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LIMIT DISTRIBUTIONS IN GENERALIZED CONVOLUTION ALGEBRAS

BY

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Abstract. In this paper we prove theorems on the accompanying laws and convergence of infinitely decomposable measures in a generalized convolution algebra, introduced by K. Urbanik [4]. These results are used to investigate the classes of s-stable and s-semi-stable measures introduced in paper [2], Chapter III.

1. Introduction. Let $\mathscr{P}$ be the class of all probability measures defined on Borel subsets of non-negative half-line. By E_a ($a \geqslant 0$) we shall denote the probability measure concentrated at the point a. For any positive number a we define a transformation T_a from $\mathscr{P}$ onto itself by means of the formula $(T_a P)(B) = P(a^{-1} B)$, where $P \in \mathscr{P}$, B is a Borel set and $a^{-1} B = \{a^{-1} x\colon\ x \in B\}$. Further, the transformation T_0 is defined by assuming $T_0 P = E_0$ for all $P \in \mathscr{P}$.

We say that a sequence $P_1, P_2, \ldots$ of probability measures is *weakly convergent* to a probability measure P, in symbols $P_n \to P$, if for every bounded continuous function f the equation

$$\lim_{n\to\infty} \int_0^\infty f(x)\, P_n(dx) = \int_0^\infty f(x)\, P(dx)$$

holds.

A commutative and associative $\mathscr{P}$-valued binary operation $\circ$ defined on $\mathscr{P}$ is called a *generalized convolution* if it satisfies the following conditions:

(i) $E_0 \circ P = P$ for all $P \in \mathscr{P}$;

(ii) $(aP+bQ)\circ R = a(P\circ R)+b(Q\circ R)$, whenever P, Q, $R \in \mathscr{P}$ and $a \geqslant 0$, $b \geqslant 0$, $a+b=1$;

(iii) $(T_a P)\circ(T_a Q) = T_a(P\circ Q)$;

(iv) if $P_n \to P$, then $P_n \circ Q \to P \circ Q$ for all $Q \in \mathscr{P}$;

(v) there exists a sequence $c_1, c_2, \ldots$ of positive numbers such that the sequence $T_{c_n} E_1^{\circ n}$ weakly converges to a measure Q different from E_0.

The power $E_a^{\circ n}$ is taken in the sense of the operation $\circ$, i.e. $E_a^{\circ 1} = E_a$, $E_a^{\circ(n+1)} = E_a^{\circ n} \circ E_a$ $(n = 1, 2, \ldots)$.

The concept of generalized convolution has been introduced and examined by Professor K. Urbanik. For the terminology and notation used here, see [4].

One of the most important example of generalized convolution is given in Kingman's work [3] (see also [4], p. 218). His example is closely connected with spherically symmetric random walks in Euclidean space.

The class $\mathscr{P}$ with a generalized convolution $\circ$ will be called a *generalized convolution algebra* and denoted by $(\mathscr{P}, \circ)$. Algebras admitting a non-trivial homomorphism into the real field are called *regular*. We say that an algebra $(\mathscr{P}, \circ)$ admits a *characteristic function* if there exists one-to-one correspondence $P \leftrightarrow \Phi_P$ between probability measures P from $\mathscr{P}$ and real-valued functions Φ_P defined on the non-negative half-line such that $\Phi_{aP+bQ} = a\Phi_P + b\Phi_Q$ $(a \geqslant 0, b \geqslant 0, a+b = 1)$, $\Phi_{P\circ Q} = \Phi_P \cdot \Phi_Q$, $\Phi_{T_a P}(t) = \Phi_P(at)$ $(a \geqslant 0, t \geqslant 0)$, and the uniform convergence in every finite interval of Φ_{P_n} is equivalent to the weak convergence of P_n. The function Φ_P is called the *characteristic function* of the probability measure P in algebra $(\mathscr{P}, \circ)$. It is proved in [4], Theorem 6, that an algebra admits a characteristic function if and only if it is regular. Moreover, each characteristic function is an integral transform

$$\Phi_P(t) = \int_0^\infty \Omega(tx)\, P(dx) \tag{1}$$

where the kernel Ω satisfies the inequality $\Omega(x) < 1$ in a neighbourhood of the origin and

$$\lim_{x\to 0} \frac{1-\Omega(tx)}{1-\Omega(x)} = t^{\varkappa} \tag{2}$$

uniformly in every finite interval. The positive constant $\varkappa$ does not depend upon a choice of characteristic function and is called a *characteristic exponent* of the algebra in question. Moreover, there exists a probability measure M called a *characteristic measure* of the algebra for which

$$\Phi_M(t) = \exp(-t^{\varkappa}) \tag{3}$$

(see [4], Theorem 7).

Troughout this paper we assume that the algebra $(\mathscr{P}, \circ)$ is regular, and Φ_P is a fixed characteristic function in $(\mathscr{P}, \circ)$.

2. Infinitely decomposable measures. This section is devoted to the study of the accompanying laws and convergence of infinitely decomposable measures. Let us recall that a measure $P \in \mathscr{P}$ is said to be *infinitely decomposable* if for every positive integer n there exists a measure $P_n \in \mathscr{P}$ such that P

$= P_n^{\circ n}$. The class of infinitely decomposable measures coincides with the class of limit distributions for sequences of the form

$$P_{n1} \circ P_{n2} \circ \ldots \circ P_{nk_n}$$

where P_{nk} ($k = 1, 2, \ldots, k_n$; $n = 1, 2, \ldots$) are *uniformly infinitesimal*, i.e., for any positive number ε,

(4) $$\lim_{n\to\infty} \max_{1\leqslant k\leqslant k_n} P_{nk}(x\colon\ x \geqslant \varepsilon) = 0$$

(see [4], Theorem 12). Moreover, one can prove an analogue of the Lévy-Khintchine representation for the characteristic functions of infinitely decomposable measures. Namely, the following theorem holds: a function Φ is a characteristic function of an infinitely decomposable measure if and only if it is of the form

(5) $$\Phi(t) = \exp \int_0^\infty \frac{\Omega(tx)-1}{\omega(x)}\, m(dx),$$

where m is finite Borel measure on the non-negative half-line,

(6) $$\omega(x) = \begin{cases} 1-\Omega(x) & \text{if } 0 \leqslant x \leqslant x_0, \\ 1-\Omega(x_0) & \text{if } x > x_0, \end{cases}$$

and x_0 is a positive fixed number such that $\Omega(x) < 1$ whenever $0 < x \leqslant x_0$. Always there exists such a number x_0 (see [4], Theorem 5). Further, representation (5) is unique, i.e. the function Φ determines the measure m (see [5], Theorem 1).

For any finite measure m on the non-negative half-line R^+ we define the *compound Poisson measure* $e(m)$ in the algebra $(\mathscr{P}, \circ)$ by the formula

$$e(m) = e^{-m(R^+)} \sum_{k=0}^{\infty} \frac{m^{\circ k}}{k!},$$

where the power $m^{\circ k}$ is taken in the sense of the operation $\circ$, and the measure m in the zero-power is equal E_0. It is easy to verify that

$$\Phi_{e(m)}(t) = \exp\left[m(R^+)\left(\Phi_{m/m(R^+)}(t)-1\right)\right].$$

Of course, the compound Poisson measures are infinitely decomposable.

THEOREM 1 (*Accompanying laws*). *Let* P_{nk} ($k = 1, 2, \ldots, k_n$; $n = 1, 2, \ldots$) *be uniformly infinitesimal probability measures and*

$$P_n = P_{n1} \circ P_{n2} \circ \ldots \circ P_{nk_n}, \quad Q_n = e\Big(\sum_{k=1}^{k_n} P_{nk}\Big).$$

Then $P_n \to P$ *if and only if* $Q_n \to P$.

Proof. Let

$$T_n = \frac{1}{k_n} \sum_{k=1}^{k_n} P_{nk}.$$

Then $T_n \in \mathscr{P}$ and

$$\Phi_{Q_n}(t) = \exp k_n(\Phi_{T_n}(t)-1) = \exp\left[\sum_{k=1}^{k_n} (\Phi_{P_{nk}}(t)-1)\right], \qquad \Phi_{P_n}(t) = \prod_{k=1}^{k_n} \Phi_{P_{nk}}(t).$$

From the elementary inequality $|\log(1+x)-x| \leqslant \frac{1}{2}|x|^2$ for $x \to 0$, we have

(7) $$\left|\sum_{k=1}^{k_n} \log \Phi_{P_{nk}}(t) / \sum_{k=1}^{k_n} (\Phi_{P_{nk}}(t)-1)^{-1}\right| \leqslant \tfrac{1}{2} \max_{1 \leqslant k \leqslant k_n} (1-\Phi_{P_{nk}}(t)).$$

Further, given a positive number $\varepsilon > 0$ and a positive number t_0, there exists a positive number δ such that $1-\Omega(tx) < \varepsilon$ whenever $0 \leqslant x \leqslant \delta$ and $0 \leqslant t \leqslant t_0$. Hence, for any number t satisfying the inequality $0 \leqslant t \leqslant t_0$ and for any integer k satisfying the inequality $1 \leqslant k \leqslant k_n$, we get

$$0 \leqslant 1-\Phi_{P_{nk}}(t) = \int_0^{\delta} (1-\Omega(tx)) P_{nk}(dx) + \int_{\delta}^{\infty} (1-\Omega(tx)) P_{nk}(dx)$$
$$\leqslant \varepsilon + 2 \max_{1 \leqslant k \leqslant k_n} P_{nk}(x\colon x \geqslant \delta)$$

which, by (4), implies

$$\max_{1 \leqslant k \leqslant k_n} (1-\Phi_{P_{nk}}(t)) \to 0$$

uniformly in every finite interval. Hence and from (7) it follows that in order that $\log \Phi_{P_n}(t) \to \log \Phi_P(t)$ uniformly in every finite interval it is necessary and sufficient that $\log \Phi_{Q_n}(t) \to \log \Phi_P(t)$ uniformly in every finite interval. Thus the Theorem is proved.

THEOREM 2. *For the convergence of a sequence $\{P_n\}$ of infinitely decomposable measures to a limit P it is necessary and sufficient that, as $n \to \infty$, $m_n \to m$, where the measures m_n and m are defined by formula* (5) *for P_n and P, respectively.*

Proof. Necessity. At first, let us remark that the class of infinitely decomposable measures in $(\mathscr{P}, \circ)$ is closed under weak limit (see [4], Theorem 11). Thus P is also infinitely decomposable. Further, let us introduce an auxiliary finite measure μ_n defined on R^+ by

(8) $$\mu_n(E) = \int_E g(x)\, m_n(dx),$$

where E es a Borel subset of R^+ and

$$g(x) = \frac{1}{\omega(x)}(1-\exp(-x^{\varkappa}))\int_0^1 (1-\Omega(ux))\,du,$$

and $\varkappa$ is the characteristic exponent of the algebra in question. The function g is positive for $x>0$ and bounded, which implies the finiteness of the measures μ_n (see [5], the proof of Theorem 1).

Further, in the same way as in the proof of Theorem 1 in [5], we get

$$\int_0^\infty \exp(-t^{\varkappa}x^{\varkappa})\,\mu_n(dx) = \int_0^\infty I_n\big((t^{\varkappa}+1)^{1/\varkappa}y\big)\,M(dy) - \int_0^\infty I_n(ty)\,M(dy),$$

where

$$I_n(t) = -\log \Phi_{P_n}(t) - \int_0^1 \log \Phi_{P_n}(u)\,du + \int_0^1\int_0^\infty \log \Phi_{P_n}(x)(E_t \circ E_u)(dx)\,du.$$

Hence it follows that the modified Laplace transforms of the measures μ_n tend to the modified Laplace transform of the measure μ such that

$$\mu(E) = \int_E g(x)\,m(dx).$$

Hence we have $\mu_n \to \mu$.

Since the function g is positive for $x>0$, continuous, bounded and $\lim g(x) = 0$ as $x \to 0$ from (8), we get

$$m_n \to m \tag{9}$$

on every Borel subset of R^+ separated from the origin. Further, let $0 < a < x_0$ be fixed. Of course

$$\int_0^\infty \frac{1-\Omega(x)}{\omega(x)}\,m_n(dx) \to \int_0^\infty \frac{1-\Omega(x)}{\omega(x)}\,m(dx) \qquad \text{as } n\to\infty.$$

Hence and from (6) there exists constant $c>0$ such that

$$m_n([0, a]) = \int_0^a \frac{1-\Omega(x)}{\omega(x)}\,m_n(dx) \leqslant c,$$

but this implies that the sequence $\{m_n\}$ is compact on $[0, a]$. Together with (9) we see that the sequence $\{m_n\}$ is compact on $[0, \infty)$. Since, for every $t\in R^+$, $\Phi_{P_n}(t) \to \Phi_P(t)$ and the spectral measure m in Levy-Khintchine representation (5) is unique, the sequence $\{m_n\}$ is weakly convergent to the measure m, and the necessity is proved.

Sufficiency. Since for any t the function $(\Omega(tx)-1)/\omega(x)$ is bounded and continuous on the half line $0 \leqslant x < \infty$, we get

$$\lim_{n\to\infty} \int_0^\infty \frac{\Omega(tx)-1}{\omega(x)}\, m_n(dx) = \int_0^\infty \frac{\Omega(tx)-1}{\omega(x)}\, m(dx).$$

The proof will be complete if it be shown that the above convergence is uniformly in every finite interval. Let us remark that for every $\delta > 0$ the measures ν_n defined by

$$\nu_n(B) = \int_{B\cap(\delta,\infty)} \frac{m_n(dx)}{\omega(x)},$$

are weakly convergent to the measure ν, where

$$\nu(B) = \int_{B\cap(\delta,\infty)} \frac{m(dx)}{\omega(x)}$$

and B is a Borel subset of R^+. Thus their characteristic functions are uniformly convergent in every finite interval. Hence

$$\int_\delta^\infty \frac{\Omega(tx)-1}{\omega(x)}\, m_n(dx) \to \int_\delta^\infty \frac{\Omega(tx)-1}{\omega(x)}\, m(dx)$$

uniformly in every finite interval.

Further, let $\varepsilon > 0$ be fixed. By (2) and (6) there exist $\eta = \eta(\varepsilon) > 0$ such that

$$\left|\frac{1-\Omega(tx)}{\omega(x)} - t^{\varkappa}\right| \leqslant \varepsilon$$

for all $0 \leqslant x \leqslant \eta$ and all $a \leqslant t \leqslant b$, where $0 \leqslant a < b < \infty$. We may assume that the interval $[0, \eta]$ is a continuity set of the measure m. Then $m_n[0, \eta] \to m[0, \eta]$ as $n \to \infty$ and there exists an $N = N(\varepsilon)$ such that $|m_n[0, \eta] - m[0, \eta]| \leqslant \varepsilon$ for $n \geqslant N$. Thus the two preceding inequalities show that

$$\left|\int_0^\eta \frac{1-\Omega(tx)}{\omega(x)}\, m_n(dx) - \int_0^\eta \frac{1-\Omega(tx)}{\omega(x)}\, m(dx)\right| \leqslant (t^{\varkappa}+\varepsilon)|m_n[0, \eta] - m[0, \eta]|$$

$$\leqslant (t^{\varkappa}+\varepsilon)\,\varepsilon$$

for $n \geq N$ and all $t \in [a, b]$. Hence, as $n \to \infty$,

$$\int_0^\infty \frac{\Omega(tx)-1}{\omega(x)} m_n(dx) \to \int_0^\infty \frac{\Omega(tx)-1}{\omega(x)} m(dx)$$

uniformly in every finite interval, and the sufficiency is proved.

In the sequel we assume that the convolution algebra $(\mathscr{P}, \circ)$ satisfies the following additional condition:

$$(*) \qquad D = \lim_{x \to 0} \frac{\omega(x)}{x^\varkappa} > 0.$$

This limit always exists and is finite. Moreover,

$$D \neq 0 \quad \text{if and only if} \quad \int_0^\infty x^\varkappa M(dx) < \infty,$$

where $\varkappa$ and M is the characteristic exponent of the characteristic measure of the convolution algebra in question (see [7] and [5], Lemma). In this case

$$D^{-1} = \int_0^\infty x^\varkappa M(dx).$$

Further, it is interesting that all known examples of generalized algebras satisfy condition $(*)$.

3. S-stable measures. Let r be a non-negative real number and U_r be a *shrinking operation*, (shortly, *s-operation*) from non-negative half-line R^+ onto itself by means of the formula

$$U_r(x) = \max(0, x-r).$$

Of course, U_r are continuous non-linear maps, the family $\{U_r: r \geq 0\}$ forms a semi-group under composition and $U_r U_s = U_{r+s}$ $(r, s \geq 0)$. Further, if $P \in \mathscr{P}$, then by $U_r P$ we mean the measure from $\mathscr{P}$ such that

$$(U_r P)(B) = P(U_r^{-1} B)$$

for all Borel subsets B of R^+.

A measure $Q \in \mathscr{P}$ will be called an *s-stable measure* in generalized convolution algebra $(\mathscr{P}, \circ)$ if there exists an increasing sequence $\{r_n\}$ of positive numbers tending to infinity and a measure $P \in \mathscr{P}$ such that

$$(10) \qquad (U_{r_n} P)^{\circ n} \to Q.$$

In [2], Chapter III, was introduced a notion of s-stability of Borel probability measures on real separable Hilbert space with ordinary convolution. In this section we give a description of the class of s-stable measures in algebras $(\mathscr{P}, \circ)$ satisfying the condition $(*)$.

Of course, for each positive ε

$$\lim_{n\to\infty} U_{r_n} P(x\colon\ x \geqslant \varepsilon) = 0,$$

thus every s-stable measure is infinitely decomposable (see [4], Theorem 12).

The following lemma will be used repeatedly and is stated here for further reference.

LEMMA 1. *Let $\{P_n\}$ and P be probability measures on positive half-line, and $\{a_n\}$, a, be positive real numbers. Then $P_n \to P$ and $a_n \to a$ implies $U_{a_n} P_n \to U_a P$.*

Proof. From the inequality

$$|U_r x - U_s x| \leqslant |r-s| \quad \text{for all } x \in R^+,$$

we get that if $x_n \to x$ and $a_n \to a$, then $U_{a_n} x_n \to U_a x$. Thus, taking into account Theorem 5.5 in [1], p. 34, we get that $U_{a_n} P_n \to U_a P$, which completes the proof of the Lemma.

The sequence $\{r_n\}$ in formula (10) we will call *norming sequence* corresponding to the s-stable measure Q. We shall give some property of norming sequence if the measure Q is not concentrated at zero.

LEMMA 2. *Let $Q \neq E_0$ be an s-stable measure. Then*

$$r_{n+1} - r_n \to 0.$$

Proof. Let $(U_{r_n} P)^{\circ n} \to Q$ and

$$\Phi_Q(t) = \exp \int_0^\infty \frac{\Omega(tx)-1}{\omega(x)}\, m(dx).$$

Then, by Theorems 1 and 2, we have

$$m_n \to m, \tag{11}$$

where

$$m_n(B) = n \int_B \omega(x)\, U_{r_n} P(dx)$$

for all Borel subsets B of R^+. Further, let us introduce the measures μ_n, μ by the formulae

$$\mu_n(B) = \int_B \frac{1}{\omega(x)}\, m_n(dx); \qquad \mu(B) = \int_B \frac{1}{\omega(x)}\, m\,(dx).$$

Since the function $1/\omega(x)$ is continuous and bounded on subsets separated from the origin, thus by (11) we get

$$\mu_n \to \mu,$$

on Borel subsets of R^+ separated from the origin.

Hence we obtain

$$nU_{r_n} P \to \mu. \tag{12}$$

Suppose that s is a limit point of the sequence $\{r_{n+1}-r_n\}$ with $0 < s \leqslant \infty$, and an interval I in $R^+ \setminus \{0\}$ is a continuity set of the measure μ. From equality

$$(n+1)\, U_{r_{n+1}} P(I) = (n+1)\, P(I+r_{n+1}) = \frac{n+1}{n}\, U_{r_{n+1}-r_n}[nU_{r_n} P(I)]$$

and from Lemma 1 we obtain

$$\mu(I) = U_s\,\mu(I) = \mu(I+s).$$

Consequently, by induction,

$$\mu(I) = \mu(I+ks) \quad (k = 1, 2, \ldots)$$

which yields $\mu(I) = 0$. Thus the measure m vanishes identically on positive half-line, i.e. on $(0, \infty)$.

In view of condition $(*)$ in section 2, we can introduce the finite Borel measures ν_n and ν by the formulae

$$\nu_n(B) = \int_B \frac{x^\varkappa}{\omega(x)}\, m_n(dx), \qquad \nu(B) = \int_B \frac{x^\varkappa}{\omega(x)}\, m(dx),$$

where B is Borel neighbourhood of the origin in R^+, and the integrand is assumed $\int_0^\infty t^\varkappa M(dt)$ if $x = 0$. Of course, by (11) we get $\nu_n \to \nu$ in every finite neighbourhood of the origin. Further, if we take the definition of the measures m_n, we get

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} n\int_0^\varepsilon x^\varkappa\, U_{r_n} P(dx) = \nu(\{0\}), \tag{13}$$

if the intervals $[0, \varepsilon]$ are continuity sets of the measure ν.

In the sequel we assume that s is a limit point of the sequence $\{r_{n+1}-r_n\}$ and $0 < s \leqslant \infty$. Let us denote

$$F_n(t) = nU_{r_n} P\{x\colon\ x > t\} \quad \text{for } t > 0, \tag{14}$$

and

(15) $$r_{k_n+1}-r_{k_n} \to s.$$

Since the measure m (and μ) vanishes on $(0, \infty)$, thus, by (12), we get

$$F_n(t) \to 0$$

for all positive t. Taking into account the formula (13) for subsequence $\{k_n+1\}$ and monotonicity of the functions F_n, by simple computation we obtain

$$\nu(\{0\}) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \left[-\varepsilon^{\varkappa} F_{k_n+1}(\varepsilon) + \varkappa \int_0^{\varepsilon} x^{\varkappa-1} F_{k_n+1}(x)\,dx \right]$$

$$= \varkappa \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{k_n+1}{k_n} \int_0^{\varepsilon} x^{\varkappa-1} F_{k_n}(x+r_{k_n+1}-r_{k_n})\,dx$$

$$\leqslant \varkappa \lim_{\varepsilon\to 0} \lim_{n\to\infty} \left\{ \frac{k_n+1}{k_n} F_{k_n}(\tfrac{1}{2}s) \int_0^{\varepsilon} x^{\varkappa-1}\,dx \right\} = 0.$$

Moreover, $m\{0\} = 0$, too.

Thus the assumption that the sequence $\{r_{n+1}-r_n\}$ has a positive limit point implies that the measure m vanishes identically on $[0, \infty)$. Hence $Q = E_0$, which contradicts the assumption in Lemma.

LEMMA 3. *If $Q \neq E_0$ is an s-stable probability measure and its representing measure m in formula* (5) *does not vanishes identically on* $(0, \infty)$, *then* $m\{0\} = 0$.

Proof. In the proof we keep on the notations used in the proof of Lemma 2. Thus

$$F_n(t) \to F(t) \overset{\mathrm{df}}{=} \mu(x\colon x > t)$$

for all positive continuity points of F, and

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} n \int_0^{\varepsilon} x^{\varkappa} U_{r_n} P(dx) = \nu(\{0\}),$$

if the intervals $[0, \varepsilon]$ are continuity sets of the measure ν (of course they are continuity sets also of the measure m).

Since the measure m does not vanish identically on $(0, \infty)$, thus the function $F(t)$ is positive in some neighbourhood of a positive real number s. By Lemma 2 for s there exists a subsequence $\{k_n\}$ such that

$$r_{k_n} - r_n \to s,$$

as $n \to \infty$. Further, if the closed interval I contained in $(0, \infty)$ is a continuity set of the measure μ, then from equality

$$k_n U_{r_{k_n}} P(I) = k_n P(I + r_{k_n}) = \frac{k_n}{n} U_{r_{k_n} - r_n}(nU_{r_n} P(I)) \tag{16}$$

and from formula (12) we get that there exists a limit of the sequence k_n/n, say c, because m (and μ) does not vanish identically on $(0, \infty)$. On the other hand, taking into account (13), (14) and Lemma 1 we obtain

$$\begin{aligned} \nu([0, \varepsilon]) &= \lim_{n\to\infty} \left[-\varepsilon^{\varkappa} F_{k_n}(\varepsilon) + \varkappa \int_0^{\varepsilon} x^{\varkappa-1} F_{k_n}(x)\,dx \right] \\ &= -\varepsilon^{\varkappa} F(\varepsilon) + \varkappa \lim_{n\to\infty} \frac{k_n}{n} \int_0^{\varepsilon} x^{\varkappa-1} F_n(x + r_{k_n} - r_n)\,dx \\ &= -\varepsilon^{\varkappa} F(\varepsilon) + \varkappa c \int_0^{\varepsilon} x^{\varkappa-1} F(x+s)\,dx. \end{aligned}$$

But $\lim\limits_{\varepsilon\to 0} \int_0^{\varepsilon} x^{\varkappa-1} F(x+s)\,dx = 0$ and hence

$$\nu(\{0\}) \leqslant \varliminf_{\varepsilon\to 0} \left(-\varepsilon^{\varkappa} F(\varepsilon)\right) \leqslant 0,$$

which implies that $m(\{0\}) = 0$. Thus the Lemma is proved.

LEMMA 4. *If $Q \neq E_0$ is s-stable probability measure and its representing measure m in formula* (5) *vanishes at zero, then there exist positive constants c and p such that*

$$m(B) = c \int_B \omega(x) e^{-px}\,dx \tag{17}$$

for all Borel subsets B of R^+.

Proof. We have

$$m_n \to m \quad \text{and} \quad m_n(B) = n \int_B \omega(x)\, U_{r_n} P(dx).$$

Moreover, on Borel subsets B of R^+ separated from the origin we get

$$nU_{r_n} P(B) \to \int_B \frac{1}{\omega(x)}\, m(dx) \stackrel{\text{df}}{=} \mu(B) \tag{18}$$

whenever the boundary of B is m-measure zero, i.e. B is a continuity set of the measure m.

By Lemma 2, for every positive t we can find a subsequence $\{k_n\}$ such that

$$r_{k_n} - r_n \to t \tag{19}$$

as $n \to \infty$. Further, by Lemma 1 and formulae (16), (18), (19), we get the existence of the limit $\lim_{n\to\infty} (k_n/n) = g(t)$ and the equation

$$\mu(B) = g(t)\, U_t\, \mu(B) = g(t)\, \mu(B+t), \tag{20}$$

because the measure m does not vanish identically. Moreover, the last equation holds for all Borel subsets B of $(0, \infty)$ and all positive t. The right-hand side of (20) is finite, thus the measure μ is finite on $(0, \infty)$. Therefore if we introduce the notation

$$f(u) = \mu(\{x \in R^+ :\ x \geqslant u\})$$

then equation (20), for $B = [u, v)$, can be rewritten in the form

$$f(u) - f(v) = g(t)\,[f(u+t) - f(v+t)] \tag{21}$$

where u, v and t are positive real numbers, $u < v$, and f is bounded non-increasing right-continuous function.

Let us remark that $g(t) > 1$ for every positive number t. In fact, in the oppositive case $g(t_0) \leqslant 1$ we would have, by induction according to (21), the inequality

$$f(u) - f(v) \leqslant f(u + t_0 k) - f(v + t_0 k) \qquad (k = 1, 2, \ldots).$$

But the right-hand side of this inequality tends to zero when $k \to \infty$. Thus f would be a constant function which would contradict the assumption that m (and also the measure μ) does not vanish identically.

Given $0 < u_0 < v_0$ with $f(u_0) - f(v_0) > 0$, we have, by (21), for every pair t_1, t_2 of positive numbers

$$\begin{aligned} f(u_0) - f(v_0) &= g(t_1)\,[f(u_0 + t_1) - f(v_0 + t_1)] \\ &= g(t_1)\, g(t_2)\,[f(u_0 + t_1 + t_2) - f(v_0 + t_1 + t_2)]. \end{aligned}$$

On the other hand,

$$f(u_0) - f(v_0) = g(t_1 + t_2)\,[f(u_0 + t_1 + t_2) - f(v_0 + t_1 + t_2)].$$

Consequently,

$$g(t_1 + t_2) = g(t_1)\, g(t_2),$$

and, by (20) and Lemma 1, the function g is continuous. It is well-known that the only solution of the last equation satisfying the condition $g(t) > 1$ is of the form $g(t) = e^{pt}$, where p is a positive constant. Furthermore, the function f being continuous outside a countable set is, by (21), continuous everywhere. Setting $v = u+t$ into (21), we get the inequality

$$f(u+2t) - 2f(u+t) + f(u) \geqslant 0.$$

Thus the function f is convex. Consequently, it is absolutely continuous. Setting

$$f(u) = \int_u^\infty h(s)\,ds$$

into formula (21), we have

$$\int_u^v h(s)\,ds = e^{pt} \int_{u+t}^{v+t} h(s)\,ds.$$

Hence we get the equation $h(t) = ce^{-pt}$ almost everywhere, c being a positive constant. Thus

$$\mu([u, v)) = f(u) - f(v) = c \int_u^v e^{-px}\,dx$$

and, by (18),

$$m(B) = c \int_B \omega(x) e^{-px}\,dx$$

for all Borel subsets B of R^+, which completes the proof.

LEMMA 5. *Each infinitely decomposable probability measure Q in $(\mathscr{P}, \circ)$ with representing measure m (in formula (5)) of the form*

$$m(B) = c \int_B \omega(x) e^{-px}\,dx,$$

where c and p are positive constant, is s-stable probability measure.

Proof. Of course, by Theorems 1 and 2 it suffices to define an increasing sequence of positive numbers $\{r_n\}$ and a probability measure P on R^+ such that the measures m_n defined by

$$m_n(B) = n \int_B \omega(x)\, U_{r_n} P(dx), \qquad (n = 1, 2, \ldots),$$

converge to m.

Put $a = m(R^+)$, $P = a^{-1} m$ and the sequence $\{r_n\}$ be such that $\exp(pr_n) = a^{-1}(1-\Omega(x_0))n$ for sufficiently large n (see (6)). Then it is easy to verify that

$$nU_{r_n} P(B) = \frac{c}{1-\Omega(x_0)} \int_B \omega(x+r_n) e^{-px}\,dx.$$

Thus, by (6), we get

$$\lim_{n\to\infty} m_n(B) = \lim_{n\to\infty} \frac{c}{1-\Omega(x_0)} \int_B \omega(x)\,\omega(x+r_n)\,e^{-px}\,dx = m(B),$$

which completes the proof.

LEMMA 6. *The characteristic measure M of the algebra $(\mathscr{P}, \circ)$ satisfying $(*)$ is an s-stable measure.*

Proof. Let us define

$$\alpha^{-1} = \int_0^\infty x^\varkappa M(dx), \qquad P(B) = (\pi/2)^{1/2} \int_B e^{-x^2/2}\,dx$$

for Borel subsets B of R^+; let r_n, for sufficiently large n, be solutions of the equations

$$x^{\varkappa+1}\exp(x^2/2) = (\pi/2)^{1/2}\,\alpha\Gamma(\varkappa+1)\,n. \tag{22}$$

Then (for every positive ε) taking into account (22) and the inequality

$$\int_a^\infty \exp(-x^2/2)\,dx \leqslant \frac{1}{a}\exp(-a^2/2) \qquad \text{for } a>0,$$

we get

$$\begin{aligned} nU_{r_n}P(x\colon\ x \geqslant \varepsilon) &= (\pi/2)^{1/2}\, n \int_{\varepsilon+r_n}^\infty \exp(-x^2/2)\,\mathrm{dx} \\ &\leqslant [\alpha\Gamma(\varkappa+1)]^{-1} r_n^{\varkappa+1}\exp(-\varepsilon r_n)(\varepsilon+r_n)^{-1}, \end{aligned}$$

and hence

$$nU_{r_n}P \to 0 \tag{23}$$

outside every neighbourhood of the zero. Further, since for every $\varepsilon > 0$

$$nU_{r_n}P(x\colon\ 0 \leqslant x \leqslant \varepsilon) = (\pi/2)^{1/2}\, n \int_0^{\varepsilon+r_n} e^{-x^2/2}\,dx,$$

thus, by (22),

$$\begin{aligned} \int_0^\varepsilon \omega(x)\,nU_{r_n}P(x) &= (\pi/2)^{1/2}\, n \int_0^\varepsilon \omega(x)\exp\left(-(\varepsilon+r_n)^2/2\right)dx \\ &= [\alpha\Gamma(\varkappa+1)]^{-1} \int_0^{\varepsilon r_n} t^\varkappa \exp(-t)\,\frac{\omega(t/r_n)}{(t/r_n)^\varkappa}\exp(-t^2/2r_n)\,dt. \end{aligned}$$

Consequently, by condition $(*)$ and Lebesgue Theorem, we get

$$\lim_{n\to\infty} n\int_0^{\varepsilon} \omega(x)\, U_{r_n} P(dx) = 1 \tag{24}$$

for every positive ε. In view of (23) and (24) we infer that the measures

$$m_n(B) = n\int_B \omega(x)\, U_{r_n} P(dx)$$

are weakly convergent to the measure E_0 and, by Theorems 1 and 2, we get that

$$(U_{r_n} P)^{\circ n} \to M,$$

which completes the proof.

Taking into account the equality

$$T_a\big(U_r(x)\big) = U_{ar}(T_a x) \quad (x \in R^+)$$

and the axiom (iii) in the definition of generalized convolution, we infer that if

$$(U_{r_n} P)^{\circ n} \to M,$$

then

$$\big(U_{ar_n}(T_a P)\big)^{\circ n} \to T_a M.$$

Hence we get

COROLLARY 1. *The measures $T_a M$, where M is the characteristic measure and a is a positive constant, are s-stable measures.*

As a simple consequence of Lemmas 3-5 and Corollary 1 we obtain the following characterization of the class of all s-stable probability measures in algebra $(\mathscr{P}, \circ)$.

THEOREM 3. *A probability measure Q in the algebra $(\mathscr{P}, \circ)$ with condition $(*)$ is s-stable measure if and only if either $Q = T_a M$, where a is a positive constant and M is the characteristic measure of the algebra $(\mathscr{P}, \circ)$, or Q is the compound Poisson measure, i.e.*

$$Q = e(m) \quad \text{and} \quad m(B) = c\int_B e^{-px}\, dx,$$

where c is a non-negative and p is a positive constant.

4. S-semi-stable measures. In this section we shall investigate limit distributions of $(U_{r_n} P)^{\circ n}$ for some subsequence of natural numbers. Let us assume that $\{l_n\}$ is an increasing subsequence of natural numbers such that

$$\lim_{n\to\infty} \frac{l_{n+1}}{l_n} = q \quad \text{for a certain finite } q. \tag{25}$$

A measure $Q \in \mathscr{P}$ will be called an *s-semi-stable measure* in algebra $(\mathscr{P}, \circ)$ if there exist an increasing sequence $\{r_n\}$ of positive numbers tending to infinity, a subsequence $\{l_n\}$ of natural numbers satisfying (25) and a measure $P \in \mathscr{P}$ such that

$$(U_{r_n} P)^{\circ l_n} \to Q. \tag{26}$$

Of course s-semi-stable measures are infinitely decomposable and s-stable measures are s-semi-stable. As before the sequences $\{r_n\}$ in (26) we call the *norming sequences* and at first we prove some properties of them.

In this section is also assumed that the convolution algebra $(\mathscr{P}, \circ)$ satisfies condition $(*)$ (see section 2).

LEMMA 7. *Let $Q \neq E_0$ be an s-semi-stable measure. Then*

(a) *$r_{n+1} - r_n \to 0$ if either $q = 1$ or $q > 1$ and the representing measure m of Φ_Q, in formula (5), is concentrated at zero;*

(b) *$r_{n+1} - r_n \to d$ and $0 < d < \infty$ if $q > 1$ and the representing measure m of Φ_Q is not concentrated at zero.*

Proof. Let us introduce the notations

$$m_n(B) = l_n \int_B \omega(x)\, U_{r_n} P(dx), \tag{27}$$

$$\mu_n(B) = \int_B \frac{1}{\omega(x)}\, m_n(dx), \qquad \mu(B) = \int_B \frac{1}{\omega(x)}\, m(dx), \tag{28}$$

$$\nu_n(B) = \int_B \frac{x^\varkappa}{\omega(x)}\, m_n(dx), \qquad \nu(B) = \int_B \frac{x^\varkappa}{\omega(x)}\, m(dx), \tag{29}$$

where B is an arbitrary Borel subset of R^+.

From (6) we get that m_n, m are finite Borel measures outside every neighbourhood of the zero and by condition $(*)$ we have that ν_n and ν are finite Borel measures on every finite neighbourhood (i.e. finite open interval) of the zero. In view of Theorems 1 and 2 we have $m_n \to m$ and hence

$$\mu_n \to \mu \tag{30}$$

outside every neighbourhood of the zero and

$$\nu_n \to \nu \tag{31}$$

on every finite interval which contain zero.

Moreover, if $F_n(t) = \mu_n(x\colon x > t)$, $t > 0$, and $F(t) = \mu(x\colon x > t)$, then

$$F_n(t) = l_n U_{r_n} P(x\colon x > t) \to F(t) \tag{32}$$

for all positive continuity point t of $F(t)$.

(a) If $q = 1$ then the proof is similar to the proof of Lemma 2 and we omit it. If $q > 1$ and the measure m is concentrated at zero, then by (27), (29), (31) and (32) we get

$$(33) \qquad \nu(\{0\}) \leqslant \lim_{\varepsilon\to 0}\lim_{n\to\infty} l_n \int_0^{\varepsilon} x^{\varkappa} U_{r_n} P(dx)$$

$$= \varkappa \lim_{\varepsilon\to 0}\varliminf_{n\to\infty} \int_0^{\varepsilon} x^{\varkappa-1} F_n(x)\,dx,$$

because $F_n(t) \to 0$. In contrary, let us assume that $0 < s < \infty$ is a limit point of the sequence $\{r_{n+1}-r_n\}$, i.e. $r_{k_n+1}-r_{k_n} \to s$. Then, by (33),

$$\nu(\{0\}) \leqslant \varkappa \lim_{\varepsilon\to 0}\varliminf_{n\to\infty} \frac{k_n+1}{k_n}\int_0^{\varepsilon} x^{\varkappa-1} F_{k_n}(x+r_{k_n+1}-r_{k_n})\,dx$$

$$\leqslant \varkappa \lim_{\varepsilon\to 0}\varliminf_{n\to\infty} \frac{k_n+1}{k_n} F_{k_n}(\tfrac{1}{2}s)\int_0^{\varepsilon} x^{\varkappa-1}\,dx = 0$$

and $m(\{0\}) = 0$, which contradicts the assumption.

It is easy to see that $s = \infty$ is not limit point of the sequence $\{r_{n+1}-r_n\}$, too.

(b) Taking into account the equality

$$(34) \qquad \mu_{n+1}(I) = \frac{l_n+1}{l_n} U_{r_{n+1}-r_n}\mu_n(I),$$

Lemma 1, formula (30) and the fact that $q > 1$ and the measure m does not vanish identically on $(0, \infty)$, we obtain that the point $s = 0$ and $s = \infty$ are not limit points of the sequence $\{r_{n+1}-r_n\}$.

Let us suppose that there exist two limit points of $\{r_{n+1}-r_n\}$, say s_1 and s_2. Let $s_1 < s_2$ and the intervals I, $I+s_1$ and $I+s_2$ be continuity sets of the measure μ. By (34) we get the formula

$$(35) \qquad \mu(I) = q\mu(I+s_1) = q\mu(I+s_2).$$

Further, by a simple reasoning we infer that the last equation holds for all intervals I in $(0, \infty)$ and the measure μ is finite. In view of (35), we see that for all intervals I contained in the half-line (s_1, ∞) the equality $\mu(I) = \mu(I+(s_2-s_1))$ holds. Consequently, by induction

$$\mu(I) = \mu(I+k(s_2-s_1)), \quad k = 1, 2, \ldots,$$

which yields $\mu(I) = 0$ for intervals I contained in (s_1, ∞). Hence and from (35) we infer that the measure μ vanishes in $(0, \infty)$. But (28) implies that also m vanishes on $(0, \infty)$, and this contradicts the assumption. Thus the Lemma is proved.

LEMMA 8. *If $Q \neq E_0$ is an s-semi-stable measure and its representing measure m, in formula* (5), *does not vanish on* $(0, \infty)$, *then* $m(\{0\}) = 0$.

Proof. In the proof we keep on the notations used in the proof of Lemma 7.

If $q = 1$, then by Lemma 7, part (a), we have $r_{n+1} - r_n \to 0$. In the same way as in the proof of Lemma 3 we get $m(\{0\}) = 0$.

Suppose that $q > 1$. By part (b) of Lemma 7 we have $r_{n+1} - r_n \to d$ and $0 < d < \infty$. Further, by (32) and Lemma 1 we get

$$\lim_{n\to\infty} l_{n+1} \int_0^{\varepsilon} x^{\varkappa} U_{r_n} P(dx) = -\varepsilon^{\varkappa} F(\varepsilon) + \varkappa \lim_{n\to\infty} \frac{l_{n+1}}{l_n} \int_0^{\varepsilon} x^{\varkappa-1} F_n(x + r_{n+1} - r_n)\, dx$$

$$= -\varepsilon^{\varkappa} F(\varepsilon) + \varkappa q \int_0^{\varepsilon} x^{\varkappa-1} F(x+d)\, dx.$$

But

$$\lim_{\varepsilon\to 0} \int_0^{\varepsilon} t^{\varkappa-1} F(x+d)\, dx = 0$$

and it implies together with (33) that $\nu(\{0\}) = 0$. Thus the measure m is concentrated on $(0, \infty)$ which completes the proof of the Lemma.

LEMMA 9. *If $Q \neq E_0$ is an s-semi-stable probability measure and its representing measure m in formula* (5) *vanishes at zero, then the measure μ defined by the formula*

$$\mu(B) = \int_B \frac{1}{\omega(x)}\, m(dx)$$

is finite on $(0, \infty)$ *and there exist real numbers* $0 < d < \infty$ *and* $0 < \tau < 1$ *such that*

$$U_d \mu = \tau\mu. \tag{36}$$

Proof. Let Q be an s-semi-stable measure and $q = 1$ (see (25)). In virtue of Lemma 7, part (a), in similar way as in the proof of Lemma 4, one can obtain that for all positive t

$$U_t \mu = e^{-pt} \mu,$$

where $0 < p < \infty$ is a constant. Thus μ is finite on $(0, \infty)$ and formula (36) holds.

If $Q \neq E_0$ is an s-semi-stable measure and $q > 1$, then we have $\mu_n \to \mu$

outside every neighbourhood of zero (see (30)). Moreover, by part (b) of Lemma 7 and by (34), we get equation

$$\mu(B) = qU_d\,\mu(B)$$

for all Borel subsets B of $(0, \infty)$. Hence formula (36) is fulfiled with $\tau = 1/q$, which completes the proof of Lemma.

LEMMA 10. *Each infinitely decomposable probability measure Q in $(\mathscr{P}, \circ)$ with representing measure m (in formula (5)) of the form*

$$m(B) = \int_B \omega(x)\,\mu(dx),$$

where μ is finite Borel measure on $(0, \infty)$, and there exist constants $0 < \tau < 1$ and $0 < d < \infty$ such that

$$U_d\,\mu(B) = \tau\mu(B),$$

for all Borel subsets B of $(0, \infty)$, is an s-semi-stable probability measure.

Proof. Of course, we may assume that the measure m does not vanish identically, because in this case the assertion is obvious.

Let us put $a^{-1} = \mu(R^+)$, $P = a\mu$, $r_n = nd$ and a sequence $\{l_n\}$ of natural numbers be such that

$$a\tau^n l_n \to 1$$

as $n \to \infty$ (for instance put $l_n = [\tau^{-n}a^{-1}]$ for sufficiently large n, where [ ] denotes the integral part of number). Then it is easy to verify that

$$l_n U_{r_n} P(I) \to \mu(I) \tag{37}$$

for all closed intervals in $(0, \infty)$. Thus

$$m_n(B) = l_n \int_B \omega(x)\,U_{r_n}P(dx) \to \int_B \omega(x)\,\mu(dx) = m(B)$$

for all m-continuity Borel sets B, and by Theorems 1 and 2 we get

$$(U_{r_n}P)^{\circ l_n} \to Q,$$

which completes the proof.

LEMMA 11. *The characteristic measure M of the algebra $(\mathscr{P}, \circ)$ satisfying the condition $(*)$ is an s-semi-stable measure.*

Proof. Let $\{l_n\}$ be an increasing subsequence of natural numbers such that $\lim_{n\to\infty} l_{n+1}/l_n = 1$. Let a sequence $\{r_n\}$ of positive real numbers be such that

$$r_n^{\varkappa+1} \exp(r_n^2/2) = (\pi/2)^{1/2}\alpha\Gamma(\varkappa+1)\,l_n,$$

where $\alpha^{-1} = \int_0^\infty x^\varkappa M(dx)$, and let the measure P be defined as follows:

$$P(B) = (\pi/2)^{1/2} \int_B \exp(-x^2/2)\, dx.$$

By the same computation as in the proof of Lemma 6 we get

$$(U_{r_n} P)^{\circ l_n} \to M,$$

and the Lemma is proved.

Using the same arguments as in the proof of Corollary 1, we obtain

COROLLARY 2. *The measures of the form $T_a M$, where a is a positive constant and M is the characteristic measure, are s-semi-stable measures.*

Now we are in position to give a full characterization of the class of all s-semi-stable measures. Namely, in view of Lemmas 8-10 and Corollary 2 we have the following

THEOREM 4. *Let the algebra $(\mathscr{P}, \circ)$ satisfies the condition $(*)$. A probability measure Q in the algebra $(\mathscr{P}, \circ)$ is an s-semi-stable measure if and only if either $Q = T_a M$, where a is a positive constant and M is the characteristic measure of the algebra $(\mathscr{P}, \circ)$, or Q is the compound Poisson measure, i.e. $Q = e(m)$, and there exist constants $0 < d < \infty$, $0 < \tau < 1$ such that*

$$U_d m(B) = \tau m(B)$$

for all Borel subsets B of $(0, \infty)$.

5. Examples. In this section we give characterizations of the class of s-stable measures in some special cases.

At first let us assume that in the set $\mathscr{P}$ of all probability measures on non-negative half-line we have the ordinary convolution, and the characteristic function $\Phi_P(t)$ is the Laplace transform of a measure P. Then, by Theorem 3, we have

COROLLARY 3. *A function Φ is Laplace transform of an s-stable measure on $[0, \infty]$ if and only if either*

$$\Phi(t) = \exp(-at),$$

a being a non-negative constant, or

$$\Phi(t) = \exp\left(-\frac{c}{p}\frac{t}{t+p}\right),$$

where c is a non-negative and p is a positive constant.

As a second example of a generalized convolution we quote the $(1, r)$-convolutions $(1 \leqslant r < \infty)$ considered by Kingman in [3].

Let us recall that the (1, 1)-*convolution* is defined by means of the formula

$$\int_0^\infty f(x)(P\circ Q)(dx) = \tfrac{1}{2}\int_0^\infty\int_0^\infty [f(x+y)+f(|x-y|)]\,P(dx)\,Q(dx),$$

where f runs over all bounded continuous functions on $[0, \infty)$.

The (1, r)-*convolution* for $r > 1$ is defined as

$$\int_0^\infty f(x)(P\circ Q)(dx)$$

$$= \frac{\Gamma\left(\frac{r}{2}\right)}{\Gamma\left(\frac{r-1}{2}\right)}\int_0^\infty\int_0^\infty\int_{-1}^{1} f(x^2+y^2+2xyz)(1-z^2)^{(r-3)/2}\,dz\,P(dx)\,Q(dx)$$

and f is arbitrary continuous bounded function on $[0, \infty)$.

All (1, r)-convolution algebras are regular. As a characteristic function in these algebras one can take the integral transformation

$$\Phi_P(t) = \Gamma\left(\frac{r}{2}\right)\int_0^\infty \left(\frac{2}{tx}\right)^{r/2-1} J_{r/2-1}(tx)\,P(dx), \tag{38}$$

where J_k is the Bessel function (see [6], p. 40).

The (1, r)-convolution is closely connected with a random walk problem in Euclidean r-space. Namely, consider a random walk in r-space given by

$$S_n = X_1+X_2+\ldots+X_n \qquad (n = 1, 2, \ldots),$$

where $X_1, X_2, \ldots$ are independent random vectors with spherical symmetry, that is, if A is a measurable subset of r-space and A' is obtained from A by rotation about the origin, then

$$\text{Prob}\,(X_k \in A) = \text{Prob}\,(X_k \in A') \qquad (k = 1, 2, \ldots).$$

The probability distribution of the length $|S_n|$ is determined by that of the length $|X_1|, |X_2|, \ldots, |X_n|$ (see [3]). More precisely, the probability distribution of $|S_n|$ is the (1, r)-convolution of the probability distributions of $|X_1|$, $|X_2|, \ldots, |X_n|$. Further, the characteristic exponent of the (1, r)-convolution is equal to 2, and the measure on $[0, \infty)$ with probability density

$$g_r(x) = 2^{-2(r-1)}\left(\Gamma\left(\frac{r}{2}\right)\right)^{-1} x^{r-1}\exp(-x^2/4),$$

corresponding to the Rayleigh distribution, plays the role of the characteristic measure, because by the Weber Theorem (see [6], p. 394, formula (4)) we get

$$2^{-2(r-1)}\int_0^\infty \left(\frac{2}{tx}\right)^{r/2-1} J_{r/2-1}(tx)\,x^{r-1}\exp(-x^2/4)\,dx = \exp(-t^2).$$

Now we can get the characterisation of s-stable measures in the $(1, r)$-convolution algebra.

Corollary 4. *A function Φ is a characteristic function of an s-stable measure in the $(1, r)$-convolution algebra if and only if either*

$$\Phi(t) = \exp(-a^2 t^2), \tag{39}$$

a being a non-negative constant, or

(40)
$$\Phi(t) = \exp\left\{\frac{c}{p}\left[\frac{p}{(p^2+t^2)^{1/2}}\left(1+(r-2)\sum_{n=1}^{\infty}\frac{(2n-1)!!}{n!\,2^n(2n+r-2)}\left(\frac{t^2}{p^2+t^2}\right)^n\right)-1\right]\right\},$$

where c is a non-negative and p is a positive constant.

Proof. In view of Theorem 3 and the arguments preceding Corollary 4 it suffices to show that the non-characteristic s-stable measure has a characteristic function of the form (40). But if the measure $Q = e(m)$ is s-stable, then, by Theorem 3, we have

$$m(B) = c\int_B e^{-px}\,dx.$$

Further, by (38) and the Hankel's formula (see [6], p. 385), we get

$$\Phi_Q(t) = \exp\left\{\frac{c}{p}\left[p\Gamma\left(\frac{r}{2}\right)\int_0^\infty\left(\frac{2}{tx}\right)^{r/2-1} J_{r/2-1}(tx)\,e^{-px}\,dx - 1\right]\right\}$$

$$= \exp\left\{\frac{c}{p}\left[\frac{p}{(p^2+t^2)^{1/2}}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{r}{2}-1\right)_n}{n!\left(\frac{r}{2}\right)_n}\left(\frac{t^2}{p^2+t^2}\right)^n - 1\right]\right\},$$

where for any α we assume $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)$. By a simple computation, from the last formula we obtain formula (40). Thus the Corollary is proved.

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