PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 35, Fasc. 1 (2015), pp. 129–142

# ALMOST SURE LIMIT POINTS OF LIGHTLY TRIMMED SUMS WHEN THE DISTRIBUTION FUNCTION BELONGS TO THE DOMAIN OF PARTIAL ATTRACTION OF A POSITIVE SEMI-STABLE LAW

BY

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*Abstract.* In this paper, we obtain the set of all almost sure limit points of lightly trimmed sums, properly normalized, when the underlying distribution function belongs to the domain of partial attraction of a semi-stable law supported on the positive half axis.

**2000 AMS Mathematics Subject Classification:** Primary: 60F15; Secondary: 60F10.

**Key words and phrases:** Lightly trimmed sums, law of the iterated logarithm, semi-stable law, almost sure limit points.

## 1. INTRODUCTION

Let $(X_n), n \geqslant 1$, be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.'s) defined over a common probability space $(\Omega, \mathcal{F}, P)$. Suppose that the distribution function (d.f.) $F$ of $X_1$ is continuous. For each $n \geqslant 1$, set $S_n = \sum_{j=1}^{n} X_j$. For any integer $r$ with $1 \leqslant r \leqslant n$, let $X_n^{(r)} = X_m$ if $|X_m|$ is the $r$-th largest among $|X_1|, |X_2|, \ldots, |X_n|$. Note that if $Z_{1,n} \leqslant Z_{2,n} \leqslant \ldots \leqslant Z_{n,n}$ are the order statistics of $|X_1|, |X_2|, \ldots, |X_n|$, then $X_n^{(r)} = Z_{n-r+1,n}, n \geqslant r$. Let ${}^{(r)}S_n = S_n - (X_n^{(1)} + X_n^{(2)} + \ldots + X_n^{(r)})$. Then ${}^{(r)}S_n$ is called a *lightly trimmed sum*. The fact that d.f. $F$ is continuous implies that the d.f. of $|X_n|$ is also continuous. Consequently, $X_n^{(j)}, 1 \leqslant j \leqslant r$, are uniquely determined except over a set of probability zero.

We now introduce some definitions and give some earlier developments in this area.

A d.f. $F$ is said to be *semi-stable* if it is infinitely divisible with the characteristic function $\phi(\cdot)$ given by

$$\log \phi(t) = i\gamma t + \int_{-\infty}^{\infty} \left( e^{itx} - 1 - \frac{itx}{1+x^2} \right) dH(x),$$

where the Lévy spectral function is of the form

$$H(x)=\begin{cases}\dfrac{-\theta_1(\log x)}{x^{\alpha}} & \text{if } x>0,\\[2mm] \dfrac{\theta_2(\log|x|)}{|x|^{\alpha}} & \text{if } x<0,\end{cases}$$

with $\alpha\in(0,2)$, and $\theta_1(\cdot),\theta_2(\cdot)$ oscillating bounded non-negative valued functions with at least one of them positive valued.

A d.f. $F$ is said to *belong to the domain of partial attraction* of an infinitely divisible d.f. $G^*$ if over a subsequence $(n_k)$ one can find sequences $(A_{n_k})$ and $(B_{n_k})$ of constants such that $\big((S_{n_k}-A_{n_k})/B_{n_k}\big)$ converges to an r.v. with d.f. $G^*$. Kruglov [6] established that if $(n_k)$ satisfies the condition

$$\lim_{k\to\infty}\frac{n_{k+1}}{n_k}=\rho,\qquad 1\leqslant\rho<\infty,$$

then the class of all limit distributions coincides with the class of all semi-stable laws (which includes stable laws). In this paper, the domain of partial attraction of semi-stable laws is defined as follows (based on Kruglov [6]).

A d.f. $F$ is said to *belong to the domain of partial attraction* (DPA) of a semi-stable law $G_\alpha^*$ if there exists a sequence $(n_k)$ satisfying

(1) $n_k<n_{k+1}, k\geqslant 1$,

(2) $n_{k+1}/n_k\to\rho$ as $k\to\infty$, $1\leqslant\rho<\infty$,

and there exist sequences $(A_{n_k})$ and $(B_{n_k})$ of constants such that

$$\frac{S_{n_k}-A_{n_k}}{B_{n_k}}\xrightarrow{d}Y_\alpha^*, \tag{1.1}$$

where $Y_\alpha^*\sim G_\alpha^*$.

In particular, if

$$\lim_{k\to\infty}\frac{n_{k+1}}{n_k}=1,$$

Kruglov [6] established the following:

(i) The limit law of $(S_{n_k})$ is a non-normal stable law.

(ii) The sequence $(S_n)$, properly normalized, will itself converge to the same stable law. In other words, the d.f. $F$ belongs to the domain of attraction of a stable law.

For other references on DPA to semi-stable laws, see Pillai [9] and Shimizu [11].

When the d.f. $F$ is a symmetric stable law with index $\alpha, 0<\alpha<2$, Chover [1] established that

$$\limsup_{n\to\infty}\left|\frac{S_n}{n^{1/\alpha}}\right|^{1/\log\log n}=e^{1/\alpha}\text{ a.s.}, \tag{1.2}$$

where a.s. means “almost surely” or “almost sure” depending on the context.

Vasudeva [12] showed that weak convergence to stable law is sufficient for the law of the iterated logarithm (LIL) to hold. He showed that whenever $(S_n/B_n)$ converges in law to a stable r.v., it follows that

$$\limsup_{n\to\infty} \left|\frac{S_n}{B_n}\right|^{1/\log\log n} = e^{1/\alpha} \text{ a.s.} \quad \text{if } \alpha \in (0,2) \tag{1.3}$$

and

$$\limsup_{n\to\infty} \left|\frac{S_n}{B_n}\right|^{1/\log\log n} = \beta \in [1, e^{1/2}] \text{ a.s.} \quad \text{if } \alpha = 2. \tag{1.4}$$

where $(B_n)$ is a solution of the equation $n\big(1 - F(B_n) + F(-B_n)\big) \simeq 1$.

Lu and Qi [7] obtained the LIL for $({}^{(r)}S_n)$ when the underlying d.f. is in the domain of attraction of a stable distribution with index $\alpha \in (0, 2]$. They established that for $\{A_n\}$ and $\{B_n\}$ with $B_n \to \infty$ as $n \to \infty$, if $\{(S_n - A_n)/B_n\}$ converges to a stable law with index $\alpha \in (0, 2]$, then

$$\limsup_{n\to\infty} \left(\frac{|{}^{(r)}S_n - A_n|}{B_n}\right)^{1/\log\log n} = e^{1/(\alpha(r+1))} \text{ a.s.} \tag{1.5}$$

when $0 < \alpha < 2$, and that there exists a $\beta \in [1, e^{1/(2(r+1))}]$ such that

$$\limsup_{n\to\infty} \left(\frac{|{}^{(r)}S_n - A_n|}{B_n}\right)^{1/\log\log n} = \beta \text{ a.s.} \tag{1.6}$$

when $\alpha = 2$.

When the d.f. $F$ belongs to the domain of partial attraction of a semi-stable law with index $\alpha, 0 < \alpha < 2$, Divanji and Vasudeva [2] established that

$$\limsup_{n\to\infty} \left|\frac{S_n}{B_n}\right|^{1/\log\log n} = e^{1/\alpha} \text{ a.s.}, \tag{1.7}$$

where $(B_n)$ satisfies the relation $n\big(1 - F(B_n) + F(-B_n)\big) \simeq 1$.

Kesten and Maller [5] obtained LIL for $({}^{(r)}S_n)$ when the underlying d.f. $F$ belongs to the domain of partial attraction of a normal distribution. They showed that there exist sequences $(A_n^*)$ and $(B_n^*) \uparrow \infty$ such that

$$-1 = \liminf_{n\to\infty} \frac{{}^{(r)}S_n - A_n^*}{B_n^*} < \limsup_{n\to\infty} \frac{{}^{(r)}S_n - A_n^*}{B_n^*} = 1 \text{ a.s.}, \tag{1.8}$$

where $A_n^*$ is a sequence chosen such that

$$\liminf_{n\to\infty} P(S_n \leqslant A_n^*) \wedge P(S_n > A_n^*) = 0. \tag{1.9}$$

Vasudeva and Srilakshminarayana [13] extended the results of Lu and Qi [7] to the class of all d.f.'s $F$ belonging to the domain of partial attraction of a semi-stable law with index $\alpha, 0 < \alpha < 2, \alpha \neq 1$. They established that

$$\limsup_{n\to\infty} \left| \frac{{}^{(r)}S_n - A_n}{B_n} \right|^{1/\log\log n} = e^{1/((r+1)\alpha)} \text{ a.s.},$$

where $B_n$ is a solution of the equation $n\big(1 - F(B_n) + F(-B_n)\big) \simeq 1$ and

$$A_n = \begin{cases} 0 & \text{if } 0 < \alpha < 1, \\ E(X_1) & \text{if } 1 < \alpha < 2. \end{cases}$$

When the d.f. has support on $[0, \infty]$ and belongs to the domain of attraction of a positive stable law with index $\alpha \in (0, 1)$, Vasudeva and Srilakshminarayana [14] obtained the large deviation results for $({}^{(r)}S_n)$, the lightly trimmed sums. They established that, for any sequence $(x_n)$ of positive constants diverging to infinity, for any $\delta > 0$ and for any integer $r, r \geqslant 1$,

(1.10)
$$\lim_{n\to\infty} x_n^{(r+\delta)\alpha} P(X_n^{(r)} \geqslant x_n B_n) = \infty, \ \lim_{n\to\infty} x_n^{(r-\delta)\alpha} P(X_n^{(r)} \geqslant x_n B_n) = 0,$$

(1.11)
$$\lim_{n\to\infty} x_n^{(r+1-\delta)\alpha} P({}^{(r)}S_n > x_n B_n) = 0, \lim_{n\to\infty} x_n^{(r+1+\delta)\alpha} P({}^{(r)}S_n > x_n B_n) = \infty.$$

In this paper, we obtain similar large deviation results when $F \in \mathrm{DPA}(\alpha)$, $0 < \alpha < 1$, and hence we will show that the set of a.s. limit points of the sequence $({}^{(r)}S_n/B_n)^{1/\log\log n}$ coincides with the interval $[1, e^{1/(\alpha(r+1))}]$ (when the d.f. $F$ has support on $(0, \infty)$). Even though many papers on LIL for trimmed sums can be found in literature, perhaps this is the first attempt to examine the a.s. limit points. It is made possible by the large deviation probability results established in this paper.

Throughout the paper, $c$, $k$ (integer) and $N$ (integer), with an index, stand for generic constants. The term 'infinitely often' will be denoted by i.o. For any $x$ positive, $[x]$ stands for the greatest integer less than or equal to $x$. A non-negative valued measurable function $L(x), x > 0$, is called a *slowly varying* (s.v.) function if for any $t > 0, L(tx)/L(x) \to 1$ as $x \to \infty$.

A point $\theta$ is said to be an *almost sure* (*a.s.*) *limit point* of a sequence $(\xi_n)$ of r.v.'s defined over a common probability space if for any given $\epsilon > 0$

$$P\left(\theta - \epsilon < \xi_n < \theta + \epsilon \text{ i.o.}\right) = 1.$$

The large deviation probability results and the a.s. limit points are obtained in the next section.

## 2. ALMOST SURE LIMIT POINTS FOR TRIMMED SUMS

In this section, we show that any point $\beta \in [1, e^{1/(\alpha(r+1))}]$ is an a.s. limit point of $({}^{(r)}S_n)$ properly normalized. We first present some lemmas and a theorem needed in establishing the main result. Lemma 2.4 and Theorem 2.1 are the extensions of large deviation results of Vasudeva and Srilakshminarayana [14] to the case when $F \in \mathrm{DPA}(\alpha), 0 < \alpha < 1$. Throughout this section, $B_n$ is a solution of the equation $n\big(1 - F(x)\big) = 1$.

LEMMA 2.1. *Let $L(\cdot)$ be a measurable s.v. function and let $(x_n)$ and $(y_n)$ be sequences of real constants, both diverging to infinity. Then for any $\delta > 0$*

$$\lim_{n\to\infty} y_n^{-\delta} \frac{L(x_n y_n)}{L(x_n)} = 0 \quad \textit{and} \quad \lim_{n\to\infty} y_n^{\delta} \frac{L(x_n y_n)}{L(x_n)} = \infty. \tag{2.1}$$

P r o o f. For a measurable s.v. function $L(\cdot)$, the Karamata representation is given by

$$L(x) = a(x) \exp\bigg(\int_c^x \frac{\epsilon(y)}{y} dy\bigg),$$

where $a(x) \to a$ ($a \in (0, \infty)$), $\epsilon(x) \to 0$ as $x \to \infty$, and $c$ is a positive constant. Consequently,

$$\frac{L(x_n y_n)}{L(x_n)} = \frac{a(x_n y_n)}{a(x_n)} \exp\bigg(\int_{x_n}^{x_n y_n} \frac{\epsilon(y)}{y} dy\bigg). \tag{2.2}$$

From the fact that $\epsilon(y) \to 0$ as $y \to \infty$ it follows that for any given $\delta_1 > 0$ one can find a $y_0 > 0$ such that $-\delta_1 < \epsilon(y) < \delta_1$ for all $y > y_0$. Consequently, for $n_0$ large such that $x_{n_0} > y_0$, $-\delta_1 < \epsilon(y) < \delta_1$ whenever $y > x_n$ with $n > n_0$. Observing that $a(x_n y_n)/a(x_n) \to 1$ as $n \to \infty$, given $\delta_2 > 0$, one can find an $n_1$ ($> n_0$) such that for all $n \geqslant n_1$

$$(1 - \delta_2) \exp\bigg(-\delta_1 \log \frac{x_n y_n}{x_n}\bigg) < \frac{L(x_n y_n)}{L(x_n)} < (1 + \delta_2) \exp\bigg(\delta_1 \log \frac{x_n y_n}{x_n}\bigg)$$

or

$$(1 - \delta_2) y_n^{-\delta_1} < \frac{L(x_n y_n)}{L(x_n)} < (1 + \delta_2) y_n^{\delta_1}. \tag{2.3}$$

From (2.3), for any $\delta > \delta_1$ one can see that

$$y_n^{\delta} \frac{L(x_n y_n)}{L(x_n)} \to \infty \quad \text{and} \quad y_n^{-\delta} \frac{L(x_n y_n)}{L(x_n)} \to 0 \quad \text{as } n \to \infty,$$

which establishes the lemma. ■

LEMMA 2.2. *Let $L(\cdot)$ be an s.v. function over $[x_0, \infty)$. Then, as $x \to \infty$,*

$$\int_{x_0}^{x} t^{\gamma} L(t)dt \sim \frac{x^{1+\gamma}}{1+\gamma} L(x)$$

*whenever $\gamma > -1$.*

P r o o f. See Feller [3]. ∎

LEMMA 2.3. *Let a d.f. $F$ be supported over $(0, \infty)$ and let $F \in \mathrm{DPA}(\alpha)$, $0 < \alpha < 1$. Then there exist an s.v. function $L(\cdot)$ and a bounded function $\theta(\cdot)$ with $0 < c_1 \leqslant \theta(x) \leqslant c_2 < \infty$, for some $c_1, c_2 > 0$, such that*

$$\lim_{n\to\infty} \frac{x^{\alpha}\big(1 - F(x)\big)}{L(x)\theta(x)} = 1.$$

P r o o f. See Divanji and Vasudeva [2]. ∎

LEMMA 2.4. *Let $F(0) = 0$ and let $F \in \mathrm{DPA}(\alpha), 0 < \alpha < 1$. Then, for any sequence $(x_n)$ of positive constants diverging to infinity, for any $\delta > 0$, and for any integer $r, r \geqslant 1$,*

$$\lim_{n\to\infty} x_n^{(r+\delta)\alpha} P(X_n^{(r)} \geqslant x_n B_n) = \infty \text{ and } \lim_{n\to\infty} x_n^{(r-\delta)\alpha} P(X_n^{(r)} \geqslant x_n B_n) = 0.$$

P r o o f. The proof follows, with minor modification in the arguments, in the way of Vasudeva and Srilakshminarayana [14]. One needs to apply Lemma 2.3 in place of the regularly varying tail of $F$. The details are omitted. ∎

THEOREM 2.1 (Large deviation results for lightly trimmed sums when $F \in \mathrm{DPA}(\alpha), 0 < \alpha < 1$). *Let $F(0) = 0$ and let $F \in \mathrm{DPA}(\alpha), 0 < \alpha < 1$. Then, for any sequence $(x_n)$ of positive constants with $x_n \to \infty$ as $n \to \infty$, and for any $\delta > 0$,*

$$\lim_{n\to\infty} x_n^{(r+1+\delta)\alpha} P({}^{(r)}S_n > x_n B_n) = \infty \tag{2.4}$$

*and*

$$\lim_{n\to\infty} x_n^{(r+1-\delta)\alpha} P({}^{(r)}S_n > x_n B_n) = 0. \tag{2.5}$$

P r o o f. The proof follows the lines of Vasudeva and Srilakshminarayana [14]. The details are omitted. ∎

THEOREM 2.2. *Let $F(0) = 0$ and let $F \in \mathrm{DPA}(\alpha), 0 < \alpha < 1$. Then the set of all a.s. limit points of $\big(({}^{(r)}S_n/B_n)^{1/\log\log n}\big)$ coincides with $[1, e^{1/((r+1)\alpha)}]$.*

P r o o f. The proof consists of two parts. In the first part we show that the set of a.s. limit points $S$ is contained in $[1, e^{1/((r+1)\alpha)}]$ and in the second, each point $\gamma \in [1, e^{1/((r+1)\alpha)}]$ is shown to be an a.s. limit point. In showing that $S \subseteq [1, e^{1/((r+1)\alpha)}]$, we establish that, for any given $\epsilon > 0$,

$$P\biggl(\Bigl(\frac{{}^{(r)}S_n}{B_n}\Bigr)^{1/\log\log n} < e^{-\epsilon} \text{ i.o.}\biggr) = 0 \tag{2.6}$$

and

$$P\biggl(\Bigl(\frac{{}^{(r)}S_n}{B_n}\Bigr)^{1/\log\log n} > e^{(1+\epsilon)/((r+1)\alpha)} \text{ i.o.}\biggr) = 0. \tag{2.7}$$

We have

$$\biggl(\Bigl(\frac{{}^{(r)}S_n}{B_n}\Bigr)^{1/\log\log n} < e^{-\epsilon}\biggr) = \biggl({}^{(r)}S_n < \frac{B_n}{(\log n)^{\epsilon}}\biggr).$$

By the relation $X_n^{(r+1)} < {}^{(r)}S_n$, (2.6) follows once we show that

$$P\biggl(X_n^{(r+1)} < \frac{B_n}{(\log n)^{\epsilon}} \text{ i.o.}\biggr) = 0. \tag{2.8}$$

Let $m_k = 2^k, k \geqslant 1$. Define

$$A_n = \biggl(X_n^{(r+1)} < \frac{B_n}{(\log n)^{\epsilon}}\biggr) \quad \text{and} \quad B_k' = \biggl(\min_{m_k \leqslant n < m_{k+1}} X_n^{(r+1)} < \frac{B_{m_{k+1}}}{(\log m_{k+1})^{\epsilon}}\biggr).$$

Observe that $(A_n \text{ i.o.}) \subseteq (B_k' \text{ i.o.})$ and that

$$\begin{aligned} P(B_k') &= P\left(X_{m_k}^{(r+1)} < \frac{B_{m_{k+1}}}{(\log m_{k+1})^{\epsilon}}\right) \\ &= P\left(X_{m_k}^{(r+1)} < \frac{B_{m_{k+1}}}{B_{m_k}} \frac{B_{m_k}}{(\log m_k)^{\epsilon}} \frac{(\log m_k)^{\epsilon}}{(\log m_{k+1})^{\epsilon}}\right). \end{aligned}$$

By Lemma 2.3 note that $B_n$ is a solution of the equation $x^{\alpha}/\bigl(L(x)\theta(x)\bigr) = n$. Hence, by Divanji and Vasudeva [2], one can see that

$$B_n \sim n^{1/\alpha} L_1(n) \eta_1(n),$$

where $L_1(\cdot)$ is an s.v. function and $\eta_1(\cdot)$ is a function bounded between two positive constants, say, $0 < a < \eta_1(x) < b < \infty$ for $x > 0$. Consequently,

$$\frac{B_{m_{k+1}}}{B_{m_k}} \sim \frac{m_{k+1}^{1/\alpha} L_1(m_{k+1}) \eta_1(m_{k+1})}{m_k^{1/\alpha} L_1(m_k) \eta_1(m_k)} \leqslant 2^{1/\alpha} \frac{L_1(2 \cdot 2^k) b}{L_1(2^k) a}.$$

By the definition of an s.v. function, we have

$$\frac{L_1(2\cdot 2^k)}{L_1(2^k)} \to 1$$

as $k \to \infty$. Hence, one can find a $k_1$ and $c_1 > 0$ such that, for all $k \geqslant k_1$,

$$\frac{B_{m_{k+1}}}{B_{m_k}} \leqslant c_1.$$

Also, since

$$\left(\frac{\log m_k}{\log m_{k+1}}\right)^{\epsilon} = \left(\frac{k}{k+1}\right)^{\epsilon} < 1,$$

for all $k \geqslant k_1$ we have

$$\frac{B_{m_{k+1}}}{B_{m_k}}\left(\frac{\log m_k}{\log m_{k+1}}\right)^{\epsilon} \leqslant c_1.$$

Let $y_{m_k} = (c_1 B_{m_k})/(\log m_k)^{\epsilon}$. Then, for all $k \geqslant k_1$,

$$P(B_k') \leqslant P(X_{m_k}^{(r+1)} < y_{m_k}) = \sum_{l=0}^{r}\binom{m_k}{l}\left(\overline{F}(y_{m_k})\right)^l\left(1-\overline{F}(y_{m_k})\right)^{m_k-l},$$

where $\overline{F}(x) = 1 - F(x), x > 0$. Since $0 < F(\cdot) < 1$, for $j = 0, 1, 2, \dots, r$ it follows that

$$F^{m_k-j}(y_{m_k}) \leqslant F^{m_k-r}(y_{m_k}).$$

Using the fact that

$$1 - F(x) \simeq x^{-\alpha}L(x)\theta(x),$$

we obtain, for $k$ large,

$$\text{(2.9)}\quad \binom{m_k}{j}\left(\overline{F}(y_{m_k})\right)^j \sim \binom{m_k}{j}\left(y_{m_k}^{-\alpha}L\left(y_{m_k}\right)\theta\left(y_{m_k}\right)\right)^j$$

$$= \frac{m_k(m_k-1)\dots(m_k-(j-1))}{j!}\left(\frac{(\log m_k)^{\epsilon\alpha}L(y_{m_k})\theta(y_{m_k})}{c_1^{\alpha}B_{m_k}^{\alpha}}\right)^j$$

$$\leqslant c_2 m_k^j\left(\frac{(\log m_k)^{\epsilon\alpha}L(y_{m_k})}{B_{m_k}^{\alpha}}\theta(y_{m_k})\right)^j,$$

where $c_2 > 0$ is a constant. Also, for $k$ large, $\overline{F}(y_{m_k}) \sim y_{m_k}^{-\alpha}L(y_{m_k})\theta(y_{m_k})$ implies that $\overline{F}(y_{m_k}) \geqslant c y_{m_k}^{-\alpha}L(y_{m_k})\theta(y_{m_k})$ for some $c > 0$. As such (for $k$ large),

$$F(y_{m_k}) \leqslant 1 - \overline{F}(y_{m_k}) \leqslant \left(1 - c y_{m_k}^{-\alpha}L(y_{m_k})\theta(y_{m_k})\right)$$

$$= \left(1 - \frac{c(\log m_k)^{\epsilon\alpha}}{B_{m_k}^{\alpha}}L(y_{m_k})\theta(y_{m_k})\right).$$

Hence, one can find a $k_2$ such that, for all $k \geqslant k_2$,

$$P(X_{m_k}^{(r+1)} < y_{m_k}) \leqslant c_2 \left(1 - \frac{c(\log m_k)^{\epsilon\alpha}}{B_{m_k}^{\alpha}} L(y_{m_k})\theta(y_{m_k})\right)^{(m_k - r)} \times \sum_{j=0}^{r} \left(\frac{m_k(\log m_k)^{\epsilon\alpha}}{B_{m_k}^{\alpha}} L(y_{m_k})\theta(y_{m_k})\right)^{j}. \tag{2.10}$$

Note that

$$\begin{aligned} L(y_{m_k})\theta(y_{m_k}) &= \frac{L(y_{m_k})\theta(y_{m_k})}{L(B_{m_k})\theta(B_{m_k})} L(B_{m_k})\theta(B_{m_k}) \\ &= \frac{L(y_{m_k})\theta(y_{m_k})}{L\big([(\log m_k)^{\epsilon}/c_1] y_{m_k}\big)\theta(B_{m_k})} L(B_{m_k})\theta(B_{m_k}). \end{aligned} \tag{2.11}$$

By Lemma 2.1 for any given $\delta_1 < \alpha$ one can find a $k_3$ such that, for all $k \geqslant k_3$,

$$(\log m_k)^{-\epsilon\delta_1} \leqslant \frac{L\big([(\log m_k)^{\epsilon}/c_1] y_{m_k}\big)}{L(y_{m_k})} \leqslant (\log m_k)^{\epsilon\delta_1}.$$

Also, from the definition of $\theta(\cdot)$, we have

$$\frac{c_1}{c_2} \leqslant \frac{\theta(y_{m_k})}{\theta(B_{m_k})} \leqslant \frac{c_2}{c_1}.$$

Hence, for a $\delta \in (\delta_1, \alpha)$ one can find a $k_4$ ($> k_3$) such that, for all $k \geqslant k_4$,

$$(\log m_k)^{-\epsilon\delta} \leqslant \frac{L(y_{m_k})\theta(y_{m_k})}{L(B_{m_k})\theta(B_{m_k})} \leqslant (\log m_k)^{\epsilon\delta}.$$

In turn, for all $k \geqslant k_4$ from (2.11) we have

$$\begin{aligned} 1 - \frac{c(\log m_k)^{\epsilon\alpha}}{B_{m_k}^{\alpha}} L(y_{m_k})\theta(y_{m_k}) &\leqslant 1 - \frac{c(\log m_k)^{(\alpha-\delta)\epsilon}}{B_{m_k}^{\alpha}} L(B_{m_k})\theta(B_{m_k}) \\ &= 1 - \frac{c(\log m_k)^{(\alpha-\delta)\epsilon}}{m_k} \frac{m_k L(B_{m_k})\theta(B_{m_k})}{B_{m_k}^{\alpha}} \end{aligned}$$

and

$$\frac{m_k(\log m_k)^{\epsilon\alpha}}{B_{m_k}^{\alpha}} L(y_{m_k})\theta(y_{m_k}) \leqslant (\log m_k)^{(\alpha+\delta)\epsilon} \frac{m_k L(B_{m_k})\theta(B_{m_k})}{B_{m_k}^{\alpha}}.$$

Consequently, from (2.10) one can see that, for all $k \geqslant k_4$,

$$\begin{aligned} P(X_{m_k}^{(r+1)} < y_{m_k}) \leqslant c_3 &\left(1 - \frac{c(\log m_k)^{(\alpha-\delta)\epsilon}}{m_k} \frac{m_k L(B_{m_k})\theta(B_{m_k})}{B_{m_k}^{\alpha}}\right)^{(m_k - r)} \\ &\times \sum_{j=0}^{r} \left((\log m_k)^{(\alpha+\delta)\epsilon} \frac{m_k L(B_{m_k})\theta(B_{m_k})}{B_{m_k}^{\alpha}}\right)^{j}. \end{aligned}$$

Since

$$\frac{m_k L(B_{m_k})\theta(B_{m_k})}{B_{m_k}^{\alpha}} \to 1$$

as $k \to \infty$, one can find a $k_5$ ($\geqslant k_4$) and $c_4 > 0, c_5 > 0$ such that, for all $k \geqslant k_5$,

$$\left(1 - \frac{c(\log m_k)^{(\alpha-\delta)\epsilon}}{m_k} \frac{m_k L(B_{m_k})\theta(B_{m_k})}{B_{m_k}^{\alpha}}\right)^{(m_k - r)} \leqslant e^{-c_4(\log m_k)^{(\alpha-\delta)\epsilon}}$$

and

$$\sum_{j=0}^{r} \left((\log m_k)^{(\alpha+\delta)\epsilon} \frac{m_k L(B_{m_k})\theta(B_{m_k})}{B_{m_k}^{\alpha}}\right)^{j} \leqslant c_5(\log m_k)^{(\alpha+\delta)(r+1)\epsilon}.$$

In turn, by (2.10) one can show that, for all $k \geqslant k_5$,

$$P(X_{m_k}^{(r+1)} < y_{m_k}) \leqslant c_6 k^{(\alpha+\delta)(r+1)\epsilon} e^{-c_7 k^{(\alpha-\delta)\epsilon}},$$

where $c_6 > 0, c_7 > 0$ are constants. From the order test we get $\sum_k P\left(B_k'\right) < \infty$. By the Borel–Cantelli lemma, $P\left(B_k' \text{ i.o.}\right) = 0$ follows. Recalling that

$$(A_n \text{ i.o.}) \subseteq (B_k' \text{ i.o.}),$$

we obtain (2.8), which implies (2.6).

In Vasudeva and Srilakshminarayana [13], (2.7) has been established by using the truncation arguments given in Mori [8]. We give an elementary proof based on large deviation results in Theorem 2.1.

Note that

$$\left(\left(\frac{{}^{(r)}S_n}{B_n}\right)^{1/\log\log n} > e^{(1+\epsilon)/((r+1)\alpha)}\right) = \left({}^{(r)}S_n > B_n(\log n)^{(1+\epsilon)/((r+1)\alpha)}\right).$$

Define

$$D_n = \left({}^{(r)}S_n > B_n(\log n)^{(1+\epsilon)/((r+1)\alpha)}\right)$$

and

$$E_k = \Big(\sup_{m_k < n \leqslant m_{k+1}} {}^{(r)}S_n > B_{m_k}(\log m_k)^{(1+\epsilon)/((r+1)\alpha)}\Big).$$

Observe that $(D_n \text{ i.o.}) \subseteq (E_k \text{ i.o.})$. Also, since $X_n$'s are positive valued, $({}^{(r)}S_n)$ is an increasing sequence. Consequently,

$$P(E_k) = P\left({}^{(r)}S_{m_{k+1}} > B_{m_{k+1}} \frac{B_{m_k}}{B_{m_{k+1}}}(\log m_k)^{(1+\epsilon)/((r+1)\alpha)}\right).$$

Let

$$x_{m_k} = \frac{B_{m_k}}{B_{m_{k+1}}}(\log m_k)^{(1+\epsilon)/((r+1)\alpha)}.$$

One can find $k_6, c_8 > 0, c_9 > 0$ such that, for all $k \geqslant k_6$, $c_8 < B_{m_k}/B_{m_{k+1}} < c_9$. Hence, by Theorem 2.1, for all $k \geqslant k_6$,

$$P(E_k) \leqslant c_9(\log m_k)^{-[(r+1-\delta)\alpha(1+\epsilon)]/[(r+1)\alpha]} = c_{10}k^{-[(r+1-\delta)\alpha(1+\epsilon)]/[(r+1)\alpha]},$$

where $c_{10} > 0$ is a constant. Choosing $\delta$ small such that

$$\left(1 - \frac{\delta}{r+1}\right)(1+\epsilon) > 1 + \frac{\epsilon}{2},$$

one can have, for $k \geqslant k_6$,

$$P(E_k) \leqslant \frac{c_{10}}{k^{1+\epsilon/2}}.$$

From the Borel–Cantelli lemma and the relation $(D_n \text{ i.o.}) \subseteq (E_k \text{ i.o.})$ we obtain (2.7). By (2.6) and (2.7), note that $S \subseteq [1, e^{1/((r+1)\alpha)}]$.

We now establish that every point in $[1, e^{1/((r+1)\alpha)}]$ is an almost sure limit point of $\big(({}^{(r)}S_n/B_n)^{1/\log\log n}\big)$, and hence complete the proof.

For any $p \in (0,1)$, define $n_k = [e^{k^{1/p}}]$. We now show that, for any $\epsilon$ with $0 < \epsilon < p$,

$$P\left(\left(\frac{{}^{(r)}S_{n_k}}{B_{n_k}}\right)^{1/\log\log n_k} > e^{(p+\epsilon)/((r+1)\alpha)} \text{ i.o.}\right) = 0 \tag{2.12}$$

and

$$P\left(\left(\frac{{}^{(r)}S_{n_k}}{B_{n_k}}\right)^{1/\log\log n_k} > e^{(p-\epsilon)/((r+1)\alpha)} \text{ i.o.}\right) = 1, \tag{2.13}$$

which together imply that $e^{p/((r+1)\alpha)}$ is an almost sure limit point of the sequence $({}^{(r)}S_{n_k}/B_{n_k})^{1/\log\log n_k}$. Note that

$$P\left(\left(\frac{{}^{(r)}S_{n_k}}{B_{n_k}}\right)^{1/\log\log n_k} > e^{(p+\epsilon)/((r+1)\alpha)}\right) = P\big({}^{(r)}S_{n_k} > B_{n_k}(\log n_k)^{(p+\epsilon)/((r+1)\alpha)}\big).$$

Taking $x_n = (\log n)^{(p+\epsilon)/((r+1)\alpha)}$ in Theorem 2.1, one can find a $k_7$ such that, for all $k \geqslant k_7$,

$$P\big({}^{(r)}S_{n_k} > B_{n_k}(\log n_k)^{(p+\epsilon)/((r+1)\alpha)}\big) \leqslant \frac{1}{(\log n_k)^{[(r+1-\delta)(p+\epsilon)]/(r+1)}} \leqslant \frac{c_{11}}{k^{(1-\delta/(r+1))(1+\epsilon/p)}},$$

where $c_{11}$ is a constant. Choosing $\delta$ small such that

$$\left(1 - \frac{\delta}{r+1}\right)\left(1 + \frac{\epsilon}{p}\right) = 1 + \epsilon_1 > 1$$

for some $\epsilon_1 > 0$, one gets, for all $k \geqslant k_7$,

$$P\left(\left(\frac{{}^{(r)}S_{n_k}}{B_{n_k}}\right)^{1/\log\log n_k} > e^{(p+\epsilon)/((r+1)\alpha)}\right) \leqslant \frac{c_{11}}{k^{1+\epsilon_1}}.$$

By the Borel–Cantelli lemma, (2.12) follows.

In order to prove (2.13), note that ${}^{(r)}S_{n_k} \geqslant X_{n_k}^{(r+1)} \geqslant M_{r+1,k}$, where $M_{r+1,k}$ is the $(r+1)$-st largest among $(X_{n_{k-1}+1}, X_{n_{k-1}+2}, \ldots, X_{n_k})$. Consequently,

$$\begin{aligned} P\big({}^{(r)}S_{n_k} > B_{n_k}(\log n_k)^{(p-\epsilon)/((r+1)\alpha)}\big) \\ &\geqslant P\big(M_{r+1,k} > B_{n_k}(\log n_k)^{(p-\epsilon)/((r+1)\alpha)}\big) \\ &= P\big(X_{n_k-n_{k-1}}^{(r+1)} > B_{n_k}(\log n_k)^{(p-\epsilon)/((r+1)\alpha)}\big). \end{aligned}$$

Applying Theorem 2.1 and using the property that $n_{k-1}/n_k \to 0$ as $k \to \infty$, one can find a $k_8 > 0$ and a $c_{12} > 0$ such that, for all $k \geqslant k_8$,

$$\begin{aligned} P\big(X_{n_k-n_{k-1}}^{(r+1)} \geqslant B_{n_k}(\log n_k)^{(p-\epsilon)/((r+1)\alpha)}\big) \\ &\geqslant \frac{1}{(\log n_k)^{[(r+1+\delta)(p-\epsilon)]/(r+1)}} \geqslant \frac{c_{12}}{k^{(1+\delta/(r+1))(1-\epsilon/p)}}. \end{aligned}$$

Choosing $\delta > 0$ such that $\big(1+\delta/(r+1)\big)\,(1-\epsilon/p) = 1-\epsilon_2$ for some $\epsilon_2 > 0$, one gets, for $k \geqslant k_8$,

$$P\big(M_{r+1,k} > B_{n_k}(\log n_k)^{(p-\epsilon)/((r+1)\alpha)}\big) \geqslant \frac{1}{k^{1-\epsilon_2}}.$$

From the fact that $(M_{r+1,k})$ are mutually independent, by the Borel–Cantelli lemma, we have

$$P\big(M_{r+1,k} > B_{n_k}(\log n_k)^{(p-\epsilon)/((r+1)\alpha)} \text{ i.o.}\big) = 1. \tag{2.14}$$

Recalling that ${}^{(r)}S_{n_k} > X_{n_k}^{(r)} > M_{r+1,k}$, we see that (2.14) implies (2.13). Points $1$ and $e^{1/((r+1)\alpha)}$ are limit points by continuity considerations. Hence $[1, e^{1/((r+1)\alpha)}]$ is the set of all a.s. limit points of $\big(({}^{(r)}S_n/B_n)^{1/\log\log n}\big)$. ∎

COROLLARY 2.1. *Let $F(0) = 0$ and let $F$ belong to the domain of attraction of a positive stable law with index $\alpha, 0 < \alpha < 1$. Then the set of all a.s. limit points of $({}^{(r)}S_n/B_n)^{1/\log\log n}$ is $[1, e^{1/((r+1)\alpha)}]$, where $B_n = n^{1/\alpha}l(n)$ and $l(\cdot)$ is slowly varying.*

### 3. AN APPLICATION

Positive semi-stable distributions (which include positive stable ones) are often considered for fitting data on loss due to natural calamities such as flood, fire, cyclone, etc. Also, these distributions are considered for fitting the claim size in non-life insurance sectors. If $X_n$ denotes the loss incurred/claim size on the $n$-th occasion, then $S_n$ stands for the total loss/total claim up to $n$ occasions. As such $S_n$ plays an important role in many policy decisions. A natural question that arises is to know how far some of the extremes/outliers effect the total loss/total claims. Based on the estimate of the index $\alpha$ of the semi-stable law, by our result, one can note that eventually the total loss on $n$ occasions, after removing the $r$ extremes, fluctuates a.s. in the interval

$$\Big(\frac{n^{1/\alpha}l(n)\eta(n)}{(\log n)^{\epsilon}}, n^{1/\alpha}l(n)\eta(n)(\log n)^{(1+\epsilon)/((r+1)\alpha)}\Big).$$

**Acknowledgments.** The authors thank the referee for a number of valuable suggestions which resulted in a better presentation of the paper.

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*Received on 5.5.2014;*
*revised version on 8.8.2014*