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LÉVY MEASURES INVOLVING A GENERALIZED FORM OF FRACTIONAL INTEGRALS

BY

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Abstract. A four-parameter fractional integral transformation $\mathcal{A}_{\alpha,p}^{q,r}$ of measures on $\mathbb{R}^d \setminus \{0\}$ is introduced and a systematic study of its properties depending on the values of the parameters is made. Descriptions of its domain, range, and effect on behaviors of measures near or far from the origin are found. A non-commutative relation with a two-parameter Upsilon transformation $\Upsilon_{\beta,\theta}$ is established in the form $\Upsilon_{\beta,\theta}\mathcal{A}_{\alpha,p}^{q,r} = \mathcal{A}_{\alpha,p}^{q,r}\Upsilon_{\beta',\theta'}$ for some β' and θ'. Then the class of infinitely divisible distributions having Lévy measures of the form $\mathcal{A}_{\alpha,p}^{q,r}\rho$ is discussed. It is represented as the class of laws of improper stochastic integrals with respect to Lévy processes if $-\infty < \alpha < 1$. For $1 \leqslant \alpha < 2$, it is the class of laws of essentially definable improper stochastic integrals.

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1. INTRODUCTION

Let $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$. We study a four-parameter fractional integral transformation

$$(1.1)\qquad (\mathcal{A}_{\alpha,p}^{q,r}\rho)(B) = c_p \int_0^\infty u^{-\alpha-1} du \int_{\mathbb{R}_0^d} 1_B(ux/|x|)(|x|^r - u^q)_+^{p-1}\rho(dx), \qquad B \in \mathcal{B}(\mathbb{R}_0^d),$$

of a measure ρ on $\mathbb{R}_0^d$, where p, q, $r > 0$ and $\alpha \in \mathbb{R}$. Here $\mathcal{B}(\mathbb{R}_0^d)$ is the class of Borel sets in $\mathbb{R}_0^d$, $c_p = \Gamma(p)^{-1}$, $s_+ = s \vee 0$, and $0^a = 0$ when $a \leqslant 0$. In our paper [9] we studied (1) two transformations $\mathcal{A}_1$ and $\mathcal{A}_2$ based on arcsine density and (2) the class of infinitely divisible distributions with Lévy measures being in

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the common range of $\mathcal{A}_1$ and $\mathcal{A}_2$. These two transformations are identical with $\mathcal{A}^{2,1}_{-1,1/2}$ and $\mathcal{A}^{2,2}_{-1,1/2}$, respectively, up to constant factors. Main results of [9] were a stochastic integral representation of the range and a new representation of the class $G(\mathbb{R}^d)$ introduced in [11]. ($G(\mathbb{R}^d)$ reduces in the symmetric case to the class of distributions called type G in [10]. In one dimension, type G means the distribution of $Z^{1/2}G$, where G is the standard Gaussian random variable, Z is a nonnegative infinitely divisible random variable, and G and Z are independent.) In this paper we will extend some of the results in [9] to the transformation $\mathcal{A}^{q,r}_{\alpha,p}$ with general parameter values. In the representation of $G(\mathbb{R}^d)$ in [9] we observed the importance of the composite transformations $\mathcal{A}_1\Upsilon_{-1,1}$ and $\Upsilon_{-2,2}\mathcal{A}_1$, where $\Upsilon_{-1,1}$ and $\Upsilon_{-2,2}$ are special cases of the transformation $\Upsilon_{\beta,\theta}$ defined as

$$(1.2)\qquad (\Upsilon_{\beta,\theta}\,\rho)(B) = \int_0^\infty \rho(t^{-1}B)\theta t^{-\beta-1}e^{-t^\theta}dt, \quad B \in \mathcal{B}(\mathbb{R}^d_0),$$

with $\beta \in \mathbb{R}$ and $\theta > 0$, which was already used in [8] up to a constant factor. We will study $\mathcal{A}^{q,r}_{\alpha,p}\Upsilon_{\beta,\theta}$ and $\Upsilon_{\beta,\theta}\mathcal{A}^{q,r}_{\alpha,p}$ and prove a non-commutative relation expressed by the identity

$$(1.3)\qquad \Upsilon_{\beta,\theta}\mathcal{A}^{q,r}_{\alpha,p} = \mathcal{A}^{q,r}_{\alpha,p}\Upsilon_{r(p-1+(\beta-\alpha)/q),\,\theta r/q}.$$

This is an identity between two composite transformations. It implies that the domains of both sides coincide. The domain of a composite is defined as usual. For instance, the domain $\mathfrak{D}(\Upsilon_{\beta,\theta}\mathcal{A}^{q,r}_{\alpha,p})$ equals the class of $\rho \in \mathfrak{D}(\mathcal{A}^{q,r}_{\alpha,p})$ satisfying $\mathcal{A}^{q,r}_{\alpha,p}\rho \in \mathfrak{D}(\Upsilon_{\beta,\theta})$. Thus our first problem is how to define the domains of $\mathcal{A}^{q,r}_{\alpha,p}$ and $\Upsilon_{\beta,\theta}$. It is too narrow for application if, as in [9], we merely consider Lévy measures. We will define the domain as the class of all locally finite measures ρ on $\mathbb{R}^d_0$ such that the right-hand side of (1.1) or (1.2) is a locally finite measure on $\mathbb{R}^d_0$. We can characterize the domains by behaviors of measures near or far from the origin. Then the effect of $\mathcal{A}^{q,r}_{\alpha,p}$ and $\Upsilon_{\beta,\theta}$ on behaviors of measures will be examined. This makes it possible to determine the domains of $\Upsilon_{\beta,\theta}\mathcal{A}^{q,r}_{\alpha,p}$ and $\mathcal{A}^{q,r}_{\alpha,p}\Upsilon_{\beta,\theta}$. The range $\mathfrak{R}(\mathcal{A}^{q,r}_{\alpha,p})$ is described by the use of fractional integral of measures developed in [16] and the range $\mathfrak{R}(\Upsilon_{\beta,\theta})$ is by the notion of complete monotonicity. Together with the one-to-one property, these results constitute Section 2. We will prove in Section 3 the identity (1.3). A necessary and sufficient condition on the parameter values in order that $\mathcal{A}^{q,r}_{\alpha,p}$ and $\Upsilon_{\beta,\theta}$ commute is given. Sections 2 and 3 can be seen as a study of properties of functions expressible as fractional (or Riemann–Liouville) integrals of measures, which are not yet well understood.

Let $I(\mathbb{R}^d)$ be the class of infinitely divisible distributions on $\mathbb{R}^d$. Let $A^q_{\alpha,p}(\mathbb{R}^d)$ be the class of distributions $\mu \in I(\mathbb{R}^d)$ such that the Lévy measure of μ belongs to $\mathfrak{R}(\mathcal{A}^{q,r}_{\alpha,p})$, which is shown to be independent of r. Section 4 is devoted to the study of the class $A^q_{\alpha,p}(\mathbb{R}^d)$. The central problem is whether it is expressible as the range of an improper stochastic integral mapping. For a function f locally square-integrable on $[0,\infty)$ and for a Lévy process $\{X^{(\mu)}_s\colon s \geqslant 0\}$ on $\mathbb{R}^d$ with distribution μ at time $s = 1$, the improper stochastic integral $\lim_{t\to\infty} \int_0^t f(s)dX_s^{(\mu)}$, or $\int_0^{\infty-} f(s)dX_s^{(\mu)}$, is defined whenever the limit exists in the sense of convergence in probability, and its distribution is denoted by $\Phi_f(\mu)$. The transformation from the Lévy measure ν of μ to the Lévy measure $\widetilde{\nu}$ of $\widetilde{\mu} = \Phi_f(\mu)$ is expressed by

$$\widetilde{\nu}(B) = \int_0^\infty ds \int_{\mathbb{R}_0^d} 1_B\big(f(s)x\big)\nu(dx), \qquad B \in \mathcal{B}(\mathbb{R}_0^d). \tag{1.4}$$

This is written as

$$\widetilde{\nu}(B) = \int_{\mathbb{R}\setminus\{0\}} \tau(du) \int_{\mathbb{R}_0^d} 1_B(ux)\nu(dx), \qquad B \in \mathcal{B}(\mathbb{R}_0^d), \tag{1.5}$$

for some measure τ on $\mathbb{R} \setminus \{0\}$. We call the transformation $\nu \mapsto \widetilde{\nu}$ in (1.5) the *Upsilon transformation* associated with τ, denoted by $\Upsilon^{(\tau)}$. This is an extension of the Upsilon transformation studied by [5] (see also [3]), where τ is a measure on $(0, \infty)$. Further we introduce a transformation $\rho \mapsto \Upsilon^{(\tau,b)}\rho$ for $b \in \mathbb{R}$ by

$$(\Upsilon^{(\tau,b)}\rho)(B) = \int_{\mathbb{R}\setminus\{0\}} \tau(du) \int_{\mathbb{R}_0^d} 1_B(ux)\, |x|^b \rho(dx), \qquad B \in \mathcal{B}(\mathbb{R}_0^d), \tag{1.6}$$

and call it the *generalized Upsilon transformation* associated with τ and b. We will give a necessary and sufficient condition, in terms of the parameter values, for $\mathcal{A}_{\alpha,p}^{q,r}$ to be equal to $\Upsilon^{(\tau)}$ with some τ, or to $\Upsilon^{(\tau,b)}$ with some τ and b. Then we will show that the class $A_{\alpha,p}^q(\mathbb{R}^d)$ can be expressed as the range of some Φ_f if $-\infty < \alpha < 1$ and as its slight extension called the range of essentially definable improper stochastic integrals with respect to Lévy processes if $1 \leqslant \alpha < 2$. The relation of $A_{\alpha,p}^q(\mathbb{R}^d)$ to the class $L(\mathbb{R}^d)$ of selfdecomposable distributions is given at the end of Section 4.

Let us mention related works. Special cases of the three-parameter class $A_{\alpha,p}^q(\mathbb{R}^d)$ appear in several papers. The class $A_{\alpha,p}^1(\mathbb{R}^d)$ is treated in Sato [16] in the notation $K_{p,\alpha}$ for $-\infty < \alpha < 1$ and $K_{p,\alpha}^{\mathrm{e}}$ for $1 \leqslant \alpha < 2$. It is deeply analyzed in [14] and [16] within the framework of the study of the ranges of a class of explicitly given stochastic integral mappings Φ_f related to extensions of selfdecomposability. The transformation $\mathcal{A}_{-1,1/2}^{2,1}$ appears also in Arizmendi et al. [2] in relation to the study of free type G distributions. Lévy measures of distributions in $A_{\alpha,p}^q(\mathbb{R}^d)$ extend those of the tempered stable distributions of Rosiński [12], where the fractional integral is replaced by completely monotone functions vanishing at infinity.

The original *Upsilon mapping* is the mapping Φ_f with $f(s) = 1_{(0,1]}(s) \log s^{-1}$ introduced by Barndorff-Nielsen and Thorbjørnsen [6], [7] as a connection between free and classical infinitely divisible distributions. The transformation of Lévy measures associated with this mapping is equal to $\Upsilon_{-1,1}$. The original Upsilon mapping is further studied in [4] and generalized in [5]. The transformation

$\Upsilon_{\beta,1}$ is the transformation of Lévy measures associated with some Φ_f given in [14] and [16]. This Φ_f is denoted by Ψ_β and its range is known. If $0 < \beta < 2$ and Gaussian part is absent, then this range is exactly equal to the class of tempered stable distributions of Rosiński [12]. He made a deep study of probabilistic properties of those distributions. The two-parameter $\Upsilon_{\beta,\theta}$ was introduced and analyzed in [8]. A special attention was given by [1] to $\Upsilon_{-\theta,\theta}$.

2. TRANSFORMATIONS $\mathcal{A}_{\alpha,p}^{q,r}$ AND $\Upsilon_{\beta,\theta}$

2.1. Definitions. A measure ρ on $\mathbb{R}_0^d$ is said to be *locally finite* on $\mathbb{R}_0^d$ if $\rho(\{x\colon a \leqslant |x| \leqslant b\}) < \infty$ for $0 < a < b < \infty$. Let $\mathfrak{M}_{lf}$ be the class of locally finite measures on $\mathbb{R}_0^d$.

DEFINITION 2.1. Let $p,\ q,\ r,\ \theta > 0$ and $\alpha,\ \beta \in \mathbb{R}$. Given a measure ρ on $\mathbb{R}_0^d$, let $\widetilde{\rho}$ be the measure on $\mathbb{R}_0^d$ defined by the right-hand side of (1.1) (resp. (1.2)). Let $\mathfrak{D}(\mathcal{A}_{\alpha,p}^{q,r})$ (resp. $\mathfrak{D}(\Upsilon_{\beta,\theta})$) be the class of $\rho \in \mathfrak{M}_{lf}$ such that $\widetilde{\rho} \in \mathfrak{M}_{lf}$. For $\rho \in \mathfrak{D}(\mathcal{A}_{\alpha,p}^{q,r})$ (resp. $\mathfrak{D}(\Upsilon_{\beta,\theta})$), define $\mathcal{A}_{\alpha,p}^{q,r}\rho = \widetilde{\rho}$ (resp. $\Upsilon_{\beta,\theta}\,\rho = \widetilde{\rho}$).

For $a,\ b \in \mathbb{R}$ and $\theta > 0$, let

$$\mathfrak{M}_\infty^a = \Big\{\rho \in \mathfrak{M}_{lf}\colon \int_{|x|>1} |x|^a \rho(dx) < \infty\Big\},$$
$$\mathfrak{M}_0^b = \Big\{\rho \in \mathfrak{M}_{lf}\colon \int_{|x|\leqslant 1} |x|^b \rho(dx) < \infty\Big\},$$
$$\mathfrak{M}_0^{\infty,\theta} = \Big\{\rho \in \mathfrak{M}_{lf}\colon \int_{|x|\leqslant 1} e^{-c|x|^{-\theta}} \rho(dx) < \infty \text{ for all } c > 0\Big\}.$$

If $a < a'$, then $\mathfrak{M}_\infty^a \supset \mathfrak{M}_\infty^{a'}$. If $b < b'$ and $0 < \theta < \theta'$, then

$$\mathfrak{M}_0^b \subset \mathfrak{M}_0^{b'} \subset \mathfrak{M}_0^{\infty,\theta} \subset \mathfrak{M}_0^{\infty,\theta'}.$$

Let $\mathbb{S} = \{\xi \in \mathbb{R}^d\colon |\xi| = 1\}$, the unit sphere in $\mathbb{R}^d$ if $d \geqslant 2$ and the two-point set $\{-1, 1\}$ if $d = 1$. A family $\{\rho_\xi\colon \xi \in \mathbb{S}\}$ of measures on $(0,\infty)$ is called a *measurable family* if $\rho_\xi(E)$ is measurable in $\xi \in \mathbb{S}$ for every $E \in \mathcal{B}\big((0,\infty)\big)$. If ρ is a σ-finite measure on $\mathbb{R}^d$ satisfying $\rho(\{0\}) = 0$, then there are a σ-finite measure λ on $\mathbb{S}$ with $\lambda(\mathbb{S}) \geqslant 0$ and a measurable family $\{\rho_\xi\colon \xi \in \mathbb{S}\}$ of σ-finite measures on $(0,\infty)$ with $\rho_\xi\big((0,\infty)\big) > 0$ such that

$$\rho(B) = \int_{\mathbb{S}} \lambda(d\xi) \int_{(0,\infty)} 1_B(u\xi)\rho_\xi(du), \qquad B \in \mathcal{B}(\mathbb{R}_0^d).$$

We call (λ, ρ_ξ) a *polar decomposition* of ρ; λ is called the *spherical component* of ρ, and ρ_ξ the *radial component*.

If $\rho \in \mathfrak{M}_{lf}$ with a polar decomposition (λ, ρ_ξ) and if $\widetilde{\rho}$ is the right-hand side of (1.1) or (1.2), respectively, then

$$\widetilde{\rho}(B) = c_p \int_{\mathbb{S}} \lambda(d\xi) \int_0^\infty 1_B(u\xi) u^{-\alpha-1} du \int_{(0,\infty)} (s^r - u^q)_+^{p-1} \rho_\xi(ds), \tag{2.1}$$

$$\widetilde{\rho}(B) = \int_{\mathbb{S}} \lambda(d\xi) \int_0^\infty \theta t^{-\beta-1} e^{-t^\theta} dt \int_{(0,\infty)} 1_B(ts\xi) \rho_\xi(ds), \tag{2.2}$$

respectively.

2.2. Domains. Let us describe the domains of $\mathcal{A}_{\alpha,p}^{q,r}$ and $\Upsilon_{\beta,\theta}$.

THEOREM 2.1. $\mathfrak{D}(\mathcal{A}_{\alpha,p}^{q,r}) = \mathfrak{M}_\infty^{r(p-1)}$.

Proof. It follows from (2.1) that $\mathfrak{D}(\mathcal{A}_{\alpha,p}^{q,r})$ does not depend on α nor q. Let $\rho \in \mathfrak{M}_{lf}$. Let $\widetilde{\rho}$ be as in (2.1) with $\alpha = -1$ and $q = 1$. Let $0 < a < b < \infty$. Then

$$\int_{a<|x|\leqslant b} \widetilde{\rho}(dx) = c_p \int_{\mathbb{S}} \lambda(d\xi) \int_{(b^{1/r},\infty)} \rho_\xi(ds) \int_a^b (s^r - u)^{p-1} du$$
$$+ c_p \int_{\mathbb{S}} \lambda(d\xi) \int_{(a^{1/r},b^{1/r}]} \rho_\xi(ds) \int_a^{s^r} (s^r - u)^{p-1} du =: I_1 + I_2.$$

Since $\int_a^b (s^r - u)^{p-1} du \asymp s^{r(p-1)}$ as $s \to \infty$, I_1 is bounded by a constant multiple of $\int_{\mathbb{S}} \lambda(d\xi) \int_{(b^{1/r},\infty)} s^{r(p-1)} \rho_\xi(ds)$. The term I_2 is always finite. It follows that if $\rho \in \mathfrak{M}_\infty^{r(p-1)}$, then $\widetilde{\rho} \in \mathfrak{M}_{lf}$. Similarly we can prove the converse. ∎

THEOREM 2.2. $\mathfrak{D}(\Upsilon_{\beta,\theta}) = \mathfrak{M}_\infty^\beta \cap \mathfrak{M}_0^{\infty,\theta}$.

Proof. Use (2.2). Note that, for $0 < a < b < \infty$,

$$\int_{a/u}^{b/u} t^{-\beta-1} e^{-t^\theta} dt \begin{cases} \sim (1/\beta)(a^{-\beta} - b^{-\beta}) u^\beta, & \beta \neq 0, \\ \to \log(b/a), & \beta = 0, \end{cases} \quad \text{as } u \to \infty,$$
$$\int_{a/u}^{b/u} t^{-\beta-1} e^{-t^\theta} dt \sim (1/\theta)(a/u)^{-\beta-\theta} e^{-(a/u)^\theta}, \quad \beta \in \mathbb{R}, \quad \text{as } u \downarrow 0.$$

The assertion follows from this. ∎

2.3. Effect on behaviors of measures. Comparison of the behaviors of $\mathcal{A}_{\alpha,p}^{q,r}\rho$ and $\Upsilon_{\beta,\theta}\rho$ with those of ρ is important. The following two theorems are basic in our discussion.

THEOREM 2.3. *Suppose that* $\rho \in \mathfrak{D}(\mathcal{A}^{q,r}_{\alpha,p})$.

(i) *Let* $b \in (\alpha, \infty)$. *Then* $\mathcal{A}^{q,r}_{\alpha,p}\, \rho \in \mathfrak{M}^b_\infty$ *if and only if* $\rho \in \mathfrak{M}^{r(p-1+(b-\alpha)/q)}_\infty$.

(ii) $\mathcal{A}^{q,r}_{\alpha,p}\, \rho \in \mathfrak{M}^\alpha_\infty$ *if and only if* $\int_{|x|>1} |x|^{r(p-1)} \log |x|\, \rho(dx) < \infty$.

(iii) *For any* $b \in (-\infty, \alpha)$, $\mathcal{A}^{q,r}_{\alpha,p}\, \rho \in \mathfrak{M}^b_\infty$.

(iv) *Let* $c \in (\alpha, \infty)$. *Then* $\mathcal{A}^{q,r}_{\alpha,p}\, \rho \in \mathfrak{M}^c_0$ *if and only if* $\rho \in \mathfrak{M}^{r(p-1+(c-\alpha)/q)}_0$.

(v) *Let* $c \in (-\infty, \alpha]$. *Then* $\mathcal{A}^{q,r}_{\alpha,p}\, \rho \notin \mathfrak{M}^c_0$ *for any* $\rho \neq 0$.

(vi) *Let* $\theta > 0$. *Then* $\mathcal{A}^{q,r}_{\alpha,p}\rho \in \mathfrak{M}^{\infty,\theta}_0$ *if and only if* $\rho \in \mathfrak{M}^{\infty,\theta r/q}_0$.

Proof. We have

$$\int_{|x|>1} |x|^b \mathcal{A}^{q,r}_{\alpha,p}\rho(dx) = c_p q^{-1} \int_{\mathbb{S}} \lambda(d\xi) \int_{(1,\infty)} s^{b'} \rho_\xi(ds) \int_{s^{-r}}^{1} (1-v)^{p-1} v^{(b-\alpha)/q-1} dv$$

with $b' = r\big(p - 1 + (b-\alpha)/q\big)$. Since $\int_0^1 (1-v)^{p-1} v^{(b-\alpha)/q-1} dv$ is finite for $b > \alpha$, (i) is true. Letting $b = \alpha$, we obtain (ii), since $\int_{s^{-r}}^1 (1-v)^{p-1} v^{-1} dv \sim r \log s$ as $s \to \infty$. Letting $b < \alpha$, we have

$$\int_{s^{-r}}^{1} (1-v)^{p-1} v^{(b-\alpha)/q-1} dv \sim \big(-q/(b-\alpha)\big) s^{-(b-\alpha)r/q} \quad \text{as } s \to \infty$$

and we obtain (iii) by Theorem 2.1. We notice that

$$\int_{|x|\leqslant 1} |x|^c \mathcal{A}^{q,r}_{\alpha,p}\, \rho(dx) =: I_1 + I_2,$$

where with $c' = r\big(p - 1 + (c-\alpha)/q\big)$

$$I_1 = c_p q^{-1} \int_{\mathbb{S}} \lambda(d\xi) \int_{(1,\infty)} s^{c'} \rho_\xi(ds) \int_0^{s^{-r}} (1-v)^{p-1} v^{(c-\alpha)/q-1} dv,$$

$$I_2 = c_p q^{-1} \int_{\mathbb{S}} \lambda(d\xi) \int_{(0,1]} s^{c'} \rho_\xi(ds) \int_0^1 (1-v)^{p-1} v^{(c-\alpha)/q-1} dv.$$

If $c > \alpha$, then I_1 is finite, since $\int_0^{s^{-r}} (1-v)^{p-1} v^{(c-\alpha)/q-1} dv \asymp s^{-r(c-\alpha)/q}$ as $s \to \infty$ and $\rho \in \mathfrak{M}^{r(p-1)}_\infty$. Hence (iv) follows. If $c \leqslant \alpha$, then we see that $I_1 = \infty$ when $\rho(\{|x| > 1\}) > 0$ and $I_2 = \infty$ when $\rho(\{|x| \leqslant 1\}) > 0$, which concludes (v).

Let us prove (vi). We have $\mathcal{A}^{q,r}_{\alpha,p}\rho \in \mathfrak{M}^{\infty,\theta}_0$ if and only if $\mathcal{A}^{q,r}_{-1,p}\rho \in \mathfrak{M}^{\infty,\theta}_0$. Therefore, it is enough to show (iv) for $\alpha = -1$. We have, for any $a > 0$,

$$\int_{|x|\leqslant 1} e^{-a|x|^{-\theta}} \mathcal{A}^{q,r}_{-1,p}\, \rho(dx) =: I_1 + I_2,$$

where

$$I_1 = c_p \int_{\mathbb{S}} \lambda(d\xi) \int_{(1,\infty)} s^{r(p-1)} \rho_\xi(ds) \int_0^1 (1 - s^{-r}v)^{p-1} e^{-av^{-\theta/q}} q^{-1} v^{1/q-1} dv,$$

which is finite, since $\rho \in \mathfrak{M}_\infty^{r(p-1)}$, and

$$I_2 = c_p \int_{\mathbb{S}} \lambda(d\xi) \int_{(0,1]} q^{-1} s^{r(p-1+1/q)} \rho_\xi(ds) \int_0^1 (1 - v)^{p-1} e^{-a(s^r v)^{-\theta/q}} v^{1/q-1} dv.$$

Note that

$$\int_0^1 (1 - v)^{p-1} e^{-a(s^r v)^{-\theta/q}} v^{1/q-1} dv = o(e^{-as^{-\theta r/q}}), \qquad s \downarrow 0.$$

If $\rho \in \mathfrak{M}_0^{\infty,\theta r/q}$, then $I_2 < \infty$ and $\mathcal{A}_{-1,p}^{q,r}\rho \in \mathfrak{M}_0^{\infty,\theta}$. Conversely, suppose that $\rho \notin \mathfrak{M}_0^{\infty,\theta r/q}$. Then, for some $a_0 > 0$, $\int_{|x|\leqslant 1} e^{-a_0|x|^{-\theta r/q}} \rho(dx) = \infty$. Choose a such that $a2^{\theta/q} = a_0/2$. Then $av^{-\theta/q} < a_0/2$ for $v > 1/2$ and

$$I_2 \geqslant \int_{\mathbb{S}} \lambda(d\xi) \int_{(0,1]} q^{-1} s^{r(p-1+1/q)} e^{-(a_0/2)s^{-\theta r/q}} \rho_\xi(ds) \int_{1/2}^1 c_p (1 - v)^{p-1} v^{1/q-1} dv.$$

There is $s_0 \in (0, 1]$ such that $s^{r(p-1+1/q)} e^{-(a_0/2)s^{-\theta r/q}} \geqslant e^{-a_0 s^{-\theta r/q}}$ for all $s \in (0, s_0]$. Thus

$$I_2 \geqslant \text{const} \int_{|x|\leqslant s_0} e^{-a_0|x|^{-\theta r/q}} \rho(dx) = \infty.$$

Hence $\int_{|x|\leqslant 1} e^{-a|x|^{-\theta}} \mathcal{A}_{-1,p}^{q,r}\, \rho(dx) = \infty$. This means that $\mathcal{A}_{-1,p}^{q,r}\rho \notin \mathfrak{M}_0^{\infty,\theta}$. ■

THEOREM 2.4. *Let $\rho \in \mathfrak{D}(\Upsilon_{\beta,\theta})$.*

(i) *Let $b \in (\beta, \infty)$. Then $\Upsilon_{\beta,\theta}\, \rho \in \mathfrak{M}_\infty^b$ if and only if $\rho \in \mathfrak{M}_\infty^b$.*

(ii) *$\Upsilon_{\beta,\theta}\, \rho \in \mathfrak{M}_\infty^\beta$ if and only if $\int_{|x|>1} |x|^\beta \log |x|\, \rho(dx) < \infty$.*

(iii) *For any $b \in (-\infty, \beta)$, $\Upsilon_{\beta,\theta}\, \rho \in \mathfrak{M}_\infty^b$.*

(iv) *Let $c \in (\beta, \infty)$. Then $\Upsilon_{\beta,\theta}\, \rho \in \mathfrak{M}_0^c$ if and only if $\rho \in \mathfrak{M}_0^c$.*

(v) *Let $c \in (-\infty, \beta]$. Then $\Upsilon_{\beta,\theta}\, \rho \notin \mathfrak{M}_0^c$ for any $\rho \neq 0$.*

P r o o f. For any $b \in \mathbb{R}$,

$$\int_{|y|>1} |y|^b\, \Upsilon_{\beta,\theta}\, \rho(dy) = \Big(\int_{|x|\leqslant 1} + \int_{|x|>1} \Big) |x|^b \rho(dx) \int_{1/|x|}^\infty \theta t^{b-\beta-1} e^{-t^\theta} dt$$
$$=: I_1 + I_2,$$

where $I_1 < \infty$ from $\rho \in \mathfrak{M}_0^{\infty,\theta}$, since

$$\int_{1/u}^{\infty} \theta t^{b-\beta-1} e^{-t^\theta} dt \sim u^{\beta-b+\theta} e^{-u^{-\theta}} \quad \text{as } u \downarrow 0.$$

If $b > \beta$, then $I_2 < \infty$ is equivalent to $\int_{|x|>1} |x|^b \rho(dx) < \infty$, since

$$\int_0^{\infty} t^{b-\beta-1} e^{-t^\theta} dt < \infty.$$

If $b = \beta$, then $I_2 < \infty$ is equivalent to $\int_{|x|>1} |x|^b \log|x| \rho(dx) < \infty$, since

$$\int_{1/u}^{\infty} t^{-1} e^{-t^\theta} dt \sim \log u \quad \text{as } u \to \infty.$$

If $b < \beta$, then $I_2 < \infty$ follows from $\rho \in \mathfrak{M}_\infty^\beta$, since

$$\int_{1/u}^{\infty} t^{b-\beta-1} e^{-t^\theta} dt \sim (\beta - b)^{-1} u^{\beta-b} \quad \text{as } u \to \infty.$$

Hence we obtain (i)–(iii). For any $c \in \mathbb{R}$,

$$\int_{|y|\leqslant 1} |y|^c \, \Upsilon_{\beta,\theta}\, \rho(dy) = \int_{\mathbb{R}_0^d} |x|^c \rho(dx) \int_0^{1/|x|} \theta t^{c-\beta-1} e^{-t^\theta} dt.$$

If $c > \beta$, then $\int_0^{1/u} t^{c-\beta-1} e^{-t^\theta} dt \sim (c-\beta)^{-1} u^{\beta-c}$ as $u \to \infty$ and we obtain (iv), using $\rho \in \mathfrak{M}_\infty^\beta$. For $c \leqslant \beta$ we obtain (v), since $\int_0^{1/u} t^{c-\beta-1} e^{-t^\theta} dt = \infty$ for all $u > 0$. ■

Theorems 2.1, 2.2, 2.3, and 2.4 combined enable us to describe the domains of $\Upsilon_{\beta,\theta} \mathcal{A}_{\alpha,p}^{q,r}$ and $\mathcal{A}_{\alpha,p}^{q,r} \Upsilon_{\beta,\theta}$. See Section 3.

2.4. Ranges.

THEOREM 2.5. $\mathfrak{R}(\mathcal{A}_{\alpha,p}^{q,r})$ *does not depend on* r.

P r o o f. Let $r' > 0$. Given ρ, define $\rho^\sharp$ by

$$\rho^\sharp(B) = \int_{\mathbb{R}_0^d} 1_B(|x|^{(r/r')-1} x) \rho(dx).$$

Then, we can see that $\rho \in \mathfrak{D}(\mathcal{A}_{\alpha,p}^{q,r})$ if and only if $\rho^\sharp \in \mathfrak{D}(\mathcal{A}_{\alpha,p}^{q,r'})$ and that, in this case, $\mathcal{A}_{\alpha,p}^{q,r} \rho = \mathcal{A}_{\alpha,p}^{q,r'} \rho^\sharp$. Hence $\mathfrak{R}(\mathcal{A}_{\alpha,p}^{q,r}) = \mathfrak{R}(\mathcal{A}_{\alpha,p}^{q,r'})$. ■

REMARK 2.1. *Let* $p=1$. *By Theorem* 2.5, $\mathfrak{R}(\mathcal{A}_{\alpha,1}^{q,r})=\mathfrak{R}(\mathcal{A}_{\alpha,1}^{q,q})$. *By* (1.1), $\mathcal{A}_{\alpha,1}^{q,q}$ *does not depend on* q. *Thus*, $A_{\alpha,1}^{q}(\mathbb{R}^d)$ *does not depend on* q.

We use the fractional integral mapping I_+^p of order $p>0$ in the notation of [16]. For a measure σ on $(0,\infty)$ let

$$\widetilde{\sigma}(du)=\big(c_p\int_{(u,\infty)}(s-u)^{p-1}\sigma(ds)\big)du.$$

Let $\mathfrak{D}(I_+^p)$ be the class of locally finite measures σ on $(0,\infty)$ such that $\widetilde{\sigma}$ is locally finite on $(0,\infty)$. For $\sigma\in\mathfrak{D}(I_+^p)$ we define $I_+^p\sigma=\widetilde{\sigma}$. Properties of I_+^p are studied in [16]. One of them is that a locally finite measure σ on $(0,\infty)$ is in $\mathfrak{D}(I_+^p)$ if and only if $\int_{(1,\infty)}u^{p-1}\sigma(du)<\infty$. Using the mapping I_+^p, we can show the following description of $\mathfrak{R}(\mathcal{A}_{\alpha,p}^{q,r})$. This gives an alternative proof of Theorem 2.5.

PROPOSITION 2.1. *A measure* $\widetilde{\rho}$ *is in* $\mathfrak{R}(\mathcal{A}_{\alpha,p}^{q,r})$ *if and only if* $\widetilde{\rho}\in\mathfrak{M}_{lf}$ *and there are a measure* λ *on* $\mathbb{S}$ *and a measurable family* $\{\sigma_\xi\colon\xi\in\mathbb{S}\}$ *of measures on* $(0,\infty)$ *such that* $\sigma_\xi\in\mathfrak{D}(I_+^p)$ *for* λ*-a.e.* ξ *and*

$$\widetilde{\rho}(B)=\int_{\mathbb{S}}\lambda(d\xi)\int_{(0,\infty)}1_B(u^{1/q}\xi)u^{-\alpha/q-1}(I_+^p\sigma_\xi)(du)\tag{2.3}$$

for $B\in\mathcal{B}(\mathbb{R}_0^d)$.

P r o o f. Suppose that $\rho\in\mathfrak{D}(\mathcal{A}_{\alpha,p}^{q,r})$ with a polar decomposition (λ,ρ_ξ) and $\widetilde{\rho}=\mathcal{A}_{\alpha,p}^{q,r}\rho$. Then, it follows from (2.1) that

$$\widetilde{\rho}(B)=c_pq^{-1}\int_{\mathbb{S}}\lambda(d\xi)\int_0^\infty 1_B(u^{1/q}\xi)u^{-\alpha/q-1}du\int_{(0,\infty)}(s-u)_+^{p-1}\rho_\xi^\sharp(ds),$$

where $\int_{(0,\infty)}1_C(s)\rho_\xi^\sharp(ds)=\int_{(0,\infty)}1_C(s^r)\rho_\xi(ds)$ for any C. Since $\rho,\widetilde{\rho}\in\mathfrak{M}_{lf}$, the measures ρ_ξ, $\rho_\xi^\sharp$, and $\big(c_p\int_{(0,\infty)}(s-u)_+^{p-1}\rho_\xi^\sharp(ds)\big)du$ are locally finite on $(0,\infty)$ for λ-a.e. ξ. Hence $\rho_\xi^\sharp\in\mathfrak{D}(I_+^p)$ for λ-a.e. ξ and we obtain (2.3) with $\sigma_\xi=q^{-1}\rho_\xi^\sharp$. Proof of the “if” part is similar. ∎

The following description of $\mathfrak{R}(\Upsilon_{\beta,\theta})$ is essentially given in [8].

PROPOSITION 2.2. *A measure* $\widetilde{\rho}$ *is in* $\mathfrak{R}(\Upsilon_{\beta,\theta})$ *if and only if* $\widetilde{\rho}\in\mathfrak{M}_{lf}$ *with a polar decomposition* $\big(\lambda,u^{-\beta-1}g_\xi(u^\theta)du\big)$, *where* $g_\xi(v)$ *is measurable in* $\xi\in\mathbb{S}$ *and completely monotone and vanishing at* ∞ *in* $v>0$.

2.5. One-to-one property.

THEOREM 2.6. *The transformation* $\mathcal{A}_{\alpha,p}^{q,r}$ *is one-to-one.*

P r o o f. Assume that $\rho, \rho' \in \mathfrak{D}(\mathcal{A}_{\alpha,p}^{q,r})$ and $\mathcal{A}_{\alpha,p}^{q,r}\,\rho = \mathcal{A}_{\alpha,p}^{q,r}\,\rho'$. Let (λ, ρ_ξ) and (λ', ρ'_ξ) be polar decompositions of ρ and ρ', respectively. Then, as in the proof of Proposition 2.1,

$$q^{-1}\int_{\mathbb{S}}\lambda(d\xi)\int_0^\infty 1_B(u^{1/q}\xi)u^{-\alpha/q-1}I_+^p(\rho_\xi^\sharp)(du) = q^{-1}\int_{\mathbb{S}}\lambda'(d\xi)\int_0^\infty 1_B(u^{1/q}\xi)u^{-\alpha/q-1}I_+^p(\rho_\xi'^\sharp)(du)$$

for $B \in \mathcal{B}(\mathbb{R}_0^d)$. Hence, for any $B \in \mathcal{B}(\mathbb{R}_0^d)$,

$$q^{-1}\int_{\mathbb{S}}\lambda(d\xi)\int_0^\infty 1_B(u\xi)u^{-\alpha/q-1}I_+^p(\rho_\xi^\sharp)(du) = q^{-1}\int_{\mathbb{S}}\lambda'(d\xi)\int_0^\infty 1_B(u\xi)u^{-\alpha/q-1}I_+^p(\rho_\xi'^\sharp)(du).$$

It follows that there is a positive finite measurable function $c(\xi)$ such that $\lambda'(d\xi) = c(\xi)\lambda(d\xi)$ and, for λ-a.e. ξ,

$$c(\xi)q^{-1}u^{-\alpha/q-1}I_+^p(\rho_\xi'^\sharp)(du) = q^{-1}u^{-\alpha/q-1}I_+^p(\rho_\xi^\sharp)(du)$$

(see [4] and [16]). Hence, for λ-a.e. ξ, we obtain $c(\xi)\rho_\xi'^\sharp = \rho_\xi^\sharp$ from the one-to-one property of I_0^p in [16], and thus $c(\xi)\rho'_\xi = \rho_\xi$. Hence $\rho = \rho'$. ∎

THEOREM 2.7. *The transformation $\Upsilon_{\beta,\theta}$ is one-to-one.*

P r o o f. Similarly to the above, use the representation of $\Upsilon_{\beta,\theta}\rho$ and the uniqueness of polar decomposition in the sense of [4], [16], and then use the uniqueness theorem in the Laplace transform theory instead of the one-to-one property of I_+^p. ∎

3. NON-COMMUTATIVE RELATIONS OF $\mathcal{A}_{\alpha,p}^{q,r}$ AND $\Upsilon_{\beta,\theta}$

THEOREM 3.1. *The identity*

$$\Upsilon_{\beta,\theta}\mathcal{A}_{\alpha,p}^{q,r} = \mathcal{A}_{\alpha,p}^{q,r}\Upsilon_{\beta',\theta'} \tag{3.1}$$

holds with

$$\beta' = r\big(p-1+(\beta-\alpha)/q\big), \quad \theta' = \theta r/q. \tag{3.2}$$

The common domain of $\Upsilon_{\beta,\theta}\mathcal{A}_{\alpha,p}^{q,r}$ and $\mathcal{A}_{\alpha,p}^{q,r}\Upsilon_{\beta',\theta'}$ equals:

(3.3) $\mathfrak{M}_0^{\infty,\theta'} \cap \mathfrak{M}_\infty^{r(p-1)}$ *if* $r(p-1) > \beta'$,

(3.4) $\big\{\sigma \in \mathfrak{M}_0^{\infty,\theta'} : \int_{|x|>1} |x|^{r(p-1)}\log(1+|x|)\sigma(dx) < \infty\big\}$ *if* $r(p-1) = \beta'$,

(3.5) $\mathfrak{M}_0^{\infty,\theta'} \cap \mathfrak{M}_\infty^{\beta'}$ *if* $r(p-1) < \beta'$.

P r o o f. S t e p 1. Let us show that $\mathfrak{D}(\Upsilon_{\beta,\theta}\mathcal{A}^{q,r}_{\alpha,p}) = \mathfrak{D}(\mathcal{A}^{q,r}_{\alpha,p}\Upsilon_{\beta',\theta'})$ and that it is described as asserted. First, note that $r(p-1) > \beta'$ (resp. $r(p-1) = \beta'$, $< \beta'$) if and only if $\beta < \alpha$ (resp. $\beta = \alpha$, $> \alpha$). If $\sigma \in \mathfrak{D}(\mathcal{A}^{q,r}_{\alpha,p})$, then write $\rho = \mathcal{A}^{q,r}_{\alpha,p}\sigma$. If $\sigma \in \mathfrak{D}(\Upsilon_{\beta',\theta'})$, then write $\eta = \Upsilon_{\beta',\theta'}\sigma$. It follows from Theorems 2.1–2.4 that

$$\begin{aligned}\sigma \in \mathfrak{D}(\Upsilon_{\beta,\theta}\mathcal{A}^{q,r}_{\alpha,p}) &\Leftrightarrow \sigma \in \mathfrak{D}(\mathcal{A}^{q,r}_{\alpha,p}),\ \rho \in \mathfrak{D}(\Upsilon_{\beta,\theta})\\ &\Leftrightarrow \int\limits_{|x|>1} |x|^{r(p-1)}\sigma(dx) + \int\limits_{|x|>1} |x|^{\beta}\rho(dx)\\ &\quad + \int\limits_{|x|\leqslant 1} e^{-a|x|^{-\theta}}\rho(dx) < \infty,\ \forall a>0\\ &\Leftrightarrow \sigma \in \text{(3.3)–(3.5)}\end{aligned}$$

and further

$$\begin{aligned}\sigma \in \mathfrak{D}(\mathcal{A}^{q,r}_{\alpha,p}\Upsilon_{\beta',\theta'}) &\Leftrightarrow \sigma \in \mathfrak{D}(\Upsilon_{\beta',\theta'}),\ \eta \in \mathfrak{D}(\mathcal{A}^{q,r}_{\alpha,p})\\ &\Leftrightarrow \int\limits_{|x|>1} |x|^{\beta'}\sigma(dx) + \int\limits_{|x|\leqslant 1} e^{-a|x|^{-\theta'}}\sigma(dx)\\ &\quad + \int\limits_{|x|>1} |x|^{r(p-1)}\eta(dx) < \infty,\ \forall a>0\\ &\Leftrightarrow \sigma \in \text{(3.3)–(3.5)}.\end{aligned}$$

S t e p 2. Let $\sigma \in \mathfrak{D}(\Upsilon_{\beta,\theta}\mathcal{A}^{q,r}_{\alpha,p}) = \mathfrak{D}(\mathcal{A}^{q,r}_{\alpha,p}\Upsilon_{\beta',\theta'})$. Write $\rho = \mathcal{A}^{q,r}_{\alpha,p}\sigma$. Then $\rho(B)$ equals the right-hand side of (1.1) with σ in place of ρ. Hence it follows from (1.2) that

$$\begin{aligned}&\Upsilon_{\beta,\theta}\rho(B) =\\ &= c_p\int\limits_0^\infty \theta t^{-\beta-1}e^{-t^\theta}dt\int\limits_0^\infty u^{-\alpha-1}du\int\limits_{\mathbb{R}^d_0} 1_B(tux/|x|)(|x|^r-u^q)_+^{p-1}\sigma(dx)\\ &= c_p\int\limits_0^\infty \theta'^{-\beta r/q-1}e^{-s^{\theta'}}ds\int\limits_0^\infty s^{\alpha r/q}v^{-\alpha-1}dv\int\limits_{\mathbb{R}^d_0} 1_B(vx/|x|)(|x|^r-s^{-r}v^q)_+^{p-1}\sigma(dx)\\ &= c_p\int\limits_0^\infty \theta'^{-\beta'-1}e^{-s^{\theta'}}ds\int\limits_{\mathbb{R}^d_0}\sigma(dx)\int\limits_0^\infty v^{-\alpha-1}1_B(vx/|x|)(|sx|^r-v^q)_+^{p-1}dv.\end{aligned}$$

This shows that

$$\begin{aligned}\Upsilon_{\beta,\theta}\rho(B) &= c_p\int\limits_{\mathbb{R}^d_0}\Upsilon_{\beta',\theta'}\sigma(dx)\int\limits_0^\infty v^{-\alpha-1}1_B(vx/|x|)(|x|^r-v^q)_+^{p-1}dv\\ &= (\mathcal{A}^{q,r}_{\alpha,p}\Upsilon_{\beta',\theta'}\sigma)(B),\end{aligned}$$

completing the proof. ■

The application of Theorem 3.1 will be found in the proof of Theorems 4.1 and 4.2.

THEOREM 3.2. $\Upsilon_{\beta,\theta}\mathcal{A}^{q,r}_{\alpha,p} = \mathcal{A}^{q,r}_{\alpha,p}\Upsilon_{\beta,\theta}$ *if and only if*

$$q = r \quad \text{and} \quad q(p-1) - \alpha = 0. \tag{3.6}$$

Proof. Define β' and θ' by (3.2). Notice that (3.6) is equivalent to $\beta' = \beta$ and $\theta' = \theta$. Thus the "if" part is a consequence of Theorem 3.1. To show the "only if" part, suppose that $\Upsilon_{\beta,\theta}\mathcal{A}^{q,r}_{\alpha,p} = \mathcal{A}^{q,r}_{\alpha,p}\Upsilon_{\beta,\theta}$. Since we have (3.1) with (3.2), it follows from Theorem 2.6 that $\Upsilon_{\beta',\theta'}\sigma = \Upsilon_{\beta,\theta}\sigma$ for all $\sigma \in \mathfrak{D}(\Upsilon_{\beta,\theta}\mathcal{A}^{q,r}_{\alpha,p})$. As the delta distributions belong to this domain, we have, for $e_1 = (1, 0, \ldots, 0)$,

$$\int_0^\infty \theta' t^{-\beta'-1} e^{-t^{\theta'}} 1_B(te_1)dt = \int_0^\infty \theta t^{-\beta-1} e^{-t^{\theta}} 1_B(te_1)dt.$$

Hence for any $c > \beta' \vee \beta$ we have $\int_0^\infty \theta' t^{c-\beta'-1} e^{-t^{\theta'}} dt = \int_0^\infty \theta t^{c-\beta-1} e^{-t^{\theta}} dt$, that is, $\Gamma\big((c-\beta')/\theta'\big) = \Gamma\big((c-\beta)/\theta\big)$. Hence $(c-\beta')/\theta' = (c-\beta)/\theta$ for any c satisfying $(c-\beta')/\theta' > 2$ and $(c-\beta)/\theta > 2$. Therefore $\theta' = \theta$ and $\beta' = \beta$. ■

It is noteworthy that condition (3.6) does not depend on β and θ.

4. CLASS $A^q_{\alpha,p}(\mathbb{R}^d)$

4.1. Definitions. As in Section 1, let $A^q_{\alpha,p}(\mathbb{R}^d)$ be the class of distributions $\mu \in I(\mathbb{R}^d)$ such that the Lévy measure of μ belongs to $\mathfrak{R}(\mathcal{A}^{q,r}_{\alpha,p})$. Let us recall that $\mathfrak{R}(\mathcal{A}^{q,r}_{\alpha,p})$ does not depend on r (Theorem 2.5). We discuss the problem what relation the class $A^q_{\alpha,p}(\mathbb{R}^d)$ has with (improper) stochastic integrals of non-random functions with respect to Lévy processes. A related problem is whether $\nu \mapsto \mathcal{A}^{q,r}_{\alpha,p}\nu$ represents the transformation of Lévy measures associated with an (improper) stochastic integral.

Any $\mu \in I(\mathbb{R}^d)$ is uniquely represented by the Lévy–Khintchine triplet (Σ, ν, γ). By this we mean that the characteristic function $\widehat{\mu}(z)$, $z \in \mathbb{R}^d$, of μ is expressed as

$$\widehat{\mu}(z) = \exp\left[-\frac{1}{2}\langle \Sigma z, z\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z,x\rangle}{1+|x|^2}\right)\nu(dx) + i\langle \gamma, z\rangle\right],$$

where Σ is the covariance matrix of the Gaussian part of μ, ν is the Lévy measure of μ, and γ is the location parameter of μ. Let $\mathfrak{M}_L(\mathbb{R}^d)$ denote the class of Lévy measures of all $\mu \in I(\mathbb{R}^d)$. That is, $\mathfrak{M}_L(\mathbb{R}^d) = \mathfrak{M}^2_0 \cap \mathfrak{M}^0_\infty$.

Let $\mu \in I(\mathbb{R}^d)$ with triplet (Σ, ν, γ). Let $f(s)$ be a locally square-integrable function on $[0, \infty)$. If $t \in (0, \infty)$, then the stochastic integral $\int_0^t f(s)dX^{(\mu)}_s$ with respect to a Lévy process $\{X_s^{(\mu)}\}$ is definable for any μ and its distribution is infinitely divisible with triplet $(\widetilde{\Sigma}_t, \widetilde{\nu}_t, \widetilde{\gamma}_t)$ expressed as

$$\widetilde{\Sigma}_t = \int_0^t f(s)^2 \Sigma \, ds, \tag{4.1}$$

$$\widetilde{\nu}_t(B) = \int_0^t ds \int_{\mathbb{R}^d} 1_B\big(f(s)x\big)\, \nu(dx), \qquad B \in \mathcal{B}(\mathbb{R}_0^d), \tag{4.2}$$

$$\widetilde{\gamma}_t = \int_0^t f(s)ds \left[\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right) \nu(dx)\right]. \tag{4.3}$$

The improper stochastic integral $\int_0^{\infty-} f(s)dX_s^{(\mu)}$ and its distribution $\Phi_f(\mu)$ are defined as in Section 1. The domain $\mathfrak{D}(\Phi_f)$ is the class of $\mu \in I(\mathbb{R}^d)$ such that the improper stochastic integral is definable. If $\mu \in \mathfrak{D}(\Phi_f)$, then $\Phi_f(\mu)$ is infinitely divisible with triplet $(\widetilde{\Sigma}, \widetilde{\nu}, \widetilde{\gamma})$ expressed as the limit of $(\widetilde{\Sigma}_t, \widetilde{\nu}_t, \widetilde{\gamma}_t)$ as $t \to \infty$. In particular, $\widetilde{\nu}$ is expressed by (1.4). We also use the following modification of improper stochastic integrals. We say that $\int_0^{\infty-} f(s)dX_s^{(\mu)}$ is *essentially definable* if, for some $\mathbb{R}^d$-valued function $h(t)$ on $[0,\infty)$, $\int_0^t f(s)dX_s^{(\mu)} - h(t)$ is convergent in probability as $t \to \infty$. The class of μ such that $\int_0^{\infty-} f(s)dX_s^{(\mu)}$ is essentially definable is called the *essential domain* of Φ_f and denoted by $\mathfrak{D}^{\mathrm{e}}(\Phi_f)$. The class of the resulting limit distributions is called the *essential range* of Φ_f and denoted by $\mathfrak{R}^{\mathrm{e}}(\Phi_f)$. See [13], [14], and [16]. If we define the τ*-measure* of f by $\tau(E) = \int_0^\infty 1_E\big(f(s)\big)ds$ for $E \in \mathcal{B}(\mathbb{R})$ as in [15], then $\tau(\{0\})$ is irrelevant, and (1.4) is written as (1.5), the right-hand side of which is denoted by $\Upsilon^{(\tau)}\nu$. The domain $\mathfrak{D}(\Upsilon^{(\tau)})$ is understood to be the class of $\nu \in \mathfrak{M}_{lf}$ such that the right-hand side of (1.5) is a measure in $\mathfrak{M}_{lf}$. The generalized Upsilon transformation associated with τ and $b \in \mathbb{R}$ is defined by (1.6) as in Section 1. Again the domain $\mathfrak{D}(\Upsilon^{(\tau,b)})$ is understood to be the class of $\rho \in \mathfrak{M}_{lf}$ such that the right-hand side of (1.6) is in $\mathfrak{M}_{lf}$. The definition of $\Upsilon_{\beta,\theta}$ shows that it is an Upsilon transformation associated with some τ.

4.2. $\mathcal{A}_{\alpha,p}^{q,r}$ and $\Upsilon_{\beta,\theta}$ producing Lévy measures. Theorems 2.3 and 2.4 imply the following two propositions.

PROPOSITION 4.1. *The class* $\{\rho \in \mathfrak{D}(\mathcal{A}_{\alpha,p}^{q,r})\colon \mathcal{A}_{\alpha,p}^{q,r}\rho \in \mathfrak{M}_L(\mathbb{R}^d)\}$ *equals*:

$\mathfrak{M}_0^{r(p-1+(2-\alpha)/q)} \cap \mathfrak{M}_\infty^{r(p-1-\alpha/q)}$ *if* $\alpha < 0$,

$\big\{\rho \in \mathfrak{M}_0^{r(p-1+2/q)}\colon \int_{|x|>1} |x|^{r(p-1)} \log|x| \rho(dx) < \infty\big\}$ *if* $\alpha = 0$,

$\mathfrak{M}_0^{r(p-1+(2-\alpha)/q)} \cap \mathfrak{M}_\infty^{r(p-1)}$ *if* $0 < \alpha < 2$,

$\{0\}$ *if* $\alpha \geqslant 2$.

Hence, if $\alpha \geqslant 2$, then the study of $A^q_{\alpha,p}(\mathbb{R}^d)$ is meaningless.

PROPOSITION 4.2. *The class* $\{\rho \in \mathfrak{D}(\Upsilon_{\beta,\theta})\colon \Upsilon_{\beta,\theta}\,\rho \in \mathfrak{M}_L(\mathbb{R}^d)\}$ *equals*:

$$\mathfrak{M}_L(\mathbb{R}^d) \quad \textit{if } \beta < 0,$$

$$\Big\{\rho \in \mathfrak{M}_L(\mathbb{R}^d)\colon \int\limits_{|x|>1} \log|x|\,\rho(dx) < \infty\Big\} \quad \textit{if } \beta = 0,$$

$$\Big\{\rho \in \mathfrak{M}_L(\mathbb{R}^d)\colon \int\limits_{|x|>1} |x|^\beta \rho(dx) < \infty\Big\} \quad \textit{if } 0 < \beta < 2,$$

$$\{0\} \quad \textit{if } \beta \geqslant 2.$$

4.3. The question when $\mathcal{A}^{q,r}_{\alpha,p}$ is a (generalized) Upsilon transformation. This question is answered by using Theorem 3.1 on the non-commutativity with $\Upsilon_{\beta,\theta}$.

THEOREM 4.1. *If* $q = r$, *then* $\mathcal{A}^{q,r}_{\alpha,p}$ *is the generalized Upsilon transformation associated with* $\tau(du) = c_p 1_{(0,1)}(u)(1-u^q)^{p-1}u^{-\alpha-1}du$ *and* $b = q(p-1) - \alpha$. *If* $q \neq r$, *then* $\mathcal{A}^{q,r}_{\alpha,p}$ *is not a generalized Upsilon transformation.*

Proof. Assume $q = r$ and let $\rho \in \mathfrak{D}(\mathcal{A}^{q,q}_{\alpha,p})$. Using (1.1), we have

$$\mathcal{A}^{q,q}_{\alpha,p}\rho(B) = \int\limits_0^1 \tau(du) \int\limits_{\mathbb{R}^d_0} 1_B(ux)|x|^b \rho(dx)$$

for $B \in \mathcal{B}(\mathbb{R}^d_0)$ with τ and b as in the statement of the theorem. Hence we have $\rho \in \mathfrak{D}(\Upsilon^{(\tau,b)})$ and $\mathcal{A}^{q,q}_{\alpha,p}\rho = \Upsilon^{(\tau,b)}\rho$. Similarly, if $\rho \in \mathfrak{D}(\Upsilon^{(\tau,b)})$, then $\rho \in \mathfrak{D}(\mathcal{A}^{q,q}_{\alpha,p})$. This proves the first half of the theorem.

To show the second half, suppose that $\mathcal{A}^{q,r}_{\alpha,p} = \Upsilon^{(\tau,b)}$ with some τ and b. Let $\sigma \in \mathfrak{D}(\Upsilon_{\beta,\theta}\mathcal{A}^{q,r}_{\alpha,p})$ and $\rho = \mathcal{A}^{q,r}_{\alpha,p}\sigma$. Let β' and θ' be as in (3.2). Then, from Theorem 3.1,

$$\begin{aligned}\Upsilon_{\beta,\theta}\rho(B) &= \mathcal{A}^{q,r}_{\alpha,p}\Upsilon_{\beta',\theta'}\sigma(B) = \Upsilon^{(\tau,b)}\Upsilon_{\beta',\theta'}\sigma(B)\\ &= \int\limits_{\mathbb{R}\setminus\{0\}} \tau(du) \int\limits_0^\infty \theta' t^{-\beta'-1}e^{-t^{\theta'}}\,dt \int\limits_{\mathbb{R}^d_0} 1_B(utx)|tx|^b \sigma(dx)\\ &= \Upsilon_{\beta'-b,\theta'}\Upsilon^{(\tau,b)}\sigma(B) = \Upsilon_{\beta'-b,\theta'}\rho(B)\end{aligned}$$

for $B \in \mathcal{B}(\mathbb{R}^d_0)$. Thus

$$\int\limits_0^\infty \theta t^{-\beta-1}e^{-t^\theta}\,dt \int\limits_{\mathbb{R}^d_0} |tx|^c \rho(dx) = \int\limits_0^\infty \theta' t^{b-\beta'-1}e^{-t^{\theta'}}\,dt \int\limits_{\mathbb{R}^d_0} |tx|^c \rho(dx)$$

for all $c \in \mathbb{R}$. Choose $\sigma = \delta_{e_1}$, where $e_1 = (1, 0, \ldots, 0)$. Then $\sigma \in \mathfrak{D}(\Upsilon_{\beta,\theta}\mathcal{A}_{\alpha,p}^{q,r})$ by Theorem 3.1. Let $c > \alpha \vee \beta \vee (\beta' - b)$. Then

$$\rho(B) = \mathcal{A}_{\alpha,p}^{q,r}\delta_{e_1}(B) = c_p \int_0^1 1_B(ue_1)u^{-\alpha-1}(1-u^q)^{p-1}du,$$

$$\int_{\mathbb{R}_0^d} |x|^c \rho(dx) = c_p \int_0^1 u^{c-\alpha-1}(1-u^q)^{p-1}du = \frac{c_p}{q} B\left(\frac{c-\alpha}{q}, p\right) < \infty.$$

Hence $\Gamma\big((c-\beta)/\theta\big) = \Gamma\big((c+b-\beta')/\theta'\big)$. It follows that $(c-\beta)/\theta = (c+b-\beta')/\theta'$ for all large c. Hence $\theta = \theta'$ and $\beta = \beta' - b$. Therefore $q = r$ and $b = q(p-1) - \alpha$. ∎

THEOREM 4.2. *If* (3.6) *holds, then $\mathcal{A}_{\alpha,p}^{q,r}$ is the Upsilon transformation associated with $\tau(du) = c_p 1_{(0,1)}(u)(1-u^q)^{p-1}u^{-\alpha-1}du$. If* (3.6) *does not hold, then $\mathcal{A}_{\alpha,p}^{q,r}$ is not an Upsilon transformation.*

Proof. The first half of the theorem follows immediately from the first half of Theorem 4.1. The second half is proved as in the proof of the second half of Theorem 4.1 with $b = 0$. ∎

4.4. Representation of $A_{\alpha,p}^{q}(\mathbb{R}^d)$ by stochastic integral. The following theorems reduce the study of $A_{\alpha,p}^{q}(\mathbb{R}^d)$ to the study of the range of a stochastic integral mapping, usual or essentially defined.

In view of Theorems 4.1 and 4.2, it is natural to use the function

$$g(t) = c_p \int_t^1 (1-u^q)^{p-1}u^{-\alpha-1}du, \qquad 0 \leqslant t \leqslant 1, \tag{4.4}$$

and its inverse $f(s)$. We have $g(0) = q^{-1}\Gamma(-\alpha q^{-1})/\Gamma(-\alpha q^{-1} + p)$ if $\alpha < 0$, and $g(0) = \infty$ if $\alpha \geqslant 0$. Let $t = f(s)$ for $0 \leqslant s < g(0)$ be defined by $s = g(t)$ for $0 < t \leqslant 1$. Define $f(s) = 0$ for $s \geqslant g(0)$ if $g(0) < \infty$. The function $f(s)$ is continuous and decreasing from 1 to 0. This $f(s)$ is denoted by $f_{\alpha,p,q}(s)$.

THEOREM 4.3. *If $-\infty < \alpha < 1$, then $A_{\alpha,p}^{q}(\mathbb{R}^d)$ equals $\mathfrak{R}(\Phi_{f_{\alpha,p,q}})$, the range of Φ_f with $f = f_{\alpha,p,q}$.*

Proof. Let $f = f_{\alpha,p,q}$. Let $\widetilde{\mu} \in A_{\alpha,p}^{q}(\mathbb{R}^d)$ with triplet $(\widetilde{\Sigma}, \widetilde{\nu}, \widetilde{\gamma})$. Let us show that $\widetilde{\mu} \in \mathfrak{R}(\Phi_f)$. It follows from Theorem 2.5 that $\widetilde{\nu} \in \mathfrak{R}(\mathcal{A}_{\alpha,p}^{q,q}) \cap \mathfrak{M}_L(\mathbb{R}^d)$. We can find a measure $\rho \in \mathfrak{D}(\mathcal{A}_{\alpha,p}^{q,q})$ such that $\widetilde{\nu} = \mathcal{A}_{\alpha,p}^{q,q}\rho$. Theorem 4.1 shows that

$$\mathcal{A}_{\alpha,p}^{q,q}\,\rho(B) = c_p \int_0^1 (1-u^q)^{p-1}u^{-\alpha-1}du \int_{\mathbb{R}_0^d} 1_B(ux)\nu(dx), \qquad B \in \mathcal{B}(\mathbb{R}_0^d), \tag{4.5}$$

with $\nu(dx) = |x|^{q(p-1)-\alpha}\rho(dx)$. Using Proposition 4.1, we see that $\nu \in \mathfrak{M}_L(\mathbb{R}^d)$. Moreover, $\int_{|x|>1} \log|x|\nu(dx) < \infty$ if $\alpha = 0$, and $\int_{|x|>1} |x|^\alpha \nu(dx) < \infty$ if $0 < \alpha < 1$. It follows from (4.5) and the definition of f that $\widetilde{\nu}$ satisfies (1.4). If $\alpha = 0$, then $f(s) \sim ce^{-\Gamma(p)s}$, $s \to \infty$, with some $c > 0$. If $0 < \alpha < 2$, then $f(s) \sim cs^{-1/\alpha}$, $s \to \infty$, with some $c > 0$. These asymptotics follow from Proposition 4.6 of [16], as $f(s) = \big(\bar{f}_{p,\alpha/q}(qs)\big)^{1/q}$ in the notation of [16]. Therefore, a result of [14] or Theorem 4.2 of [16] says that

$$\mathfrak{D}(\Phi_f) = \begin{cases} \big\{\mu \in I(\mathbb{R}^d)\colon \int\limits_{|x|>1} \log|x|\mu(dx) < \infty\big\} & \text{if } \alpha = 0, \\ \big\{\mu \in I(\mathbb{R}^d)\colon \int\limits_{|x|>1} |x|^\alpha \mu(dx) < \infty\big\} & \text{if } 0 < \alpha < 1. \end{cases} \tag{4.6}$$

If $0 \leqslant \alpha < 1$, then, noting that $\int_0^\infty f(s)ds$ and $\int_0^\infty f(s)^2 ds$ are finite, determine Σ and γ from $\widetilde{\Sigma}$, $\widetilde{\gamma}$, and ν in such a way that $\widetilde{\Sigma} = \int_0^\infty f(s)^2 \Sigma ds$ and $\widetilde{\gamma} = \lim_{t\to\infty} \widetilde{\gamma}_t$ with $\widetilde{\gamma}_t$ of (4.3). If $\alpha < 0$, then we can simply determine Σ and γ from $\widetilde{\Sigma}$, $\widetilde{\gamma}$, and ν, using (4.1) and (4.3) with $t = g(0)$. Let μ be the infinitely divisible distribution with triplet (Σ, ν, γ). Then $\mu \in \mathfrak{D}(\Phi_f)$ and $\Phi_f(\mu) = \widetilde{\mu}$. Hence $A^q_{\alpha,p}(\mathbb{R}^d) \subset \mathfrak{R}(\Phi_f)$.

Conversely, let $\widetilde{\mu} \in \mathfrak{R}(\Phi_f)$, that is, $\widetilde{\mu} = \Phi_f(\mu)$ with some $\mu \in \mathfrak{D}(\Phi_f)$. Let $\widetilde{\nu}$ and ν be the Lévy measures of $\widetilde{\mu}$ and μ. Then they satisfy (1.4). Let $\rho(dx) = |x|^{\alpha-q(p-1)}\nu(dx)$. Then $\int (1 \wedge |x|^2)|x|^{q(p-1)-\alpha}\rho(dx) < \infty$. It follows from (4.6) that ρ satisfies the condition in Proposition 4.1 with $q = r$. Hence $\rho \in \mathfrak{D}(\mathcal{A}^{q,q}_{\alpha,p})$ and $\mathcal{A}^{q,q}_{\alpha,p}\rho \in \mathfrak{M}_L(\mathbb{R}^d)$. Since we have (4.5), we see that $\mathcal{A}^{q,q}_{\alpha,p}\rho = \widetilde{\nu}$. Hence $\widetilde{\mu} \in A^q_{\alpha,p}(\mathbb{R}^d)$. This means that $\mathfrak{R}(\Phi_f) \subset A^q_{\alpha,p}(\mathbb{R}^d)$. ∎

THEOREM 4.4. *If* $-\infty < \alpha < 2$, *then* $A^q_{\alpha,p}(\mathbb{R}^d) = \mathfrak{R}^{\mathrm{e}}(\Phi_{f_{\alpha,p,q}})$, *where* $\mathfrak{R}^{\mathrm{e}}(\Phi_{f_{\alpha,p,q}})$ *is the essential range of* $\Phi_{f_{\alpha,p,q}}$. *If* $1 \leqslant \alpha < 2$, *then* $A^q_{\alpha,p}(\mathbb{R}^d) \supsetneqq \mathfrak{R}(\Phi_{f_{\alpha,p,q}})$.

Proof. Let $f = f_{\alpha,p,q}$ and $-\infty < \alpha < 2$. Then $\int_0^\infty f(s)^2 ds < \infty$, and hence $\mathfrak{R}^{\mathrm{e}}(\Phi_f)$ is the class of $\widetilde{\mu} \in I(\mathbb{R}^d)$ with Lévy measure $\widetilde{\nu}$ satisfying (1.4) for some $\nu \in \mathfrak{M}_L(\mathbb{R}^d)$, as Proposition 3.27 of [16] says. Now, the first five lines of the proof of Theorem 4.3 show that $A^q_{\alpha,p}(\mathbb{R}^d) \subset \mathfrak{R}^{\mathrm{e}}(\Phi_f)$. The converse inclusion is established in the same way as in the last paragraph of the proof of Theorem 4.3. Hence $A^q_{\alpha,p}(\mathbb{R}^d) = \mathfrak{R}^{\mathrm{e}}(\Phi_f)$.

As in the proof of the preceding theorem, we can show that, as $s \to \infty$, $f(s) \sim \big(\alpha\Gamma(p)s\big)^{-1/\alpha}$ and further $f(s) = \big(\Gamma(p)s\big)^{-1}\big(1 + O(s^{-q}) + O(s^{-1})\big)$ if $\alpha = 1$ and $q \neq 1$, and $f(s) = \big(\Gamma(p)s\big)^{-1}\big(1 + O(s^{-1}\log s)\big)$ if $\alpha = q = 1$. Hence we can use Theorems 4.2 and 4.4 of [16] and conclude that $A^q_{\alpha,p}(\mathbb{R}^d) = \mathfrak{R}^{\mathrm{e}}(\Phi_f) \supsetneqq \mathfrak{R}(\Phi_f)$ if $1 \leqslant \alpha < 2$. ∎

EXAMPLE 4.1. Let $\alpha = -q$. Then (4.4) gives $g(t) = (c_{p+1}/q)(1 - t^q)^p$, $0 \leqslant t \leqslant 1$, with $g(0) = c_{p+1}/q$. Hence, we obtain an explicit expression

$$f_{-q,p,q}(s) = \big(1 - (qs/c_{p+1})^{1/p}\big)^{1/q}, \quad 0 \leqslant s \leqslant c_{p+1}/q.$$

REMARK 4.1. *It is an open problem whether or not $A^q_{\alpha,p}(\mathbb{R}^d)$ with $1 \leqslant \alpha < 2$ is equal to $\mathfrak{R}(\Phi_f)$ for a locally square-integrable function f on $[0, \infty)$. A relevant question is whether the class $\mathfrak{R}(\mathcal{A}^{q,q}_{\alpha,p}) \cap \mathfrak{M}_L(\mathbb{R}^d)$ is related to $\Upsilon^{(\tau,b)}$ for some (τ, b) other than the one described in Theorem* 4.1.

4.5. One-to-one property of $\Phi_{f_{\alpha,p,q}}$. A stochastic integral mapping is not necessarily one-to-one, as Barndorff-Nielsen et al. [5] show. So the following result is meaningful.

THEOREM 4.5. *Let $-\infty < \alpha < 2$. The mapping $\Phi_{f_{\alpha,p,q}}$ is one-to-one.*

Proof. Let $\mu \in \mathfrak{D}(\Phi_{f_{\alpha,p,q}})$ and $\widetilde{\mu} = \Phi_{f_{\alpha,p,q}}\mu$. Let (Σ, ν, γ) and $(\widetilde{\Sigma}, \widetilde{\nu}, \widetilde{\gamma})$ be the triplets of μ and $\widetilde{\mu}$, respectively. It is straightforward that $\widetilde{\Sigma}$ determines Σ. Recalling the proof of Theorem 4.3, we see by Theorem 2.6 that $\widetilde{\nu}$ determines ν. Then γ is determined by $\widetilde{\gamma}$ and $\widetilde{\nu}$ as in the proof of Theorem 4.23 of [16]. ∎

4.6. Comparison of $A^q_{\alpha,p}(\mathbb{R}^d)$ with $L(\mathbb{R}^d)$. Let $L(\mathbb{R}^d)$ be the class of selfdecomposable distributions on $\mathbb{R}^d$. A distribution $\mu \in I(\mathbb{R}^d)$ belongs to $L(\mathbb{R}^d)$ if and only if the Lévy measure ν of μ has polar decomposition $\big(\lambda, h_\xi(u)du\big)$, where $h_\xi(u)$ is measurable in (ξ, u) and $uh_\xi(u)$ is decreasing in $u > 0$. The class of Lévy measures ν of $\mu \in L(\mathbb{R}^d)$ is denoted by $\mathfrak{M}^L_L(\mathbb{R}^d)$.

THEOREM 4.6. *The following inclusions are true:*

$$A^q_{\alpha,p}(\mathbb{R}^d) \supset L(\mathbb{R}^d) \quad \text{if } 0 < p \leqslant 1 \text{ and } \alpha \leqslant 0, \tag{4.7}$$

$$A^q_{\alpha,p}(\mathbb{R}^d) \subset L(\mathbb{R}^d) \quad \text{if } p \geqslant 1 \text{ and } 0 \leqslant \alpha < 2. \tag{4.8}$$

Proof. (4.7): Assume that $0 < p \leqslant 1$ and $\alpha \leqslant 0$. To show (4.7), it is enough to prove $\mathfrak{M}^L_L(\mathbb{R}^d) \subset \mathfrak{R}(\mathcal{A}^{q,r}_{\alpha,p})$. Let $\nu \in \mathfrak{M}^L_L(\mathbb{R}^d)$ with polar decomposition $\big(\lambda, h_\xi(u)du\big)$. We can and do assume that, for each $\xi \in \mathbb{S}$, $uh_\xi(u)$ is right-continuous in u and decreases to zero as $u \to \infty$. Let $j_\xi(u) = u^{(\alpha+1)/q}h_\xi(u^{1/q})$. Then $j_\xi(u)$ is also right-continuous and decreasing to zero. If we define $\widetilde{\sigma}_\xi$ by $q^{-1}j_\xi(u)du = \big(\int_{(u,\infty)} \widetilde{\sigma}_\xi(dv)\big)du$, then $\widetilde{\sigma}_\xi \in \mathfrak{D}(I^1_+)$ and $q^{-1}j_\xi(u)du = (I^1_+\widetilde{\sigma}_\xi)(du)$. Let $\sigma_\xi = I^{1-p}_+\widetilde{\sigma}_\xi$ with the understanding that I^0_+ is the identity. Then $\{\sigma_\xi\}$ is a measurable family (Propositions 2.15 and 2.16 of [16]). Since $I^p_+ I^{p'}_+ = I^{p+p'}_+$ for $p, p' > 0$ (Proposition 2.4 of [16]), we have $\sigma_\xi \in \mathfrak{D}(I^p_+)$ and $q^{-1}j_\xi(u)du = (I^p_+\sigma_\xi)(du)$. Hence Proposition 2.1 applies and $\nu \in \mathfrak{R}(\mathcal{A}^{q,r}_{\alpha,p})$ for any r.

(4.8): Assume that $p \geqslant 1$ and $0 \leqslant \alpha < 2$. Let $\nu \in \mathfrak{R}(\mathcal{A}_{\alpha,p}^{q,r}) \cap \mathfrak{M}_L(\mathbb{R}^d)$. To prove (4.8), it is enough to show $\nu \in \mathfrak{M}_L^L(\mathbb{R}^d)$. We have $\nu = \mathcal{A}_{\alpha,p}^{q,q}\rho$ for some ρ. Let (λ, ρ_ξ) be a polar decomposition of ρ and let $\varphi(t) = (1-t^q)^{p-1}t^{-\alpha}$. Then $\varphi(t)$ is decreasing on $(0,1)$. Using Theorem 4.1 we have

$$\nu(B) = \int_{\mathbb{S}} \lambda(d\xi) \int_0^\infty 1_B(s\xi)s^{-1}k_\xi(s)ds,$$

where

$$k_\xi(s) = c_p \int_{(s,\infty)} u^{q(p-1)-\alpha}\varphi(s/u)\rho_\xi(du).$$

Now, we see that $k_\xi(s)$ is decreasing in s and measurable in (ξ, s) and that $\big(\lambda, s^{-1}k_\xi(s)ds\big)$ is a polar decomposition of ν. Hence $\nu \in \mathfrak{M}_L^L(\mathbb{R}^d)$. ∎

REMARK 4.2. *It follows from* (4.7) *and* (4.8) *that* $A_{0,1}^q(\mathbb{R}^d) = L(\mathbb{R}^d)$. *Since* $f_{0,1,q}(s) = e^{-s}$, *Theorem* 4.3 *gives the well-known stochastic integral representation of* $L(\mathbb{R}^d)$ *as a special case.*

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