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AUTOREGRESSIVE STRUCTURES AND DECOMPOSABILITY SEMIGROUPS

BY

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Abstract. A linear operator A is said to be *admissible* for a probability measure μ on a finite-dimensional vector space if there exists a stationary sequence X_n $(n = 0, \pm 1, \ldots)$ of random vectors with the probability distribution μ such that $X_{n+1} = AX_n + U_n$, where random vectors U_n are independent and identically distributed. The aim of this paper is to give a characterization of admissible operators for any probability measure in terms of its decomposability semigroup.

Throughout this paper we shall work with a finite-dimensional vector space $\mathscr{X}$ over the field of real or complex numbers. By a *probability measure* μ on $\mathscr{X}$ we shall understand a countably additive non-negative set function μ on the class of Borel subsets of $\mathscr{X}$ with the property that $\mu(\mathscr{X}) = 1$. A probability measure is said to be *full* if its support is not contained in any proper hyperplane of $\mathscr{X}$. Further, by δ_a $(a \in \mathscr{X})$ we shall denote the probability measure concentrated at the point a.

In the study of limit probability distributions [4] the author introduced the concept of *decomposability semigroup* $D(\mu)$ of linear operators associated with the probability μ. Namely, $D(\mu)$ consists of all linear operators A on $\mathscr{X}$ for which the equation $\mu = A\mu * \nu$ holds for a certain probability measure ν. The asterisk denotes here the convolution of measures and $(A\mu)(E) = \mu(A^{-1}(E))$ for all Borel subsets E of $\mathscr{X}$. We note that the zero operator 0 and the identity operator I always belong to $D(\mu)$. It has been shown that some purely probabilistic properties of μ are equivalent to some algebraic and topological properties of its decomposability semigroup $D(\mu)$.

Let $\{X_n\}$ $(n = 0, \pm 1, \ldots)$ be a stationary sequence of $\mathscr{X}$-valued random variables with a common distribution μ. It has the *first order autoregressive structure* if for any n the relation

$$X_{n+1} = AX_n + U_n \tag{1}$$

holds, where A is a linear operator on $\mathscr{X}$ and U_n form a sequence of independent and identically distributed random variables. The operator A in (1) will be called *admissible* for μ. The set of all admissible operators for μ will be denoted by $A(\mu)$. It is clear that the study of $A(\mu)$ for arbitrary probability measures can be reduced to the case of full probability measures. The purpose of this paper is to give a description of $A(\mu)$ in terms of the decomposability semigroup $D(\mu)$ for full probability measures μ on $\mathscr{X}$. All that has been done so far is to describe the set $A(\mu)$ for some special measures μ on the real line: Gaussian, gamma and exponential distribution (A. J. Lawrance in a paper presented at the Colloquium on Point Processes and Queueing Theory, Keszthely, Hungary, September 4-8, 1978), and uniform distribution over the unit interval (J. Łukaszewicz [3]).

Given a linear operator A on a complex vector space $\mathscr{X}$, we have the Jordan decomposition of $\mathscr{X}$ into a direct sum

$$\mathscr{X} = \bigotimes_{\lambda} \mathscr{X}_\lambda,$$

where the summation runs over all eigenvalues λ of A. The subspaces $\mathscr{X}_\lambda$ are invariant under the operator A and in a suitably chosen basis $e_1, e_2, \ldots, e_p$ of $\mathscr{X}_\lambda$ we have

$$Ae_1 = \lambda e_1 + e_2, \quad Ae_2 = \lambda e_2 + e_3, \ldots, Ae_{p-1} = \lambda e_{p-1} + e_p, \quad Ae_p = \lambda e_p. \tag{2}$$

Moreover,

$$A^n e_k = \sum_{j=0}^{p-k} \lambda^{\max(n-j,0)} \binom{n}{j} e_{j+k} \quad (k = 1, 2, \ldots, p). \tag{3}$$

The Jordan decomposition defines uniquely three projectors P_A, Q_A, and R_A from $\mathscr{X}$ onto A-invariant subspaces $\bigotimes_{|\lambda|<1} \mathscr{X}_\lambda$, $\bigotimes_{|\lambda|>1} \mathscr{X}_\lambda$, and $\bigotimes_{|\lambda|=1} \mathscr{X}_\lambda$, respectively. Of course, A, P_A, Q_A, and R_A commute with one another and

$$A^n P_A \to 0 \tag{4}$$

as $n \to \infty$. The operators A^n $(n = 1, 2, \ldots)$ restricted to $\bigotimes_{|\lambda|>1} \mathscr{X}_\lambda$ and $\bigotimes_{|\lambda|=1} \mathscr{X}_\lambda$ are invertible. Their inverses will be denoted by $A^{-n}Q_A$ and $A^{-n}R_A$, respectively. Moreover,

$$A^{-n}Q_A \to 0 \tag{5}$$

as $n \to \infty$. Further, from (3) we get the following statement:

(6) $\sup_n \|A^n R_A\| < \infty$ *if and only if* A *on* $\bigoplus_{|\lambda|=1} \mathscr{X}_\lambda$ *has a diagonal form, i.e., for all* λ *with* $|\lambda| = 1$ *we have* $p = 1$ *in* (2).

Our aim is to prove the following result:

THEOREM. *Let* $\mathscr{X}$ *be a finite-dimensional complex vector space and let* μ *be a full probability measure on* $\mathscr{X}$. *Then* $A \in A(\mu)$ *if and only if* Q_A, $A^{-1}Q_A$, $A(I-Q_A) \in D(\mu)$.

It should be noted that the Theorem remains true in the case of a real vector space $\mathscr{X}$. In fact, taking a complexification $\mathscr{X}_c$ of $\mathscr{X}$: $\mathscr{X}_c = \{x+iy\colon x, y \in \mathscr{X}\}$, we can extend to $\mathscr{X}_c$ every linear operator A on $\mathscr{X}$ by setting $A(x+iy) = Ax+iAy$. It is clear that to every non-real eigenvalue λ of A there corresponds a conjugate eigenvalue $\bar{\lambda}$ with the same multiplicity. Hence $\mathscr{X}$ is invariant under the projectors P_A, Q_A, and R_A defined by the Jordan decomposition of $\mathscr{X}_c$ generated by A. Furthermore, each full measure on $\mathscr{X}$ restricted to $\mathscr{X}_c$ is also full. Consequently, the Theorem remains true for vector spaces over the field of real numbers.

Before proceeding to prove the Theorem we shall establish some corollaries.

COROLLARY 1. *If* $A \in D(\mu)$, *then* $Q_A = 0$. *Moreover,*

$$A(\mu) \cap \{A\colon Q_A = 0\} = D(\mu). \tag{7}$$

Indeed, by the compactness of $D(\mu)$ (Proposition 1.1, [4], p. 121), the sequence $\{A^n Q_n\}$ is conditionally compact, which, by (5), yields $Q_A = 0$. Therefore, formula (7) is a consequence of the Theorem.

Suppose now that $\dim \mathscr{X} = n \geqslant 2$ and $A(\mu)$ contains projectors $P_1, P_2, \ldots, P_n, Q_1, Q_2$ satisfying the following conditions: $P_i P_j = 0$ for $i \neq j$ $(i, j = 1, 2, \ldots, n)$, $Q_1 Q_2 = Q_2 Q_1 = 0$, and $Q_k P_i \neq 0$ $(i = 1, 2, \ldots, n;\ k = 1, 2)$. Then by (7) all projectors $P_1, P_2, \ldots, P_n, Q_1, Q_2$ belong to $D(\mu)$ and, by Skitovich-Darmois results ([5], Theorem 3, p. 533), the measure μ is Gaussian.

Given an operator A on $\mathscr{X}$, we denote by $\Pi(A)$ the set of all projectors from $\mathscr{X}$ onto A-invariant subspaces $\bigoplus_{\lambda \in \Lambda} \mathscr{X}_\lambda$, where Λ is an arbitrary subset of the set of all eigenvalues of A different from 0 and of modulus less than 1.

COROLLARY 2. *Under the assumptions of the Theorem we have the formula*

$$A(\mu) = \{B(I-Q)+B^{-1}Q\colon B \in D(\mu),\ Q \in \Pi(B) \cap D(\mu)\}.$$

Indeed, if $A = B(I-Q)+B^{-1}Q$, where $B \in D(\mu)$ and $Q \in \Pi(B) \cap D(\mu)$, then, by Corollary 1, $Q_B = 0$. Consequently, $Q_A = Q$, which yields $Q_A \in D(\mu)$. Since, by Proposition 1.1 in [6] (p. 284), $I-Q \in D(\mu)$, we have $A(I-Q_A) = B(I$

$-Q) \in D(\mu)$. Finally, $A^{-1}Q_A = BQ \in D(\mu)$, which proves the relation $A \in A(\mu)$. Conversely, if $A \in A(\mu)$, then, by the Theorem, $A(I-Q_A)$, $A^{-1}Q_A \in D(\mu)$. Applying Proposition 1.2 from [6] (p. 284), we infer that

$$B = A(I-Q_A)+A^{-1}Q_A \in D(\mu).$$

Moreover, $Q_A \in \Pi(B) \cap D(\mu)$ and, by a simple calculation,

$$A = B(I-Q_A)+B^{-1}Q_A,$$

which completes the proof.

In the one-dimensional case we have either $\Pi(A) = \{0, I\}$ or $\Pi(A) = \{0\}$. Consequently, $\Pi(A) \subset D(\mu)$. Thus Corollary 2 yields

COROLLARY 3. *If* $\dim \mathscr{X} = 1$, *then*

$$A(\mu) = D(\mu) \cup \{A:\ A^{-1} \in D(\mu)\}.$$

The above formula reduces the characterizing problem of $A(\mu)$ to that of $D(\mu)$. My conjecture is that on the real line the following conditions characterize the decomposability semigroups S associated with full probability measure: S is a compact subsemigroup of the multiplicative semigroup of real numbers of modulus less than or equal to 1 and S contains both numbers 0 and 1. The necessity of these conditions is evident. All that has been done so far concerning their sufficiency shows that decomposability semigroups form a dense subset of the set consisting of all semigroups satisfying our conditions. More precisely, T. Rajba proved that for every semigroup S satisfying our conditions and for every open set V containing S there exists a full probability measure μ such that $S \subset D(\mu) \subset V$. For symmetric probability measures the characterizing problem has been solved by Iljinskij in [2]. Namely, a compact subsemigroup of the multiplicative semigroup of real numbers of modulus less than or equal to 1 is the decomposability semigroup for a symmetric probability measure if and only if it contains both numbers 0 and -1. Consequently, for every such semigroup S there exists a symmetric probability measure μ on the real line such that

$$A(\mu) = S \cup \{A:\ A^{-1} \in S\}.$$

Finally, we quote an example in which probability measures are characterized in terms of the set $A(\mu)$. Let $\{Y_n\}$ be a sequence of independent random vectors and let $\{A_n\}$ and $\{a_n\}$ be sequences of invertible operators and vectors, respectively, such that $A_n Y_k$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ form a uniformly infinitesimal triangular array. Suppose that the distributions of

$$A_n \sum_{k=1}^{n} Y_k + a_n$$

converge to a probability measure. This limit distribution is called a *Lévy's measure*. We refer the reader to [4] for an account of the family of all Lévy's measures.

COROLLARY 4. *Let μ be a full probability measure on a Euclidean space. Then μ is a Lévy's measure if and only if $A(\mu)$ contains a one-parameter operator semigroup e^{tB} $(t \geqslant 0)$ with the property $e^{tB} \to 0$ as $t \to \infty$.*

It is clear that e^{tB} $(t > 0)$ has no eigenvalue of modulus greater than 1. Consequently, $Q_{e^{tB}} = 0$ for all $t \geqslant 0$ and our statement is a consequence of formula (7) and Theorem 5.1 in [4] (p. 136).

Now we shall prove some auxiliary propositions for finite-dimensional complex vector spaces $\mathscr{X}$.

Let $\{X_n\}$ $(n = 0, \pm 1, \ldots)$ be a sequence of $\mathscr{X}$-valued random variables satisfying the condition (1). Put $X'_n = X_n + a$ and $U'_n = U_n + b$, where $a, b \in \mathscr{X}$. Then $X'_{n+1} = AX'_n + U'_n$ if and only if

$$b = (I - A)a. \tag{8}$$

Hence we get the following

LEMMA 1. *$A \in A(\mu)$ if and only if $A \in A(\mu * \delta_a)$ for every $a \in \mathscr{X}$.*

We note that equation (8) has a solution for a given vector b if and only if

$$R'_A b = 0, \tag{9}$$

where R'_A is the projector from $\mathscr{X}$ onto the subspace $\bigoplus_{\lambda = 1} \mathscr{X}_\lambda$. Further, from (1) we get the equations

$$X_n = \sum_{j=0}^{r} A^j U_{n-j-1} + A^{r+1} X_{n-r-1} \qquad (r = 0, 1, \ldots), \tag{10}$$

$$Q_A X_n = A^{-1} Q_A X_{n+1} - A^{-1} Q_A U_n,$$

$$Q_A X_n = -\sum_{j=0}^{r} A^{-j-1} Q_A U_{j+n} + A^{-r-1} Q_A X_{r+1} \qquad (r = 0, 1, \ldots).$$

Hence, taking into account (4), (5), and the identical distribution of random vectors X_n $(n = 0, \pm 1, \ldots)$, we get the following

LEMMA 2. *If $\{X_n\}$ fulfils condition* (1), *then*

$$P_A X_n = \sum_{j=0}^{\infty} A^j P_A U_{n-j-1} \quad \text{and} \quad Q_A X_n = -\sum_{j=0}^{\infty} A^{-j-1} Q_A U_{j+n}.$$

LEMMA 3. *Let μ be a full probability measure and $A \in A(\mu)$. Then*

$$\sup_{n \geqslant 1} \|A^n R_A\| < \infty.$$

Proof. Suppose the contrary. Then, by (6), there exists an eigenvalue λ such that $|\lambda| = 1$ and $\dim \mathscr{X}_\lambda \geqslant 2$, i.e., $p \geqslant 2$ in (2). Let R be the projector from $\mathscr{X}$ onto $\mathscr{X}_\lambda$. Of course, A, R_A, and R commute with one another. Further, let M_n be the space of all $\mathscr{X}$-valued random variables generated by the sequence $RU_{n+1}, RU_{n+2}, \ldots$, i.e., the space consisting of all $\mathscr{X}$-valued random variables measurable with respect to the σ-field induced by RU_{n+1}, $RU_{n+2}, \ldots$ Put

$$RX_0 = \sum_{j=1}^{p} \xi_j e_j, \tag{11}$$

$$RU_n = \sum_{j=1}^{p} \eta_{j,n} e_j, \tag{12}$$

where ξ_j, $\eta_{j,n}$ ($j = 1, 2, \ldots, p$; $n = 1, 2, \ldots$) are complex-valued random variables. By (10) we have the equation

$$RX_{n+1} = A^{n+1} RX_0 + \sum_{j=0}^{n} A^{n-j} RU_j. \tag{13}$$

Further, for every pair n, m of positive integers satisfying the inequality $n > m$ we put

$$Y_{n,m} = \sum_{j=m+1}^{n} A^{n-j} RU_j.$$

Obviously,

$$Y_{n,m} \in M_m \quad (n > m) \tag{14}$$

and, by (13),

$$Y_{n,m} = RX_{n+1} - A^{n+1} RX_0 - \sum_{j=0}^{m} A^{n-j} RU_j. \tag{15}$$

Since the random vectors RX_n are identically distributed and $p \geqslant 2$, we have the relation $n^{1-p} RX_{n+1} \to 0$ in probability as $n \to \infty$. Moreover, from (3), (11) and (12) we get the convergence in probability

$$(p-1)!\, \lambda^{p-2-n} n^{1-p} A^{n+1} RX_0 \to \xi_1 e_p,$$

$$(p-1)!\, \lambda^{p-2-n} n^{1-p} A^{n-j} RU_j \to \lambda^{-j-1} \eta_{1,j} e_p \quad (j = 0, 1, \ldots)$$

as $n \to \infty$, which by (15) yields the convergence in probability

$$Z_m = \lim_{n \to \infty} (p-1)!\, \lambda^{p-2-n} n^{1-p} Y_{n,m} = -\xi_1 e_p - \sum_{j=0}^{m} \lambda^{-j-1} \eta_{1,j} e_p. \tag{16}$$

Moreover, by (14),

$$Z_m \in M_m \quad (m = 1, 2, \ldots). \tag{17}$$

Thus the random vectors Z_m and $\sum_{j=0}^{m} \lambda^{-j-1} \eta_{j,j} e_p$ are independent. Further, from (16) and the independence of $\eta_{1,j}$ $(j = 0, 1, \ldots)$ we infer, according to Theorem 2.8 in [1] (p. 119), that the series $\sum_{j=0}^{\infty} \lambda^{-j-1} \eta_{1,j}$ converges with probability 1 when centered. Consequently, there exist complex numbers c_j such that $\lambda^{-j-1} \eta_{1,j} - c_j \to 0$ in probability. Since $|\lambda| = 1$ and $\eta_{1,j}$ are identically distributed, the last relation yields $\eta_{1,j} = c$ with probability 1, where c is a constant. Thus, by (17),

$$Z_m + \sum_{j=0}^{m} \lambda^{-j-1} \eta_{1,j} e_p \in M_m,$$

which by (16) implies the relation $\xi_1 e_p \in M_m$ $(m = 1, 2, \ldots)$. Consequently, by the zero-one law (Theorem 1.1, [1], p. 102), ξ_1 is constant with probability 1, which by (11) shows that the probability distribution of X_0 is concentrated on a proper hyperplane of $\mathscr{X}$. But this contradicts the assumption, which completes the proof of the lemma.

LEMMA 4. *Let $\{X_n\}$ be a stationary sequence satisfying* (1). *If the probability distribution of X_0 is full, then $R_A U_n$ is constant with probability* 1. *Moreover, $R'_A U_n = 0$ with probability* 1.

Proof. By (10) we have the equation

$$R_A X_0 = \sum_{j=0}^{r} A^j R_A U_{-j-1} + A^{r+1} R_A X_{-r-1} \qquad (r = 0, 1, \ldots). \tag{18}$$

Further, by Lemma 3, the sequence of probability distributions of the random vectors $A^{r+1} R_A X_{-r-1}$ $(r = 0, 1, \ldots)$ is conditionally compact in the sense of weak convergence. Consequently, from (18) it follows that the sequence of probability distributions of $\sum_{j=0}^{r} A^j R_A U_{-j-1}$ is also conditionally compact. Hence, according to Theorem 2.7 in [1] (p. 115), the series $\sum_{j=0}^{\infty} A^j R_A U_{-j-1}$ converges with probability 1 when centered. Thus, there exist constants $a_n \in \mathscr{X}$ such that $A^n R_A U_{-n-1} - a_n \to 0$ in probability. By Lemma 3 and (6), the sequence $\{A^{-n} R_A\}$ is also conditionally compact. Therefore, the last relation can be written in the form $R_A U_{-n-1} - b_n \to 0$ in probability, where $b_n = A^{-n} R_A a_n$. But the random vectors U_n $(n = 0, \pm 1, \ldots)$ have the same distribution, which shows that $R_A U_n$ is constant with probability 1. Setting $R_A U_n = b$, by (1) we have

$$R_A X_{n+1} = A R_A X_n + b. \tag{19}$$

Since $R'_A = R'_A R_A = R'_A A$, the last equation implies $R'_A X_{n+1} = R'_A X_n + R'_A b$.

Consequently, $R'_A b = 0$ because $R'_A X_n$ $(n = 0, \pm 1, \ldots)$ are identically distributed. The lemma is thus proved.

Remark. By Lemma 4, condition (9) is fulfilled. Thus, taking into account Lemma 1, we may assume in the sequel without loss of generality that $R_A U_n = 0$ with probability 1 or, in other words, by (19),

$$R_A X_{n+1} = AR_A X_n \quad (n = 0, \pm 1, \ldots). \tag{20}$$

LEMMA 5. *If $\{X_n\}$ fulfils* (1) *and the probability distribution of X_0 is full, then $R_A X_0$ and $\{U_n\}$ are independent.*

Proof. By (6) and Lemma 3 the operator AR_A has a diagonal form on $\bigoplus_{|\lambda|=1} \mathscr{X}_\lambda$. Consequently, there exists a sequence of integers $0 = r_0 < r_1 < \ldots$ such that

$$A^{r_n} R_A \to R_A \tag{21}$$

as $n \to \infty$. By the Remark to Lemma 4 we may assume without loss of generality that formula (20) holds. Thus

$$R_A X_n = A^n R_A X_0 \quad (n = 0, \pm 1, \ldots). \tag{22}$$

Let E and F be arbitrary Borel subsets of $\mathscr{X}$ and $\mathscr{X}^{2m+1}$ $(m = 1, 2, \ldots)$, respectively, and let c_E and c_F be their indicators. Put $Y = c_E(R_A X_0)$ and

$$V_n = c_F(U_{(2r_n-1)m+r_n}, \ldots, U_{(2r_n+1)m+r_n}) - P((U_{-m}, \ldots, U_m) \in F).$$

By the stationarity of $\{X_n\}$ the joint probability distribution of $R_A X_{k+n}, U_{k+n_1}, \ldots, U_{k+n_s}$ does not depend upon k. Consequently, by (21) and (22),

$$\mathrm{E} Y \bar{V}_n \to \mathrm{E} Y \bar{V}_0, \tag{23}$$

where E stands for the expectation. Moreover, $\{V_n\}$ are independent, identically distributed with zero mean and a finite variance. The random vector Y has also a finite variance. Thus, $\mathrm{E} Y \bar{V}_n$, being the coefficients in the orthogonal expansion of Y with respect to $\{V_n\}$, tend to 0 as $n \to \infty$, which by (23) yields $\mathrm{E} Y \bar{V}_0 = 0$. The last equation can be written in the form

$$P(R_A X_0 \in E, (U_{-m}, \ldots, U_m) \in F) = P(R_A X_0 \in E) P((U_{-m}, \ldots, U_m) \in F).$$

Hence, since m, E, and F are arbitrary, we get the assertion of the lemma.

Combining Lemmas 2 and 5 we obtain the following

COROLLARY. *If $\{X_n\}$ fulfils* (1) *and the probability distribution of X_0 is full, then the random vectors $P_A X_0$, $Q_A X_0$, and $R_A X_0$ are independent.*

We proceed now to proving the Theorem.

Necessity. Suppose that $A \in A(\mu)$, μ being a full probability measure. Let $\{X_n\}$ be a stationary sequence with property (1) such that μ is the probability distribution of each random vector X_n. By the Remark to Lemma 4 we may assume without loss of generality that formula (20) holds. Thus

$$R_A\mu = AR_A\mu. \tag{24}$$

Further, by the Corollary to Lemma 5 we have the equation

$$\mu = P_A\mu * Q_A\mu * R_A\mu. \tag{25}$$

Hence, in particular, it follows that

$$Q_A \in D(\mu). \tag{26}$$

As a consequence of Lemma 2 we have the equations

$$P_A X_0 = AP_A\Big(\sum_{j=1}^{\infty} A^j P_A U_{-j-1}\Big) + P_A U_{-1},$$

$$Q_A X_0 = -A^{-1}Q_A\Big(\sum_{j=1}^{\infty} A^{-j-1} Q_A U_j\Big) + A^{-1}Q_A U_0.$$

Since $\{U_n\}$ are independent and identically distributed, the last equations imply

$$P_A\mu = AP_A\mu * \nu_1, \tag{27}$$

$$Q_A\mu = A^{-1}Q_A\mu * \nu_2, \tag{28}$$

where ν_1 and ν_2 are probability distributions of $P_A U_{-1}$ and $A^{-1}Q_A U_0$, respectively. Combining (25) and (28) we get the relation

$$A^{-1}Q_A \in D(\mu). \tag{29}$$

Finally, from the conditions $I - Q_A = P_A + R_A$ and $P_A R_A = R_A P_A = 0$ we obtain the equation

$$A(I-Q_A)\mu = AP_A\mu * AR_A\mu$$

which by (24), (25), and (27) yields

$$\mu = A(I-Q_A)\mu * Q_A\mu * \nu_1.$$

Consequently,

$$A(I-Q_A) \in D(\mu),$$

which together with (26) and (29) completes the proof of the necessity of the conditions of the Theorem.

Sufficiency. Suppose now that μ is a full probability measure on $\mathscr{X}$, A is a linear operator on $\mathscr{X}$, and

$$Q_A,\ A^{-1}Q_A,\ A(I-Q_A) \in D(\mu). \tag{30}$$

The semigroup $D(\mu)$ is compact (Proposition 1.1, [4], p. 121). Consequently, the sequence $A^n(I-Q_A)$ $(n = 1, 2, \ldots)$ is conditionally compact. Since $A^n(I-Q_A) = A^n P_A + A^n R_A$, we infer, by virtue of (4), that the sequence $A^n R_A$ is conditionally compact and all its limit points belong to $D(\mu)$. Further, by (6), the operator AR_A has a diagonal form on $\bigoplus_{|\lambda|=1} \mathscr{X}_\lambda$. Consequently, there exists a sequence $r_1 < r_2 < \ldots$ of positive integers such that $A^{r_n} R_A \to R_A$. Moreover,

$$A^{r_n - 1} R_A \to A^{-1} R_A \quad \text{and} \quad A^{r_n + 1} R_A \to AR_A,$$

which yields

$$R_A,\ AR_A,\ A^{-1} R_A \in D(\mu). \tag{31}$$

Hence, by Proposition 1.3 in [4] (p. 122), we get the equation

$$R_A \mu = AR_A \mu * \delta_a \tag{32}$$

for a certain vector $a \in R_A X$. Since $AR'_A = R'_A = R'_A R_A$, where R'_A is the projector from $\mathscr{X}$ onto $\bigoplus_{\lambda=1} \mathscr{X}_\lambda$, by (32) we have $R'_A \mu = R'_A \mu * \delta_{a'}$, where $a' = R'_A a$. But the last equation holds for $a' = 0$ only. Thus $R'_A a = 0$ in (32). It is clear that the operator $A - I$ is invertible on the subspace $(\bigoplus_{|\lambda|=1} \mathscr{X}_\lambda) \ominus (\bigoplus_{\lambda=1} \mathscr{X}_\lambda)$. Denoting this inverse by $(A-I)^{-1}(R_A - R'_A)$ and setting $c = (A-I)^{-1}(R_A - R'_A)a$ we have, by (32),

$$R_A(\mu * \delta_c) = AR_A(\mu * \delta_c).$$

Since $BP_A = AP_A$ and $BQ_A = A^{-1} Q_A$, from (35) and (36) we get the erality, passing to the measure $\mu * \delta_c$ if necessary, that the equation

$$R_A \mu = AR_A \mu \tag{33}$$

is fulfilled. Further, from (30) and (31), by virtue of Lemma 1.2 in [6] (p. 284) we infer that both operators $B = A(I - Q_A) + A^{-1} Q_A$ and $Q_A + R_A$ belong to $D(\mu)$. Consequently, by Lemma 1.1 in [6] (p. 284), $P_A = I - Q_A - R_A \in D(\mu)$, which yields the equation

$$\mu = P_A \mu * Q_A \mu * R_A \mu. \tag{34}$$

Moreover, there exists a probability measure γ on $\mathscr{X}$ such that

$$\mu = B\mu * \gamma. \tag{35}$$

Setting $\nu = P_A \gamma * (-AQ_A)\gamma$ we have the equations

$$P_A \nu = P_A \gamma, \quad (-A^{-1} Q_A)\nu = Q_A \gamma, \quad R_A \nu = \delta_0. \tag{36}$$

Since $BP_A = AP_A$ and $BQ_A = A^{-1} Q_A$, from (35) and (36) we get the equations

$$P_A \mu = AP_A \mu * P_A \nu, \quad Q_A \mu = A^{-1} Q_A \mu * (-A^{-1} Q_A)\nu$$

which imply

$$P_A\mu = \mathop{*}_{j=0}^{n} A^j P_A \nu * A^{n+1} P_A\mu, \qquad Q_A\mu = \mathop{*}_{j=0}^{n} (-A^{-j-1}Q_A)\nu * A^{-n-1}Q_A\mu.$$

By (4) and (5) we have $A^n P_A\mu \to \delta_0$ and $A^{-n}Q_A\mu \to \delta_0$. Thus the last equations yield

$$P_A\mu = \mathop{*}_{j=0}^{\infty} A^j P_A \nu, \tag{37}$$

$$Q_A\mu = \mathop{*}_{j=0}^{\infty} (-A^{-j-1}Q_A)\nu. \tag{38}$$

By (36) the measure ν is concentrated on the subspace $(P_A+Q_A)\mathscr{X}$. Let $\{U_n\}$ $(n = 0, \pm 1, \ldots)$ be a sequence of $(P_A+Q_A)\mathscr{X}$-valued independent random vectors with the same probability distribution ν and let V be an $R_A\mathscr{X}$-valued random vector with the probability distribution $R_A\mu$ independent of all random vectors $\{U_n\}$. It is clear that the series

$$Y_n = \sum_{j=0}^{\infty} A^j P_A U_{n-j-1} \quad \text{and} \quad Z_n = -\sum_{j=0}^{\infty} A^{-j-1} Q_A U_{j+n}$$

$(n = 0, \pm 1, \ldots)$ converge with probability 1. Put $T_n = A^n R_A V$ and $X_n = Y_n + Z_n + T_n$ $(n = 0, \pm 1, \ldots)$. We can easily verify the equation

$$X_{n+1} = AX_n + U_n \quad (n = 0, \pm 1, \ldots).$$

Moreover, $Y_n + Z_n$, being a moving average of independent identically distributed random variables, is stationary. By (33), T_n is also stationary. By (33), (37), and (38) the random vectors Y_n, Z_n, and T_n have the probability distributions $P_A\mu$, $Q_A\mu$, and $R_A\mu$, respectively. Since for all pairs n, m of integers the random vectors Y_n, Z_n, and T_m are independent, we infer that the sequence $\{X_n\}$ is stationary and, by (34), μ is the probability distribution of each X_n. Thus $A \in A(\mu)$, which completes the proof.

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