n,k}| = n$ for $k = 0$. Therefore,

$$U_n(\nu,\omega) \leqslant 4!\, 32\Delta^4 \sum_{k=1}^{n-1} 3(n-k)4k4k\, \alpha^{\delta/(2+\delta)}(k) + 4!\, 32\Delta^4 n$$
$$\leqslant 4!\, 1536\Delta^4 n \sum_{k=1}^{n-1} k^2\, \alpha^{\delta/(2+\delta)}(k) + 4!\, 32\Delta^4 n \leqslant 4!\, 32\, \Delta^4 n(48K+1).$$

Hence,

$$\text{(6.5)} \quad \lim_{n\to\infty} \big| \operatorname{cov}\big(\hat{P}_n(\nu_1,\omega_1), \hat{P}_n(\nu_2,\omega_2)\big) - P_n(\nu_1,\nu_2)\overline{P_n(\omega_1,\omega_2)} - P_n(\nu_1, 2\pi-\omega_2)\overline{P_n(\nu_2, 2\pi-\omega_1)}\big| = 0.$$

Notice that, by Remark 2.1, all the assumptions of Theorems 3.1 and 3.2 hold. So, the limit of the sequence $P_n(\nu_1,\nu_2)\overline{P_n(\omega_1,\omega_2)} - P_n(\nu_1, 2\pi-\omega_2)\overline{P_n(\nu_2, 2\pi-\omega_1)}$ exists. Therefore, we may conclude that

$$\text{(6.6)} \quad \lim_{n\to\infty} \operatorname{cov}\big(\hat{P}_n(\nu_1,\omega_1), \hat{P}_n(\nu_2,\omega_2)\big) = P(\nu_1,\nu_2)\overline{P(\omega_1,\omega_2)} + P(\nu_1, 2\pi-\omega_2)\overline{P(\nu_2, 2\pi-\omega_1)}.$$

This completes the proof. ■

Proof of Corollary 4.1. The corollary follows immediately from Theorem 4.1. ■

Proof of Theorem 4.2. Let us consider the following decomposition:

$$\text{(6.7)} \quad \hat{G}_n(\nu,\omega) = \hat{G}_n^*(\nu,\omega) - \hat{R}_n(\nu,\omega),$$

where

$$\text{(6.8)} \quad \hat{G}_n^*(\nu,\omega) = \frac{1}{2\pi n} \sum_{\tau=-L_n}^{L_n} \sum_{j=1}^{n} K_n(\tau) X_j X_{j+\tau} e^{-i(\nu-\omega)j} e^{-i\nu\tau}$$

and

$$\text{(6.9)} \quad \hat{R}_n(\nu,\omega) = \frac{1}{2\pi n} \sum_{\tau=-L_n}^{-1} \sum_{j=1}^{-\tau} K_n(\tau) X_j X_{j+\tau} e^{-i(\nu-\omega)j} e^{-i\nu\tau} + \frac{1}{2\pi n} \sum_{\tau=1}^{L_n} \sum_{j=n-\tau+1}^{n} K_n(\tau) X_j X_{j+\tau} e^{-i(\nu-\omega)j} e^{-i\nu\tau}.$$

By the inequality $|z_1+z_2|^2 \leqslant 2(|z_1|^2+|z_2|^2)$, which is true for any complex-valued numbers z_1, z_2, we get

$$E|\hat{G}_n(\nu,\omega)-P(\nu,\omega)|^2 = E|\hat{G}_n^*(\nu,\omega)-P(\nu,\omega)-\hat{R}_n(\nu,\omega)|^2 \leqslant 2\big(E|\hat{G}_n^*(\nu,\omega)-P(\nu,\omega)|^2+E|\hat{R}_n(\nu,\omega)|^2\big). \tag{6.10}$$

Therefore, to show the convergence $\hat{G}_n(\nu,\omega) \xrightarrow{L_2} P(\nu,\omega)$ it is sufficient to prove that $\hat{G}_n^*(\nu,\omega) \xrightarrow{L_2} P(\nu,\omega)$ and $\hat{R}_n(\nu,\omega) \xrightarrow{L_2} 0$. We split the proof into two steps.

Step 1. In the first step we show that $\hat{R}_n(\nu,\omega) \xrightarrow{L_2} 0$. Notice that from the Minkowski inequality, the equation $|z|^2 = |z\overline{z}|$, which is true for any complex number z, and from Assumption 4.1 (A1) we have

$$\begin{aligned}\sqrt{E|\hat{R}_n(\nu,\omega)|^2} &\leqslant \Big\|\frac{1}{2\pi n}\sum_{\tau=-L_n}^{-1} K_n(\tau)\sum_{j=1}^{-\tau} X_jX_{j+\tau}e^{-i(\nu-\omega)j}e^{-i\nu\tau}\Big\|_2 \\ &+ \Big\|\frac{1}{2\pi n}\sum_{\tau=1}^{L_n} K_n(\tau)\sum_{j=n-\tau+1}^{n} X_jX_{j+\tau}e^{-i(\nu-\omega)j}e^{-i\nu\tau}\Big\|_2 \\ &\leqslant \frac{M}{2\pi n}\Big(\sum_{\tau_1=-L_n}^{-1}\sum_{j_1=1}^{-\tau_1}\sum_{\tau_2=-L_n}^{-1}\sum_{j_2=1}^{-\tau_2} E|X_{j_1}X_{j_1+\tau_1}X_{j_2}X_{j_2+\tau_2}|\Big)^{1/2} \\ &+ \frac{M}{2\pi n}\Big(\sum_{\tau_1=1}^{L_n}\sum_{j_1=n-\tau_1+1}^{n}\sum_{\tau_2=1}^{L_n}\sum_{j_2=n-\tau_2+1}^{n} E|X_{j_1}X_{j_1+\tau_1}X_{j_2}X_{j_2+\tau_2}|\Big)^{1/2}.\end{aligned}$$

In the next step we use the inequality

$$E|X_{j_1}X_{j_1+\tau_1}X_{j_2}X_{j_2+\tau_2}| \leqslant \|X_{j_1}\|_4\|X_{j_1+\tau_1}\|_4\|X_{j_2}\|_4\|X_{j_2+\tau_2}\|_4 \leqslant \Delta^4,$$

which is true for any $j_1, j_2, \tau_1, \tau_2 \in \mathbb{Z}$ and follows from the Hölder inequality. We get

$$\begin{aligned}\sqrt{E|\hat{R}_n(\nu,\omega)|^2} &\leqslant \frac{M\Delta^2}{2\pi n}\Big(\Big(\sum_{\tau_1=-L_n}^{-1}\sum_{\tau_2=-L_n}^{-1}|\tau_1\tau_2|\Big)^{1/2}+\Big(\sum_{\tau_1=1}^{L_n}\sum_{\tau_2=1}^{L_n}|\tau_1\tau_2|\Big)^{1/2}\Big) \\ &\leqslant \frac{M\Delta^2}{2\pi n}2\sum_{\tau=1}^{L_n}|\tau| = \frac{M\Delta^2 L_n(L_n+1)}{2\pi n} \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Consequently, $\hat{R}_n(\nu,\omega) \xrightarrow{L_2} 0$.

Step 2. In this step we show that $\hat{G}_n^*(\nu,\omega) \xrightarrow{L_2} P(\nu,\omega)$. To see this we prove that the absolute value of the bias term and the variance of the estimator $\hat{G}_n^*(\nu,\omega)$ tends to zero as $n\to\infty$. This follows from the equation

$$E|Z_n - z|^2 = \mathrm{var}(Z_n) + |E(Z_n)-z|^2, \tag{6.11}$$

which is true for any complex-valued random sequence Z_n and a complex number z. Let us consider two cases for the bias term.

Case 1. Take any $(\nu,\omega)\in(0,2\pi]^2$ such that $\nu-\omega=\lambda$ and $\lambda\in\Lambda$. Then we obtain

$$
\begin{aligned}
&2\pi\big|E\big(\hat{G}^*_n(\nu,\omega)\big)-g_\lambda(\nu)\big|\\
&=\Big|\sum_{\tau=-L_n}^{L_n}K_n(\tau)\Big(\frac{1}{n}\sum_{j=1}^{n}B(j,\tau)e^{-i\lambda j}\Big)e^{-i\nu\tau}-\sum_{\tau=-\infty}^{\infty}a(\lambda,\tau)e^{-i\nu\tau}\Big|\\
&\leqslant\Big|\sum_{\tau=-L_n}^{L_n}\Big(K_n(\tau)\frac{1}{n}\sum_{j=1}^{n}B(j,\tau)e^{-i\lambda j}\Big)e^{-i\nu\tau}-\sum_{\tau=-L_n}^{L_n}a(\lambda,\tau)e^{-i\nu\tau}\Big|+\sum_{|\tau|>L_n}|a(\lambda,\tau)|\\
&\leqslant\Big|\sum_{\tau=-L_n}^{L_n}\Big(\Big(K_n(\tau)\frac{1}{n}\sum_{j=1}^{n}B(j,\tau)e^{-i\lambda j}\Big)-a(\lambda,\tau)\Big)e^{-i\nu\tau}\Big|+\sum_{|\tau|>L_n}|a(\lambda,\tau)|\\
&\leqslant\sum_{\tau=-L_n}^{L_n}|K_n(\tau)|\Big|\Big(\frac{1}{n}\sum_{j=1}^{n}B(j,\tau)e^{-i\lambda j}\Big)-a(\lambda,\tau)\Big|\\
&\quad+\sum_{\tau=-L_n}^{L_n}\big|\big(K_n(\tau)-1\big)a(\lambda,\tau)\big|+\sum_{|\tau|>L_n}|a(\lambda,\tau)|.
\end{aligned}
$$

Using Lemma 6.2 (ii) and (2.4) for the first term of the last inequality and (2.4) for the second and third term we get

$$
\begin{aligned}
&2\pi\big|E\big(\hat{G}^*_n(\nu,\omega)\big)-g_\lambda(\nu)\big|\\
&\leqslant\frac{8\Delta_1^2B(|\Lambda|-1)}{n}\sum_{\tau=-L_n}^{L_n}|K_n(\tau)|\alpha^{\delta/(2+\delta)}(|\tau|)\\
&\quad+8\Delta_1^2\sum_{\tau=-L_n}^{L_n}|K_n(\tau)-1|\alpha^{\delta/(2+\delta)}(|\tau|)+16\Delta_1^2\sum_{\tau=L_n+1}^{\infty}\alpha^{\delta/(2+\delta)}(\tau)\\
&\leqslant\frac{8\Delta_1^2B\,(|\Lambda|-1)\,M\,2K_1}{n}+8\Delta_1^2\sum_{\tau=-L_n}^{L_n}|K_n(\tau)-1|\alpha^{\delta/(2+\delta)}(|\tau|)+16\Delta_1^2o(1)\\
&=o(1)+8\Delta_1^2\sum_{\tau=-L_n}^{L_n}|K_n(\tau)-1|\alpha^{\delta/(2+\delta)}(|\tau|)\\
&=o(1)+8\Delta_1^2\sum_{\tau=-a_n}^{a_n}|K_n(\tau)-1|\alpha^{\delta/(2+\delta)}(|\tau|)\\
&\quad+8\Delta_1^2\sum_{a_n<|\tau|\leqslant L_n}|K_n(\tau)-1|\alpha^{\delta/(2+\delta)}(|\tau|)
\end{aligned}
$$

$$\leqslant o(1) + 8\Delta_1^2 \sup_{|\tau|\leqslant a_n} |K_n(\tau) - 1| \sum_{\tau=-a_n}^{a_n} \alpha^{\delta/(2+\delta)}(|\tau|)$$
$$+ 8\Delta_1^2(M+1) \sum_{a_n<|\tau|\leqslant L_n} \alpha^{\delta/(2+\delta)}(|\tau|)$$
$$\leqslant o(1) + 8\Delta_1^2 \sup_{|\tau|\leqslant a_n} |K_n(\tau) - 1|\, 2K_1 + 8\Delta_1^2(M+1)\, o(1) \to 0 \quad \text{as } n \to \infty,$$

where the last convergence follows from Assumption 4.1 (A2).

Case 2. Take any $(\nu,\omega) \in (0,2\pi]^2$ such that $(\nu,\omega) \notin S$. Then using Lemma 6.2 (i) and inequality (2.4) we have

$$\big|E\big(\hat{G}_n^*(\nu,\omega)\big)\big| \leqslant \sum_{\tau=-L_n}^{L_n} |K_n(\tau)| \left|\frac{1}{2\pi n}\sum_{j=1}^{n} B(j,\tau)e^{-i(\omega-\nu)j}\right|$$
$$\leqslant M \sum_{\tau=-L_n}^{L_n} \left(\frac{1}{2\pi n} D(\nu-\omega) \sum_{\lambda\in\Lambda} |a(\lambda,\tau)|\right)$$
$$\leqslant \frac{8M\Delta_1^2 D(\nu-\omega)|\Lambda|}{2\pi n} \sum_{\tau=-L_n}^{L_n} \alpha^{\delta/(2+\delta)}(|\tau|) \leqslant \frac{8M\Delta_1^2 D(\nu-\omega)|\Lambda|}{2\pi n} 2K_1 \to 0$$

as $n \to \infty$. This completes the proof that the bias term tends to zero. Therefore, to prove consistency in mean square sense for the estimator $\hat{G}_n^*(\nu,\omega)$ it is sufficient to show that variance of this estimator vanishes when $n \to \infty$. Notice that

$$\big|\mathrm{var}\big(\hat{G}_n^*(\nu,\omega)\big)\big| \leqslant \frac{M^2}{4\pi^2 n^2} \sum_{\tau_1=-L_n}^{L_n} \sum_{\tau_2=-L_n}^{L_n} \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} |\mathrm{cov}(X_{j_1}X_{j_1+\tau_1}, X_{j_2}X_{j_2+\tau_2})|.$$

Define the following sets:

$$A_n = \{(j_1,j_2) \in \mathbb{N}^2 \colon |j_1 - j_2| \leqslant 2L_n,\ 1 \leqslant j_1 \leqslant n,\ 1 \leqslant j_2 \leqslant n\},$$
$$A_n' = \{(j_1,j_2) \in \mathbb{N}^2 \colon |j_1 - j_2| > 2L_n,\ 1 \leqslant j_1 \leqslant n,\ 1 \leqslant j_2 \leqslant n\},$$

and write $c(\tau_1,\tau_2,j_1,j_2) = |\mathrm{cov}(X_{j_1}X_{j_1+\tau_1}, X_{j_2}X_{j_2+\tau_2})|$. Notice that using repeatedly the Jensen and Hölder inequalities we get

$$\begin{aligned} (6.12)\qquad & c(\tau_1,\tau_2,j_1,j_2) \\ & \leqslant E|X_{j_1}X_{j_1+\tau_1}X_{j_2}X_{j_2+\tau_2}| + E|X_{j_1}X_{j_1+\tau_1}|E|X_{j_2}X_{j_2+\tau_2}| \\ & \leqslant \|X_{j_1}\|_4\|X_{j_1+\tau_1}\|_4\|X_{j_2}\|_4\|X_{j_2+\tau_2}\|_4 \\ & \quad + \|X_{j_1}\|_2\|X_{j_1+\tau_1}\|_2\|X_{j_2}\|_2\|X_{j_2+\tau_2}\|_2 \leqslant 2\Delta_1^4 \end{aligned}$$

for any integers j_1,τ_1,j_2,τ_2. If we assume that $|\tau_1| \leqslant L_n$, $|\tau_2| \leqslant L_n$ and $(j_1,j_2) \in A_n'$, then using successively (2.4), the Hölder inequality and properties of α-mixing sequence we get

$$(6.13)\qquad c(\tau_1,\tau_2,j_1,j_2) \leqslant 8\Delta_1^4\alpha^{\delta/(2+\delta)}(|j_1-j_2| - |\tau_1| - |\tau_2|).$$

Simple decompositions and the inequalities (6.12) and (6.13) give

$$
\begin{aligned}
\left|\mathrm{var}\big(\hat{G}_n^*(\nu,\omega)\big)\right| &\leqslant \frac{M^2}{4\pi^2 n^2}\sum_{\tau_1=-L_n}^{L_n}\sum_{\tau_2=-L_n}^{L_n}\Big(\sum_{(j_1,j_2)\in A_n}+\sum_{(j_1,j_2)\in A_n'}\Big)c(\tau_1,\tau_2,j_1,j_2)\\
&\leqslant \frac{M^2}{4\pi^2 n^2}\sum_{\tau_1=-L_n}^{L_n}\sum_{\tau_2=-L_n}^{L_n}\sum_{(j_1,j_2)\in A_n}2\Delta_1^4\\
&\quad+\frac{M^2}{4\pi^2 n^2}\sum_{\tau_1=-L_n}^{L_n}\sum_{\tau_2=-L_n}^{L_n}\sum_{(j_1,j_2)\in A_n'}8\Delta_1^4\alpha^{\delta/(2+\delta)}(|j_1-j_2|-|\tau_1|-|\tau_2|)\\
&\leqslant \frac{M^2 2\Delta_1^4}{4\pi^2 n^2}\sum_{\tau_1=-L_n}^{L_n}\sum_{\tau_2=-L_n}^{L_n}(2L_n+1)n\\
&\quad+\frac{M^2 8\Delta_1^4}{4\pi^2 n^2}\sum_{\tau_1=-L_n}^{L_n}\sum_{\tau_2=-L_n}^{L_n}\sum_{k=2L_n+1}^{n-1}\sum_{|j_1-j_2|=k}\alpha^{\delta/(2+\delta)}(|j_1-j_2|-|\tau_1|-|\tau_2|)\\
&\leqslant \frac{M^2 2\Delta_1^4(2L_n+1)^3 n}{4\pi^2 n^2}\\
&\quad+\frac{M^2 8\Delta_1^4}{4\pi^2 n^2}\sum_{\tau_1=-L_n}^{L_n}\sum_{\tau_2=-L_n}^{L_n}\sum_{k=2L_n+1}^{n-1}\sum_{|j_1-j_2|=k}\alpha^{\delta/(2+\delta)}(|j_1-j_2|-2L_n)\\
&\leqslant o(1)+\frac{M^2 8\Delta_1^4}{4\pi^2 n^2}\sum_{\tau_1=-L_n}^{L_n}\sum_{\tau_2=-L_n}^{L_n}\sum_{k=1}^{n-2L_n-1}\sum_{|j_1-j_2|-2L_n=k}\alpha^{\delta/(2+\delta)}(k)\\
&\leqslant o(1)+\frac{M^2 8\Delta_1^4}{4\pi^2 n^2}\sum_{\tau_1=-L_n}^{L_n}\sum_{\tau_2=-L_n}^{L_n}\sum_{k=1}^{n-2L_n-1}n\,\alpha^{\delta/(2+\delta)}(k)\\
&\leqslant o(1)+\frac{M^2 8\Delta_1^4}{4\pi^2 n^2}(2L_n+1)^2 n\sum_{k=1}^{n-2L_n-1}\alpha^{\delta/(2+\delta)}(k)\\
&\leqslant o(1)+\frac{M^2 8\Delta_1^4}{4\pi^2 n}(2L_n+1)^2 K_1\to 0\quad \text{as } n\to\infty.
\end{aligned}
$$

This completes the proof of Step 2. ∎

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