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SOME PROPERTIES OF QUANTUM LÉVY AREA IN FOCK AND NON-FOCK QUANTUM STOCHASTIC CALCULUS

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Dedicated to the memory of Slava Belavkin

Abstract. We consider the analogue of Lévy area, defined as an iterated stochastic integral, obtained by replacing two independent component one-dimensional Brownian motions by the mutually non-commuting momentum and position Brownian motions P and Q of either Fock or non-Fock quantum stochastic calculus, which are also stochastically independent in a certain sense. We show that the resulting quantum Lévy area is trivially distributed in the Fock case, but has a non-trivial distribution in non-Fock quantum stochastic calculus which, after rescaling, interpolates between the trivial distribution and that of classical Lévy area in the "infinite temperature" limit. We also show that it behaves differently from the classical Lévy area under a kind of time reversal, in both the Fock and non-Fock cases.

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1. INTRODUCTION

Lévy's stochastic area [8] can be defined in terms of iterated stochastic integration as

$$\frac{1}{2}\int_{0\leqslant x<y<t}\big(dX(x)dY(y)-dY(x)dX(y)\big),$$

where X and Y are independent one-dimensional Brownian motions. The *Lévy area formula* for the characteristic function of this is

$$\mathbb{E}\left[\exp\left\{\frac{i\lambda}{2}\int_{0\leqslant x<y<t}\big(dX(x)dY(y)-dY(x)dX(y)\big)\right\}\right]=\operatorname{sech}\frac{1}{2}\lambda t. \tag{1.1}$$

It has many interesting connotations. For example, the corresponding distribution is one of the Meixner family [9] and there are connections to integrable systems, Bernoulli and Euler polynomials, values of the Riemann zeta function, etc. For a recent review see [7].

In quantum stochastic calculus [10] the "momentum" and "position" Brownian motions do not commute so that it is not a physically meaningful question to ask whether they are independent. Nevertheless we shall show in Theorem 2.1 below that they enjoy a property that in classical probability is tantamount to independence. The purpose of this article is to explore what becomes of Lévy area when X and Y are replaced by P and Q.

For an alternative approach to this quantum Lévy area motivated by Lévy's original martingale definition, see [4].

2. THE MOMENTUM AND POSITION PROCESSES IN FOCK AND NON-FOCK QUANTUM STOCHASTIC CALCULUS

The usual form of quantum stochastic calculus (see [10]) takes place in the Fock space, denoted by $\mathcal{H}$, over the Hilbert space $\mathfrak{h} = L^2(\mathbb{R}^+)$, which may conveniently be defined as the closed linear span of the so-called *exponential vectors* $\big(e(f), f \in \mathfrak{h}\big)$ satisfying

$$\langle e(f), e(g)\rangle_{\mathcal{H}} = \exp\langle f, g\rangle_{\mathfrak{h}}.$$

The *Weyl operators*, $W(f), f \in \mathfrak{h}$, are the unitary operators on $\mathcal{H}$ whose actions on exponential vectors are

$$W(f)e(g) = \exp\left(-\frac{1}{2}\|f\|^2 - \langle f, g\rangle\right) e(f+g)$$

(we use a probabilistic normalisation in which in effect Planck's constant h is set equal to 4π rather than the physicist's normalisation 2π, so that our Weyl operator $W(f)$ is the physically normalised $W_{\text{phys}}(\sqrt{2}f)$). The Weyl operators satisfy the *canonical commutation relations* (ccr)

$$W(f)W(g) = \exp\left(-i\operatorname{Im}\langle f, g\rangle\right) W(f+g). \tag{2.1}$$

The *momentum* and *position* processes $P = \big(P(t)\big)_{t\in\mathbb{R}^+}$ and $Q = \big(Q(t)\big)_{t\in\mathbb{R}^+}$, respectively, are the families of self-adjoint operators in $\mathcal{H}$ defined by

$$\exp\big(ixP(t)\big) = W(x\chi_{[0,t]}), \quad \exp\big(iyQ(t)\big) = W(iy\chi_{[0,t]}), \quad x, y \in \mathbb{R}, \tag{2.2}$$

where $\chi_{[0,t]} \in \mathfrak{h}$ denotes the indicator function of the interval $[0,t] \subset \mathbb{R}^+$. Each process is commutative in the sense that for arbitrary $x, y \in \mathbb{R}$ and $t, u \in \mathbb{R}^+$

$$\exp\big(ixP(t)\big)\exp\big(iyP(u)\big) = \exp\big(iyP(u)\big)\exp\big(ixP(t)\big),$$
$$\exp\big(ixQ(t)\big)\exp\big(iyQ(u)\big) = \exp\big(iyQ(u)\big)\exp\big(ixQ(t)\big).$$

Furthermore, the processes P and Q are both standard Brownian motions in the *vacuum state* $\Omega = e(0)$ in the sense that, for example, P satisfies the following:

- the vacuum charateristic function

$$\mathbb{E}_{\Omega}\big[\exp\big(ixP(t)\big)\big] = \exp\bigg(-\frac{t}{2}x^2\bigg), \tag{2.3}$$

where for an operator K on $\mathcal{H}$, $\mathbb{E}_{e(0)}[K] = \langle e(0), Ke(0)\rangle$, so that the quantum random variable $P(t)$ is normally distributed with mean zero and variance t;

- it begins anew independently of the past at each time $s \in \mathbb{R}^+$ in the sense that

$$\mathbb{E}_{\Omega}\Big[K\exp\Big(ix\big(P(s+t)-P(s)\big)\Big)\Big] = \mathbb{E}_{\Omega}[K]\,\mathbb{E}_{\Omega}\big[\exp\big(ixP(t)\big)\big],$$

whenever K is an element of the *pre-s* von Neumann algebra generated by the Weyl operators $W(f)$ for which $f = f\chi_{[0,s]}$.

But the two Brownian motions P and Q do not commute with each other; in fact, they satisfy the commutation relation

$$[P(t), Q(u)] = -2i(t\wedge u)I,$$

where $t\wedge u$ denotes the minimum of t and u, and I is the identity operator, in the sense that

$$\begin{aligned}\exp\big(ixP(t)\big)&\exp\big(iyQ(u)\big)\\ &= \exp(2ixyt\wedge u)\exp\big(iyQ(u)\big)\exp\big(ixP(t)\big),\end{aligned} \tag{2.4}$$

as follows from (2.1).

We shall also use another *non-Fock* pair of mutually non-commuting Brownian motions P_σ and Q_σ satisfying the same commutation relations constructed as follows. Let there be given a real number $\sigma > 1$ called the *variance*. In the tensor product $\mathcal{H}\otimes\bar{\mathcal{H}}$, where $\bar{\mathcal{H}}$ denotes the Hilbert dual space of $\mathcal{H}$, define operators

$$W_\sigma(f) = W\bigg(\frac{\sqrt{\sigma^2+1}}{2}f\bigg)\otimes\bigg(W\bigg(\frac{\sqrt{\sigma^2-1}}{2}f\bigg)\bigg)^{-}, \quad f\in\mathfrak{h},$$

where for a bounded operator K on $\mathcal{H}$, $\bar{K}$ denotes the operator $\bar{K}\bar{\psi} = (K\psi)^-$, and $\bar{\psi}$ denotes the element $\chi\mapsto\langle\psi,\chi\rangle$ of $\bar{\mathcal{H}}$. Then the family $W_\sigma = \big(W_\sigma(f), f\in\mathfrak{h}\big)$ satisfies the ccr (2.1). Moreover, in the state $\Omega_\sigma = \Omega\otimes\bar{\Omega}$, the processes P_σ and Q_σ got by replacing W in (2.2) by W_σ are Brownian motions of variance σ^2 in the sense that instead of (2.3) we have

$$\mathbb{E}_{\Omega_\sigma}\big[\exp\big(ixP_\sigma(t)\big)\big] = \mathbb{E}_{\Omega_\sigma}\big[\exp\big(ixQ_\sigma(t)\big)\big] = \exp\bigg(-\frac{t\sigma^2}{2}x^2\bigg). \tag{2.5}$$

We shall find another realisation [1] of the non-Fock quantum Brownian motions P_σ and Q_σ useful. Denote by $(\mathcal{C}^2, \mathcal{F}, \mathbb{P})$ the Wiener space realisation of two-dimensional standard classical Brownian motion, so that $\mathcal{C}^2$ is the space of continuous $\mathbb{R}^2$-valued functions $\omega = (\omega_1, \omega_2)$ on $\mathbb{R}^+$ with $\omega(0) = 0$, $\mathcal{F}$ is the σ-field of subsets of $\mathcal{C}^2$ generated by the evaluations $X(t,\omega)=\omega_1(t), Y(t,\omega)=\omega_2(t), \omega=(\omega_1,\omega_2)\in\mathcal{C}^2$, and $\mathbb{P}$ is two-dimensional Wiener measure, which makes the processes X and Y into independent standard unit-variance Brownian motions. In the Hilbert space tensor product $\mathcal{H}_\sigma=\mathcal{H}\otimes L^2(\mathcal{C}^2,\mathcal{F},\mathbb{P})$, equipped with the unit vector $\Omega\otimes 1$, where 1 is the constant function $1(\omega)=1$ on $\mathcal{C}^2$, we may define

$$P_\sigma(t)=P(t)\otimes I+\sqrt{(\sigma^2-1)}I\otimes \mathrm{mult}_{X(t)}, \tag{2.6}$$

$$Q_\sigma(t)=Q(t)\otimes I+\sqrt{(\sigma^2-1)}I\otimes \mathrm{mult}_{Y(t)}, \tag{2.7}$$

mult_F denoting the operator of multiplication by the function F on $L^2(\mathcal{C}^2,\mathcal{F},\mathbb{P})$. More rigorously we may define the corresponding Weyl operators as

$$\begin{aligned}\exp\big(ixP_\sigma(t)\big)&=\exp\big(ixP(t)\big)\otimes\exp\big(ix\sqrt{(\sigma^2-1)}\,\mathrm{mult}_{X(t)}\big),\\ \exp\big(iyQ_\sigma(t)\big)&=\exp\big(iyQ(t)\big)\otimes\exp\big(ix\sqrt{(\sigma^2-1)}\,\mathrm{mult}_{Y(t)}\big).\end{aligned} \tag{2.8}$$

Then it can be verified that both P_σ and Q_σ again satisfy (2.5), where now Ω_σ is the unit vector $\Omega\otimes 1_{\mathcal{C}^2}$, as well as the commutation relations (2.4).

In the rest of the paper we shall denote by the same symbol (P,Q) the pair consisting either of the Fock momentum and position Brownian motions or the pair (P_σ,Q_σ) as defined in either of the alternative ways above, making clear which is intended where necessary. The same symbol $\mathbb{E}$ will similarly denote either $\mathbb{E}_\Omega$ or $\mathbb{E}_{\Omega_\sigma}$.

Because of their mutual non-commutativity, in orthodox quantum theory it is not possible to measure P and Q simultaneously and it is therefore meaningless from the point of view of quantum physics to speak of their stochastic independence. Nevertheless, they retain a property that in classical probability is tantamount to independence, namely factorization of joint characteristic functions, in the sense of Theorem 2.1 which follows below.

We give first a rigorous definition, for arbitrary real numbers $\lambda_1,\lambda_2,\dots,\lambda_m$ and $\mu_1,\mu_2,\dots,\mu_n$, and nonnegative numbers $s_1,s_2,\dots,s_m$ and $t_1,t_2,\dots,t_n$, of the unitary operator $\exp\big\{i\big(\sum_{j=1}^m\lambda_jP(s_j)+\sum_{k=1}^n\mu_kQ(t_k)\big)\big\}$. It is defined to be the value W_1 of the one-parameter unitary group $(W_x)_{x\in\mathbb{R}}$, where

$$\begin{aligned}W_x&=\exp(i\tau x^2)\prod_{j=1}^m\exp\{ix\lambda_jP(s_j)\}\prod_{k=1}^n\exp\{ix\mu_kQ(t_k)\}\\ &=\exp(-i\tau x^2)\prod_{k=1}^n\exp\{ix\mu_kQ(t_k)\}\prod_{j=1}^m\exp\{ix\lambda_jP(s_j)\}\end{aligned} \tag{2.9}$$

and

$$\tau=\sum_{j=1}^m\sum_{k=1}^n\lambda_j\mu_k(s_j\wedge t_k).$$

That the two forms of W_x are equal and that the group relation $W_xW_y=W_{x+y}$ holds are consequences of (2.4). That the infinitesimal generator of this one-parameter group acts on the appropriate intersection of domains as $\sum_{j=1}^{m} \lambda_j P(s_j) + \sum_{k=1}^{n} \mu_k Q(t_k)$ may be verified by differentiating either form with respect to x and setting $x = 0$.

THEOREM 2.1. *For arbitrary real numbers* $\lambda_1, \lambda_2, \dots, \lambda_m, \mu_1, \mu_2, \dots, \mu_n$ *and nonnegative* $s_1, s_2, \dots, s_m, t_1, t_2, \dots, t_n$,

$$\begin{aligned}\mathbb{E}\big[\exp\big\{i\big(\sum_{j=1}^{m} \lambda_j P(s_j) + \sum_{k=1}^{n} \mu_k Q(t_k)\big)\big\}\big] \\ = \mathbb{E}\big[\exp\big\{i\sum_{j=1}^{m} \lambda_j P(s_j)\big\}\big]\mathbb{E}\big[\exp\big\{i\sum_{k=1}^{n} \mu_k Q(t_k)\big\}\big].\end{aligned} \tag{2.10}$$

P r o o f. We identify $\exp\big\{i\big(\sum_{j=1}^{m} \lambda_j P(s_j) + \sum_{k=1}^{n} \mu_k Q(t_k)\big)\big\}$ as the Weyl operator $W\big(\sum_{j=1}^{m} \lambda_j \chi_{[0,s_j]} + i\sum_{k=1}^{n} \mu_k \chi_{[0,t_k]}\big)$ (see [5] and [6]). Using the Weyl operator vacuum expectation value

$$\mathbb{E}\,[W(f)] = \exp\left(-\frac{\sigma^2}{2}\,\|f\|^2\right),$$

we have

$$\begin{aligned}&\mathbb{E}\big[\exp\big\{i\big(\sum_{j=1}^{m} \lambda_j P(s_j) + \sum_{k=1}^{n} \mu_k Q(t_k)\big)\big\}\big] \\ &= \mathbb{E}\big[W\big(\sum_{j=1}^{m} \lambda_j \chi_{[0,s_j]} + i\sum_{k=1}^{n} \mu_k \chi_{[0,t_k]}\big)\big] \\ &= \exp\left(-\frac{\sigma^2}{2}\big\|\sum_{j=1}^{m} \lambda_j \chi_{[0,s_j]} + i\sum_{k=1}^{n} \mu_k \chi_{[0,t_k]}\big\|^2\right) \\ &= \exp\left(-\frac{\sigma^2}{2}\int_0^\infty \big|\sum_{j=1}^{m} \lambda_j \chi_{[0,s_j]}(x) + i\sum_{k=1}^{n} \mu_k \chi_{[0,t_k]}(x)\big|^2\,dx\right) \\ &= \exp\left(-\frac{\sigma^2}{2}\int_0^\infty \big|\sum_{j=1}^{m} \lambda_j \chi_{[0,s_j]}(x)\big|^2\,dx\right)\exp\left(-\frac{\sigma^2}{2}\int_0^\infty \big|\sum_{k=1}^{n} \mu_k \chi_{[0,t_k]}(x)\big|^2\,dx\right) \\ &= \mathbb{E}\big[W\big(-i\sum_{j=1}^{m} \lambda_j \chi_{[0,s_j]}\big)\big]\mathbb{E}\big[W\big(\sum_{k=1}^{n} \mu_k \chi_{[0,t_k]}\big)\big] \\ &= \mathbb{E}\big[\exp\big\{i\sum_{j=1}^{m} \lambda_j P(s_j)\big\}\big]\mathbb{E}\big[\exp\big\{i\sum_{k=1}^{n} \mu_k Q(t_k)\big\}\big]. \quad \blacksquare\end{aligned}$$

3. QUANTUM LÉVY AREA

We define the quantum Lévy area process, in either Fock or non-Fock quantum stochastic calculus, to be the linear combination of iterated quantum stochastic integrals [6]

$$L(t) = \frac{1}{2} \int\limits_{0\leqslant x<y<t} \big(dP(x)dQ(y) - dQ(x)dP(y)\big).$$

Thus

$$L(t) = J_0^t\left(\frac{1}{2}\{dP\otimes dQ\} - \frac{1}{2}\{dQ\otimes dP\}\right),$$

where we extend by linearity the notation for iterated stochastic integrals

$$\int\limits_{a\leqslant x_1<x_2<\ldots<x_m<b} d\Lambda_1(x_1)d\Lambda_2(x_2)\ldots d\Lambda_m(x_m) = J_a^b\left(\{d\Lambda_1\otimes d\Lambda_2\otimes\ldots\otimes d\Lambda_m\}\right), \tag{3.1}$$

$d\Lambda_1, d\Lambda_2, \ldots, d\Lambda_m \in \{dP, dQ, dT\}$ being differentials either of the processes P or Q, or of the time process $T(t) = tI$.

THEOREM 3.1. $\mathbb{E}\big[\int_{a\leqslant x_1<x_2<\ldots<x_m<b} d\Lambda_1(x_1)d\Lambda_2(x_2)\ldots d\Lambda_m(x_m)\big] = 0$ *unless* $d\Lambda_1 = d\Lambda_2 = \ldots = d\Lambda_m = dT$.

Proof. By the martingale property of stochastic integrals against either dP or dQ we have

$$\mathbb{E}\Big[\int\limits_{a\leqslant x_1<x_2<\ldots<x_m<b} d\Lambda_1(x_1)d\Lambda_2(x_2)\ldots d\Lambda_m(x_m)\Big] = 0$$

unless $d\Lambda_m = dT$. But then, by the same argument,

$$\begin{aligned}\mathbb{E}\Big[&\int\limits_{a\leqslant x_1<x_2<\ldots<x_m<b} d\Lambda_1(x_1)d\Lambda_2(x_2)\ldots d\Lambda_{m-1}(x_{m-1})dT(x_m)\Big]\\ &= \int\limits_a^b \Big\{\mathbb{E}\Big[\int\limits_{a\leqslant x_1<x_2<\ldots<x_m} d\Lambda_1(x_1)d\Lambda_2(x_2)\ldots d\Lambda_{m-1}(x_{m-1})\Big]\Big\}dx_m\\ &= 0\end{aligned}$$

unless $d\Lambda_{m-1} = dT$, and by repetition we find successively that the original expectation vanishes unless $d\Lambda_m = d\Lambda_{m-1} = d\Lambda_{m-2} = \ldots = d\Lambda_1 = dT$. ■

To find the distribution of the quantum random variable $L(t)$ we investigate its moments by combining Theorem 3.1 with the fact that products of arbitrary iterated integrals of form (3.1) are themselves linear combinations of such integrals.

To see this we introduce the algebra of Itô differentials $\mathcal{I}_\sigma = \mathbb{C}\langle dP, dQ, dT\rangle$, equipped with the quantum Itô multiplication rule

(3.2)

	dP	dQ	dT
dP	$\sigma^2 dT$	$-idT$	0
dQ	idT	$\sigma^2 dT$	0
dT	0	0	0

We equip the complex vector space

$$\mathcal{T}(\mathcal{I}_\sigma) = \mathbb{C} \oplus \mathcal{I}_\sigma \oplus (\mathcal{I}_\sigma \otimes \mathcal{I}_\sigma) \oplus (\mathcal{I}_\sigma \otimes \mathcal{I}_\sigma \otimes \mathcal{I}_\sigma) \oplus \dots$$

of arbitrary tensors over $\mathcal{I}_\sigma$ with the *sticky shuffle product* $\alpha\beta = \gamma$ (see [2] and [3]) where the components of rank $0, 1, 2, 3, \dots$ of the tensor

$$\gamma = \gamma_0 \oplus \gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \dots$$

are defined in terms of those of α and β by

$$\gamma_N = \sum_{A \cup B = \{1,2,\dots,N\}} \alpha^A_{|A|} \beta^B_{|B|}.$$

Here the sum is over all ordered pairs (A, B) of not necessarily disjoint subsets whose union is $\{1, 2, \dots, N\}$. The symbol $\alpha^A_{|A|}$ indicates that the component $\alpha_{|A|}$ of rank $|A|$ of the tensor α is to occupy the tensor product of those $|A|$ copies of $\mathcal{I}_\sigma$ within $\otimes^N \mathcal{I}_\sigma$ labelled by the $|A|$ integers in A, so that, with $\beta^B_{|B|}$ defined in the same way, all N integers in $\{1, 2, \dots, N\}$ are occupied, and when $A \cap B$ is nonempty, double occupancies are reduced to single by using the multiplication rule (3.2) in $\mathcal{I}_\sigma$. Then we have

$$J^b_a(\alpha) J^b_a(\beta) = J^b_a(\alpha\beta),$$

in the weak sense that, in the Fock case,

$$\langle J^b_a(\alpha^\dagger)e(f), J^b_a(\beta)e(g)\rangle = \langle e(f), J^b_a(\alpha\beta)e(g)\rangle, \tag{3.3}$$

where $\alpha^\dagger$ is the natural adjoint of α in $\mathcal{T}(\mathcal{I}_\sigma)$. In the non-Fock case we may replace the exponential vectors in (3.3) by the action of the Weyl operators $W_\sigma(f)$, $W_\sigma(g)$ on the vector Ω_σ.

By iteration of the sticky shuffle product each positive power $\big(L(t)\big)^n$ of the quantum Lévy area can be expressed as a linear combination of iterated integrals of form (3.2) where the interval $[a, b[= [0, t[$. In view of Theorem 3.1, only the "purely sticky" terms, in which each occurrence of either dP or dQ is multiplied by another term dP or dQ in accordance with the quantum Itô table (3.2) to give dT, can contribute to the nth moment $\mu_n = \mathbb{E}\big[\big(L(t)\big)^n\big]$. In particular, μ_n must vanish when n is odd.

4. TRIVIALITY OF QUANTUM LÉVY AREA IN THE FOCK CASE

THEOREM 4.1. *In the Fock case $\sigma^2 = 1$, all positive moments μ_n vanish.*

Proof. We introduce the creation and annihilation processes

$$dA^\dagger = \frac{1}{2}(Q - iP), \quad dA = \frac{1}{2}(Q + iP).$$

Using the basis $\left(dA^\dagger, dA, dT\right)$ of the Itô algebra, in the Fock case the quantum Itô table (3.2) becomes

$$
\begin{array}{c|ccc}
 & dA^\dagger & dA & dT \\
\hline
dA^\dagger & 0 & 0 & 0 \\
dA & dT & 0 & 0 \\
dT & 0 & 0 & 0
\end{array}
\tag{4.1}
$$

while the quantum Lévy area is given by

$$L(t) = J_0^t\big(i(\{dA^\dagger \otimes dA\} - \{dA \otimes dA^\dagger\})\big).$$

Since $\mathbb{C}\langle dA^\dagger, dA, dT\rangle = \mathbb{C}\langle dP, dQ, dT\rangle$, Theorem 3.1 implies that expectations of the form

$$\mathbb{E}[J_0^t(\{d\Lambda_1 \otimes d\Lambda_2 \otimes \ldots \otimes d\Lambda_m\})],$$

where each $d\Lambda_j \in \{dA^\dagger, dA, dT\}$ vanishes, unless each $d\Lambda_j = dT$. In forming contributions of iterated sticky shuffle powers of $(\{dA^\dagger \otimes dA\} - \{dA \otimes dA^\dagger\})$ involving only dT we must "stick", by multiplication using the table (4.1), both components of both the linearly independent terms $dA^\dagger \otimes dA$ and $dA \otimes dA^\dagger$occurring in the first copy to either $dA^\dagger$ or dA occurring in the sticky shuffle product of the remaining copies. But since both $dA^\dagger dA$ and $dA^\dagger dA^\dagger$ are equal to zero, the result of sticking either term $dA^\dagger \otimes dA$ or $dA \otimes dA^\dagger$ in this way will necessarily be zero. ■

Thus, in the Fock case, $L(t)$ has the atomic distribution concentrated at zero. But it has a nontrivial distribution in the non-Fock case $\sigma^2 > 1$. In particular, since the rescaled standard Brownian motions $\sigma^{-1}P_\sigma$ and $\sigma^{-1}Q_\sigma$ satisfy

$$[\sigma^{-1}P_\sigma(s), \sigma^{-1}Q_\sigma(t)] = -2i\sigma^{-2}\,(t \wedge u)\,I,$$

they become commutative in the limit as $\sigma^2 \to \infty$, so, when rescaled in the same way, this distribution interpolates between the trivial distribution and that of the classical Lévy area formula with characteristic function (1.1) as σ^2 increases from one to infinity.

5. TIME REVERSAL

The sticky shuffle product algebra $\mathcal{T}\,(\mathcal{I}_\sigma)$ becomes a Hopf algebra [3] when equipped with the *coproduct* $\Delta : \mathcal{T}\,(\mathcal{I}_\sigma) \to \mathcal{T}\,(\mathcal{I}_\sigma) \otimes \mathcal{T}\,(\mathcal{I}_\sigma)$ for which

$$\big(\Delta(\alpha)\big)_{m,n} = \alpha_{m+n}.$$

The coproduct corresponds to the independent increments property of Brownian motion in that, for arbitrary $a \leqslant b \leqslant c$ and $\alpha \in \mathcal{T}(\mathcal{I}_\sigma)$,

$$J_a^c(\alpha) = J_a^b(\alpha) \otimes J_b^c(\alpha). \tag{5.1}$$

Here the operator on the right-hand side is the tensor product of operators on the past and future Fock spaces (or tensor products of Fock spaces with their duals in the case $\sigma^2 > 1$) resulting from splitting at time b.

As a Hopf algebra $\mathcal{T}(\mathcal{I}_\sigma)$ also has an *antipode* $S : \mathcal{T}(\mathcal{I}_\sigma) \to \mathcal{T}(\mathcal{I}_\sigma)$ whose action on first and second rank homogeneous elements of $\mathcal{T}(\mathcal{I}_\sigma)$ is given by

$$S\{d\Lambda\} = -\{d\Lambda\},\ S\{d\Lambda_1 \otimes d\Lambda_2\} = \{d\Lambda_2 \otimes d\Lambda_1\} + \{d\Lambda_1 d\Lambda_2\} \tag{5.2}$$

(see [3] for the action on a general element which will not be needed here). The antipode can be interpreted in terms of time reversal as follows. Let us agree to define the map J_a^b when $a > b$ by

$$J_a^b = J_b^a \circ S.$$

Then the relation (5.1) continues to hold even when the condition $a \leqslant b \leqslant c$ is violated, modulo ampliations.

In view of (5.2), in both the Fock and non-Fock cases,

$$S\{dP \otimes dQ\} = \{dQ \otimes dP\} - i\{dT\}, \quad S\{dQ \otimes dP\} = \{dP \otimes dQ\} + i\{dT\},$$

so that

$$S(\{dP \otimes dQ\} - \{dQ \otimes dP\}) = -(\{dP \otimes dQ\} - \{dQ \otimes dP\}) - 2i\{dT\}$$

and

$$S^2(\{dP \otimes dQ\} - \{dQ \otimes dP\}) = \{dP \otimes dQ\} - \{dQ \otimes dP\} + 4i\{dT\}.$$

Consequently, quantum Lévy area is not only not invariant under this notion of time reversal; it is not even invariant under double time reversal! Thus, it appears to contain within itself an intrinsic arrow of time.

This is an essentially quantum phenomenon. It is well known that the antipode S of a Hopf algebra must satisfy

$$S^2 = \mathrm{id} \tag{5.3}$$

if it is either commutative or cocommutative. Thus the failure of the condition (5.3) cannot be seen in the commutative Hopf algebra corresponding to classical Itô calculus, in particular on classical Lévy area. Nor can it be seen in the quantum case on the cocommutative sub-Hopf algebra of symmetric tensors of the Hopf algebra $\mathcal{T}(\mathcal{I}_\sigma)$ whose elements generate polynomials in the increments over the interval $[a, b[$ of the basic processes P, Q and T under action of the iterated integral map J_a^b. An interpretation of this intriguing phenomenon may perhaps be found in the so-called "eventum mechanics" of V. P. Belavkin which interprets quantum measurement as a continuous transfer of information from a random future to a determined past.

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