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A NOTE ON EXTREMES OF COMPOUND POISSON PROCESSES

BY

DANIEL BEREND, MICHAEL BRAVERMAN* AND ILYA GOLDSTEIN (BEER SHEVA)

Abstract. We investigate the conditions on a compound Poisson process $\big(X(t)\big)_{0\leqslant t\leqslant 1}$, under which the right tail of $\sup_{0\leqslant t\leqslant 1} X(t)$ is equivalent to the tail of $X(1)$. New examples for which this relation does not hold are given.

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1. INTRODUCTION

A classical theme in the theory of stochastic processes is the study of extremes. One of the problems in this realm is about the relation between the tail of the supremum over a finite interval, and the tail of the process at the right end of the interval. The first result in this direction is due to Lévy and states that, if $X\,(t)$ is a Brownian motion and $X\,(0) = 0$, then

$$P\Big(\sup_{0\leqslant t\leqslant 1} X\,(t) > u\Big) = 2P\big(X\,(1) > u\big), \qquad u > 0.$$

(See, for example, [7], Chapter 6.) This question was investigated later for other classes of Lévy processes (see [2]–[5], [8]–[10]). One of these studies (see [10]) deals with the class of long-tailed distribution functions,

$$\mathcal{L} = \left\{ F \text{ d.f.} \,|\, \lim_{t\to\infty} \frac{1 - F\,(t-y)}{1 - F\,(t)} = 1,\ \forall y \in \mathbb{R} \right\},$$

and shows in particular that a Lévy process $X\,(t)$, satisfying $F_{X(1)} \in \mathcal{L}$, has the property

$$P\Big(\sup_{0\leqslant t\leqslant 1} X\,(t) > u\Big) \sim P\big(X\,(1) > u\big), \qquad u \to \infty. \tag{1.1}$$

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Compound Poisson processes form an important class of Lévy processes. They are defined by $X(t) = \sum_{i=1}^{N(t)} X_i$, where $(X_i)_{i=1}^{\infty}$ are i.i.d. random variables, and $N(t)$ is a standard Poisson process with a rate $\lambda > 0$. There are many known distributions which do not belong to $\mathcal{L}$, but for which the limit

$$\lim_{u\to\infty} \frac{P\left(\sup_{0\leqslant t\leqslant 1} X(t) > u\right)}{P\big(X(1) > u\big)} \tag{1.2}$$

exists (see [2], [4], [5]). All these results are proved under certain conditions on the *right* tail of the jumps of the process. On the other hand, if the jumps are non-negative, then the limit (1.2) clearly exists and equals 1. Thus, the following question arises naturally: does the negative tail of the jumps affect the existence of the limit in (1.2)? We show that, in general, the influence of a negative tail may be quite significant. Namely, we find a distribution F_1 on $\mathbb{R}_+$ such that, for every distribution F_2 on $\mathbb{R}_-$ and $0 < \alpha < 1$, the limit in (1.2) for the corresponding compound Poisson process for the distribution $F = \alpha F_1 + (1-\alpha)F_2$ does not exist.

It should be mentioned that examples of compound Poisson processes with negative drift for which the limit in (1.2) does not exist were given in [4], but what may happen in the absence of drift was unclear.

2. RESULTS

Let $X(t) = \sum_{i=1}^{N(t)} X_i$, where $(X_i)_{i=1}^{\infty}$ are i.i.d. random variables with distribution function F, and $N(t)$ is a standard Poisson process with a rate $\lambda > 0$.

Our first result demonstrates a strange phenomenon.

THEOREM 2.1. *There exists a distribution F_1 on $\mathbb{R}_+$ such that, if $F = \alpha F_1 + (1-\alpha)F_2$, where F_2 is any distribution on $\mathbb{R}_-$ with $F_2(0-) > 0$ and $0 < \alpha < 1$, then the process X satisfies*

$$\limsup_{u\to\infty} \frac{P\left(\sup_{0\leqslant t\leqslant 1} X(t) \geqslant u\right)}{P\big(X(1) \geqslant u\big)} > \liminf_{u\to\infty} \frac{P\left(\sup_{0\leqslant t\leqslant 1} X(t) \geqslant u\right)}{P\big(X(1) \geqslant u\big)} = 1.$$

It can be derived from Theorem 4.2 of [2] that if $F(0-) > 0$ and

$$\liminf_{u\to\infty} \frac{1 - F(x)}{1 - F * F(x)} > 0, \tag{2.1}$$

then the following assertions are equivalent:

(i) F is long-tailed.

(ii) $\displaystyle\lim_{u\to\infty} \frac{P\left(\sup_{0\leqslant t\leqslant 1} X(t) > u\right)}{P\big(X(1) > u\big)} = 1.$

The following two results show that, if (2.1) fails to hold, and F is not long-tailed, then (1.1) may fail and may hold.

THEOREM 2.2. *There exists a distribution F such that*

$$\liminf_{u\to\infty} \frac{1-F(x)}{1-F*F(x)} = 0$$

and

$$\limsup_{u\to\infty} \frac{P\left(\sup_{0\leqslant t\leqslant 1} X(t) \geqslant u\right)}{P\big(X(1) \geqslant u\big)} > 1.$$

THEOREM 2.3. *There exists a non-long-tailed distribution F such that*

$$F(0-) > 0, \qquad \limsup_{u\to\infty} \frac{1-F(x)}{1-F*F(x)} > 0,$$

and

$$\lim_{u\to\infty} \frac{P\left(\sup_{0\leqslant t\leqslant 1} X(t) > u\right)}{P\big(X(1) > u\big)} = 1.$$

While the long-tail property and (2.1) may hold simultaneously, neither of these conditions implies the other. Indeed, it was shown in [6] that there exists a long-tailed distribution F such that

$$\liminf_{u\to\infty} \frac{1-F(u)}{1-F*F(u)} = 0.$$

On the other hand, one can easily verify that the distribution function

$$F(t) = \begin{cases} 0, & t < 1, \\ 1-2^{-k}, & 2^k \leqslant t < 2^{k+1}, \end{cases}$$

where $k \geqslant 0$, is a non-long-tailed distribution function, and satisfies

$$\liminf_{u\to\infty} \frac{1-F(u)}{1-F*F(u)} > 0.$$

3. PROOFS

In this section we prove Theorems 2.1, 2.3 and 2.2, by introducing appropriate examples. First, we prove Theorem 2.1. To this end, we introduce a distribution F_1, and prove that it satisfies the properties stated in the theorem. The distribution function is defined by

$$F_1(x) = \begin{cases} 0, & x < 2, \\ 1 - \dfrac{1}{n!}, & n! \leqslant x < (n+1)!, \ \ n = 2, 3, \ldots \end{cases}$$

We start with several lemmas. Put

$$M = \max_{1\leqslant i\leqslant N(1)} X_i \quad \text{and} \quad Y = \sup_{0\leqslant t\leqslant 1} X(t).$$

LEMMA 3.1. *If* $F = \alpha F_1 + (1-\alpha)F_2$, *where* F_2 *is a distribution on* $\mathbb{R}_-$, $0 < \alpha < 1$, *and* $(a_n)_{n=1}^{\infty}$ *is a sequence satisfying* $3 < a_n < n+1$ *for all* $n \geqslant 3$, *then*

$$\lim_{n\to\infty} P\left(M > n! \,|\, Y > a_n \cdot n!\right) = \lim_{n\to\infty} P\big(M > n! \,|\, X(1) > a_n \cdot n!\big) = 1.$$

Proof. The equalities are equivalent to

$$\lim_{n\to\infty} \frac{P\left(M \leqslant n!, Y > a_n \cdot n!\right)}{P\left(Y > a_n \cdot n!\right)} = \lim_{n\to\infty} \frac{P\big(M \leqslant n!, X(1) > a_n \cdot n!\big)}{P\big(X(1) > a_n \cdot n!\big)} = 0.$$

Obviously,

$$\begin{aligned} (3.1) \quad & P\left(M \leqslant n!, Y > a_n \cdot n!\right) \\ & \qquad \leqslant P\big(N(1) > 2n\big) + P\big(M \leqslant n!, Y > a_n \cdot n!, N(1) \leqslant 2n\big). \end{aligned}$$

Since $N(1) \sim P(\lambda)$, we have $P\big(N(1) > 2n\big) = O\left(\lambda^{2n}/(2n)!\right)$. For $n \geqslant 3$, the event $\{M \leqslant n!, Y > a_n \cdot n!, N(1) \leqslant 2n\}$ is contained in the event where at least two of the variables $(X_i)_{i=1}^{N(1)}$ assume the value $n!$, and thus

$$\begin{aligned} & P\big(M \leqslant n!, Y > a_n \cdot n!, N(1) \leqslant 2n\big) \\ & \qquad \leqslant \binom{2n}{2} \left(\frac{\alpha}{(n-1)!}\right)^2 = O\bigg(\frac{1}{\big((n-2)!\big)^2}\bigg). \end{aligned}$$

Returning to (3.1), we find that

$$\begin{aligned} & P(M \leqslant n!, Y > a_n \cdot n!) \\ & \qquad = O\left(\frac{\lambda^{2n}}{(2n)!}\right) + O\bigg(\frac{1}{\big((n-2)!\big)^2}\bigg) = O\bigg(\frac{C^n}{\big((n-2)!\big)^2}\bigg), \end{aligned}$$

where $C = \max\left\{1, \lambda^2\right\}$, and thus

$$P\big(M \leqslant n!, X(1) > a_n \cdot n!\big) \leqslant P\left(M \leqslant n!, Y > a_n \cdot n!\right) = O\bigg(\frac{C^n}{\big((n-2)!\big)^2}\bigg).$$

On the other hand,

$$P\big(X(1) > a_n \cdot n!\big) \geqslant P\big(N(1) = 1\big) P\big(X_1 \geqslant (n+1)!\big) = \lambda e^{-\lambda} \frac{\alpha}{n!} \geqslant \frac{C_1}{n!},$$

and thus

$$P\left(Y > a_n \cdot n!\right) \geqslant \frac{C_2}{n!}$$

for some $C_1, C_2 > 0$. Altogether

$$\frac{P\big(M \leqslant n!, X(1) > a_n \cdot n!\big)}{P\big(X(1) > a_n \cdot n!\big)} = O\bigg(\frac{C^n n!}{\big((n-2)!\big)^2}\bigg) = O\left(\frac{C^n}{(n-4)!}\right)$$

and

$$\frac{P(M \leqslant n!, Y > a_n \cdot n!)}{P(Y > a_n \cdot n!)} = O\bigg(\frac{C^n n!}{\big((n-2)!\big)^2}\bigg) = O\left(\frac{C^n}{(n-4)!}\right),$$

which completes the proof. ■

LEMMA 3.2. *If $F = \alpha F_1 + (1-\alpha)F_2$, where F_2 is a distribution on $\mathbb{R}_-$ and $0 < \alpha < 1$, then*

$$\liminf_{n\to\infty} P\big(N(1) = 2 \mid M > n!\big) > 0.$$

P r o o f. Obviously, the required inequality is equivalent to

$$\limsup_{n\to\infty} \frac{P(M > n!)}{P\big(N(1) = 2, M > n!\big)} < \infty.$$

Now,

$$P(M > n!) = \sum_{k=1}^{\infty} P(\max_{1\leqslant i\leqslant k} X_i > n!) P\big(N(1) = k\big) \leqslant \sum_{k=1}^{\infty} k \cdot \frac{\alpha}{n!} e^{-\lambda} \frac{\lambda^k}{k!} = \frac{\alpha\lambda}{n!}$$

and, on the other hand,

$$P\big(N(1) = 2, M > n!\big) \geqslant P(X_1 > n!)\, P\big(N(1) = 2\big) = \frac{\alpha}{n!} \cdot e^{-\lambda} \frac{\lambda}{2}.$$

Thus

$$\frac{P(M > n!)}{P\big(N(1) = 2, M > n!\big)} \leqslant 2e^{\lambda},$$

which completes the proof. ■

LEMMA 3.3. *If $F = \alpha F_1 + (1-\alpha)F_2$, where F_2 is a distribution on $\mathbb{R}_-$ and $0 < \alpha < 1$, then*

$$\lim_{n\to\infty} P\big(X(1) > n \cdot n! \mid M > n!\big) = 1.$$

Proof. Clearly,

$$\begin{aligned} P\big(X(1) \leqslant n \cdot n!, M > n!\big) &= \sum_{k=1}^{\infty} P\big(X(1) \leqslant n \cdot n!, M > n!, N(1) = k\big) \\ &\leqslant \sum_{k=1}^{\infty} k \cdot \frac{\alpha}{n!} P\left(S_{k-1} \leqslant -n!\right) e^{-\lambda} \frac{\lambda^k}{k!} \\ &= \lambda \frac{\alpha}{n!} \sum_{k=1}^{\infty} P\left(S_{k-1} \leqslant -n!\right) e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!} \\ &= \lambda \frac{\alpha}{n!} P\big(X(1) \leqslant -n!\big). \end{aligned}$$

On the other hand,

$$P(M > n!) \geqslant P\big(N(1) = 1\big) \frac{\alpha}{n!} = e^{-\lambda} \lambda \frac{\alpha}{n!}.$$

Thus

$$\frac{P\big(X(1) \leqslant n \cdot n!, M > n!\big)}{P(M > n!)} \leqslant e^{\lambda} P\big(X(1) \leqslant -n!\big)$$

and, consequently,

$$\lim_{n\to\infty} \frac{P\big(X(1) \leqslant n \cdot n!, M > n!\big)}{P(M > n!)} \leqslant e^{\lambda} \lim_{n\to\infty} P\big(X(1) \leqslant -n!\big) = 0,$$

which completes the proof. ■

Proof of Theorem 2.1. Since $F_2(0-) > 0$, there is an $\varepsilon_1 > 0$ such that $F(-2\varepsilon_1) > 0$. By Lemma 3.1 we know that

$$\limsup_{n\to\infty} \frac{P\big(Y > (n+1)! - \varepsilon_1\big)}{P\big(X(1) > (n+1)! - \varepsilon_1\big)} = \limsup_{n\to\infty} \frac{P\big(M > n!, Y > (n+1)! - \varepsilon_1\big)}{P\big(M > n!, X(1) > (n+1)! - \varepsilon_1\big)}$$

and

$$\liminf_{n\to\infty} \frac{P(Y > n \cdot n!)}{P\big(X(1) > n \cdot n!\big)} = \liminf_{n\to\infty} \frac{P(M > n!, Y > n \cdot n!)}{P\big(M > n!, X(1) > n \cdot n!\big)}.$$

Therefore,

$$\limsup_{n\to\infty} \frac{P\big(Y > (n+1)! - \varepsilon_1\big)}{P\big(X(1) > (n+1)! - \varepsilon_1\big)} = \limsup_{n\to\infty} \frac{P\big(Y > (n+1)! - \varepsilon_1 \mid M > n!\big)}{P\big(X(1) > (n+1)! - \varepsilon_1 \mid M > n!\big)}$$

and

$$\liminf_{n\to\infty} \frac{P\left(Y > n\cdot n!\right)}{P\big(X\left(1\right) > n\cdot n!\big)} = \liminf_{n\to\infty} \frac{P\left(Y > n\cdot n! \,|\, M > n!\right)}{P\big(X\left(1\right) > n\cdot n! \,|\, M > n!\big)}.$$

According to Lemma 3.3,

$$1 \geqslant \liminf_{n\to\infty} P\left(Y > n\cdot n! \,|\, M > n!\right) \geqslant \lim_{n\to\infty} P\big(X\left(1\right) > n\cdot n! \,|\, M > n!\big) = 1.$$

Thus

$$\liminf_{n\to\infty} \frac{P\left(Y > n\cdot n!\right)}{P\big(X\left(1\right) > n\cdot n!\big)} = 1. \tag{3.2}$$

On the other hand, we have

$$P\big(X_1 = (n+1)!, X_2 < -\varepsilon_1\big) \geqslant \left(\frac{\alpha}{n!} - \frac{\alpha}{(n+1)!}\right) F\left(-2\varepsilon_1\right)$$

and

$$P\big(M > n! \,|\, N\left(1\right) = 2\big) \leqslant \frac{2\alpha}{n!},$$

which implies

$$P\big(X_1 = (n+1)!, X_2 < -\varepsilon_1 \,|\, M > n!, N\left(1\right) = 2\big) \geqslant \frac{1}{2}\left(1 - \frac{1}{n+1}\right) F\left(-2\varepsilon_1\right).$$

Thus, for every $n \geqslant 1$,

$$\begin{aligned} &P\big(Y > (n+1)! - \varepsilon_1 \,|\, M > n!, N\left(1\right) = 2\big) \\ &\qquad > P\big(X\left(1\right) > (n+1)! - \varepsilon_1 \,|\, M > n!, N\left(1\right) = 2\big) + \frac{1}{4} F\left(-2\varepsilon_1\right). \end{aligned} \tag{3.3}$$

Since

$$\begin{aligned} &P\big(Y > (n+1)! - \varepsilon_1 \,|\, M > n!, N\left(1\right) = k\big) \\ &\qquad \geqslant P\big(X\left(1\right) > (n+1)! - \varepsilon_1 \,|\, M > n!, N\left(1\right) = k\big) \end{aligned}$$

for every $k \in N$, by (3.3) we have

$$\begin{aligned} &P\big(Y > (n+1)! - \varepsilon_1 \,|\, M > n!\big) \\ &\geqslant P\big(X\left(1\right) > (n+1)! - \varepsilon_1 \,|\, M > n!\big) + \frac{1}{4} F\left(-2\varepsilon_1\right) P\big(N\left(1\right) = 2 \,|\, M > n!\big). \end{aligned}$$

Thus

$$\frac{P\bigl(Y > (n+1)! - \varepsilon_1 \,|\, M > n!\bigr)}{P\bigl(X(1) > (n+1)! - \varepsilon_1 \,|\, M > n!\bigr)} \geqslant 1 + \frac{1}{4} F(-2\varepsilon_1)\, P\bigl(N(1) = 2 \,|\, M > n!\bigr), \qquad n \geqslant 1.$$

According to Lemma 3.2, this implies

$$\limsup_{n\to\infty} \frac{P\bigl(Y > (n+1)! - \varepsilon_1 \,|\, M > n!\bigr)}{P\bigl(X(1) > (n+1)! - \varepsilon_1 \,|\, M > n!\bigr)} > 1. \tag{3.4}$$

Joining (3.2) and (3.4), we complete the proof. ■

Proof of Theorem 2.2. We shall use a distribution F similar to that in Theorem 2.1, obtained from the same F_1, and specific F_2 and α. Let F_2 be the distribution defined by

$$F_2(x) = \frac{-1}{x}, \qquad -\infty < x \leqslant -1,$$

and take

$$F(x) = \frac{1}{2} F_1 + \frac{1}{2} F_2, \qquad -\infty < x < \infty.$$

By Theorem 2.1, the process X satisfies

$$\limsup_{u\to\infty} \frac{P\left(\sup_{0\leqslant t\leqslant 1} X(t) \geqslant u\right)}{P\bigl(X(1) \geqslant u\bigr)} > 1.$$

On the other hand,

$$1 - F * F(n!) \geqslant \frac{1}{4(n-1)!}, \qquad n \geqslant 1.$$

Thus

$$\liminf_{n\to\infty} \frac{1 - F(n!)}{1 - F * F(n!)} = \liminf_{n\to\infty} \frac{1/2n!}{1 - F * F(n!)} \leqslant \liminf_{n\to\infty} \frac{2}{n} = 0,$$

and, in particular,

$$\liminf_{u\to\infty} \frac{1 - F(u)}{1 - F * F(u)} = 0.$$

This completes the proof. ■

Next we turn to prove Theorem 2.3. We prove two lemmas first.

LEMMA 3.4. *If*

$$\lim_{u\to\infty}\frac{P\left(S_n>u\right)}{P\big(X\left(1\right)>u\big)}=0,\qquad n\geqslant 1,$$

then

$$\lim_{u\to\infty}\frac{P\left(\sup_{0\leqslant t\leqslant 1}X\left(t\right)>u\right)}{P\big(X\left(1\right)>u\big)}=1.$$

P r o o f. Let $\varepsilon>0$. Since $N\left(1\right)\sim P\left(\lambda\right)$, there is a K such that

$$\frac{P\big(N\left(1\right)\geqslant k\big)}{P\big(N\left(1\right)=k\big)}<1+\frac{\varepsilon}{2},\qquad k\geqslant K.$$

Obviously,

$$P\big(\sup_{0\leqslant t\leqslant 1}X\left(t\right)>u\big)\leqslant\sum_{k=1}^{K}P\left(S_k>u\right)+\sum_{k=K+1}^{\infty}P\left(S_k>u\right)P\big(N\left(1\right)\geqslant k\big),$$

and

$$P\big(X\left(1\right)>u\big)\geqslant\sum_{k=K+1}^{\infty}P\left(S_k>u\right)P\big(N\left(1\right)=k\big). \tag{3.5}$$

By the choice of K we have

$$\sum_{k=K+1}^{\infty}P\left(S_k>u\right)P\big(N\left(1\right)\geqslant k\big)<\left(1+\frac{\varepsilon}{2}\right)\sum_{k=K+1}^{\infty}P\left(S_k>u\right)P\big(N\left(1\right)=k\big),$$

which, together with (3.5), yields

$$\frac{\sum_{k=K+1}^{\infty}P\left(S_k>u\right)P\big(N\left(1\right)\geqslant k\big)}{P\big(X\left(1\right)>u\big)}<1+\frac{\varepsilon}{2}.$$

On the other hand, the condition of the lemma provides a $u_0>0$ such that

$$\sum_{k=1}^{K}\frac{P\left(S_n>u\right)}{P\big(X\left(1\right)>u\big)}<\frac{\varepsilon}{2},\qquad u>u_0.$$

Thus for $u>u_0$

$$\begin{aligned}&\frac{P\left(\sup_{0\leqslant t\leqslant 1}X\left(t\right)>u\right)}{P\big(X\left(1\right)>u\big)}\\&\quad\leqslant\frac{\sum_{k=1}^{K}P\left(S_n>u\right)}{P\big(X\left(1\right)>u\big)}+\frac{\sum_{k=K+1}^{\infty}P\left(S_k>u\right)P\big(N\left(1\right)\geqslant k\big)}{P\big(X\left(1\right)>u\big)}<1+\varepsilon,\end{aligned}$$

which proves the lemma. ∎

LEMMA 3.5. *If*

$$\lim_{u\to\infty}\frac{P(X_1>u)}{P(S_n>u)}=0 \tag{3.6}$$

for some $n>0$, *then*

$$\lim_{u\to\infty}\frac{P\left(\sup_{0\leqslant t\leqslant 1}X(t)>u\right)}{P\big(X(1)>u\big)}=1.$$

Proof. Let $k>0$, $\varepsilon>0$, and let G be the distribution function of S_{k-1}. If $S_k>u$, then $X_i\geqslant u/k$ for some $1\leqslant i\leqslant k$. Thus

$$P(S_k>u)\leqslant k\int_{-\infty}^{\infty}P\left(X_1>\max\left\{\frac{u}{k},u-t\right\}\right)G(dt). \tag{3.7}$$

Obviously, $S_{n+k-1}=S_n+(S_{n+k-1}-S_n)$. The variable $S_{n+k-1}-S_n$ is a sum of $k-1$ i.i.d. variables having the distribution F. Thus, the distribution function of $S_{n+k-1}-S_n$ is G, which implies

$$\begin{aligned}P(S_{n+k-1}>u)&\geqslant\int_{-\infty}^{\infty}P(S_n>u-t)\,G(dt)\\&\geqslant\int_{-\infty}^{\infty}P\left(S_n>\max\left\{\frac{u}{k},u-t\right\}\right)G(dt).\end{aligned} \tag{3.8}$$

Condition (3.6) provides a u_0 such that

$$\frac{P(X_1>u)}{P(S_n>u)}<\frac{\varepsilon}{k},\qquad u>u_0.$$

Thus

$$P\left(X_1>\max\left\{\frac{u}{k},u-t\right\}\right)<\frac{\varepsilon}{k}P\left(S_n>\max\left\{\frac{u}{k},u-t\right\}\right),\qquad u>ku_0,\ t\in\mathbb{R},$$

which implies

$$\begin{aligned}&k\int_{-\infty}^{\infty}P\left(X_1>\max\left\{\frac{u}{k},u-t\right\}\right)G(dt)\\&\qquad<\varepsilon\int_{-\infty}^{\infty}P\left(S_n>\max\left\{\frac{u}{k},u-t\right\}\right)G(dt),\qquad u>ku_0,\end{aligned}$$

and thus, by (3.7) and (3.8),

$$\frac{P(S_k>u)}{P(S_{n+k-1}>u)}<\varepsilon,\qquad u>ku_0.$$

This implies

$$\lim_{u\to\infty} \frac{P(S_k > u)}{P(S_{n+k-1} > u)} = 0.$$

Obviously, $P\big(X(1) > u\big) \geqslant P(S_{n+k-1} > u)\, P\big(N(1) = n+k-1\big)$. Hence

$$\lim_{u\to\infty} \frac{P(S_k > u)}{P\big(X(1) > u\big)} \leqslant \lim_{u\to\infty} \frac{P(S_k > u)}{P(S_{n+k-1} > u)\, P\big(N(1) = n+k-1\big)} = 0$$

for every $k \in N$. It follows that the condition in Lemma 3.4 is fulfilled, which completes the proof of this lemma. ∎

P r o o f o f T h e o r e m 2.3. Let F be the distribution function corresponding to the probability function

$$P(X_1 = \pm 3^n) = \frac{2^{-4^n}}{c}, \qquad n \in \mathbb{N},$$

where $c = 2\sum_{n=1}^{\infty} 2^{-4^n}$. Obviously, F is non-long-tailed, $F(0-) = \frac{1}{2}$, and

$$P(X_1 + X_2 \geqslant 3^n) \leqslant 2P(X_1 \geqslant 3^n).$$

Thus

$$\limsup_{u\to\infty} \frac{1 - F(x)}{1 - F * F(x)} \geqslant \limsup_{n\to\infty} \frac{P(X_1 \geqslant 3^n)}{P(X_1 + X_2 \geqslant 3^n)} \geqslant \frac{1}{2}.$$

Clearly, $P(S_3 = 3^n) \geqslant P(X_1 = X_2 = X_3 = 3^{n-1}) = (2^{-4^{n-1}}/c)^3$ for $n > 1$, so that

$$\lim_{n\to\infty} \frac{P(X_1 = 3^n)}{P(S_3 = 3^n)} \leqslant \lim_{n\to\infty} c^2 2^{-4^{n-1}} = 0.$$

This yields

$$\lim_{u\to\infty} \frac{P(X_1 > u)}{P(S_3 > u)} = 0,$$

and applying Lemma 3.5, we obtain

$$\lim_{u\to\infty} \frac{P\left(\sup_{0\leqslant t\leqslant 1} X(t) > u\right)}{P\big(X(1) > u\big)} = 1.$$

Thus, F satisfies the required properties. ∎

REFERENCES

[1] J. M. P. Albin, *Extremes of totally skewed stable motion*, Statist. Probab. Lett. 16 (1993), pp. 219–224.
[2] J. M. P. Albin and M. Sunden, *On the asymptotic behaviour of Lévy processes. Part I: Subexponential and exponential processes*, Stochastic Process. Appl. 119 (2009), pp. 281–304.
[3] S. Berman, *The supremum of a process with stationary independent symmetric increments*, Stochastic Process. Appl. 23 (1986), pp. 281–290.
[4] M. Braverman, *Suprema of compound Poisson processes with light tails*, Stochastic Process. Appl. 90 (2000), pp. 145–156.
[5] M. Braverman and G. Samorodnitsky, *Functionals of infinitely divisible stochastic processes with exponential tails*, Stochastic Process. Appl. 56 (1995), pp. 207–231.
[6] P. Embrechts and C. M. Goldie, *On closure and factorization properties of subexponential and related distributions*, J. Aust. Math. Soc. Ser. A 29 (1980), pp. 243–256.
[7] I. Gikhman and A. Skorohod, *Introduction to the Theory of Stochastic Processes*, Dover, Mineola, NY, 1996.
[8] M. B. Marcus, *ξ-radical Processes and Random Fourier Series*, Mem. Amer. Math. Soc. 368, Amer. Math. Soc., Providence, RI, 1987.
[9] J. Rosiński and G. Samorodnitsky, *Distributions of subadditive functionals of sample paths of infinitely divisible processes*, Ann. Probab. 21 (1993), pp. 996–1014.
[10] E. Willekens, *On the supremum of an infinitely divisible process*, Stochastic Process. Appl. 26 (1987), pp. 173–175.

Departments of Mathematics
and of Computer Science
Ben-Gurion University of the Negev
Beer-Sheva, 84105, Israel
E-mail: berend@math.bgu.ac.il

Departments of Mathematics
Ben-Gurion University of the Negev
Beer-Sheva, 84105, Israel
E-mail: braver@math.bgu.ac.il

Departments of Mathematics
Ben-Gurion University of the Negev
Beer-Sheva, 84105, Israel
E-mail: bulya2@hotmail.com

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