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LÉVY PROCESSES AS HEAVY TRAFFIC LIMITS OF TANDEM QUEUES WITH HEAVY TAILS

BY

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Abstract. We study the asymptotic behavior of the vectors of sojourn times in tandem queues in heavy traffic. The interarrival and service sequences are assumed to be stationary. The distributions of the terms can have either light or heavy tails.

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1. INTRODUCTION

In this paper we study the asymptotic behavior of the vectors of sojourn times in tandem queues in heavy traffic. A tandem queue, which is the simplest example of a queueing network, consists of m servers (processing stations) in series. We assume that each of the servers has an infinite waiting room, initially empty. Units (or customers) arrive at the first station according to an arrival process described by a stationary sequence $\{u_k, k \geqslant 1\}$, where u_k stands for the interarrival time between the k-th and $(k+1)$-st units. At each server, the arriving unit is either served immediately (if the waiting room is empty) or joins the queue. Furthermore, we assume that the units waiting in the queues are attended to in order of their arrivals; the service process is described by stationary random sequences $\{v_k^i, k \geqslant 1\}$, $1 \leqslant i \leqslant m$, where v_k^i represents the service time of the k-th unit at the i-th server. More precisely, units arrive at the first server according to the arrival process and, having completed service there, immediately proceed to the second server. Afterwards, having completed service at the i-th server, they immediately proceed to the $(i+1)$-st server until they reach the last server, whereupon they exit the entire system. Thus, tandem queues under consideration in this paper are completely described by the stationary sequence

$$\{\zeta_k := (v_k^1, \ldots, v_k^m, u_k), k \geqslant 1\}$$

of random vectors in $\mathbb{R}^{m+1}$ with nonnegative components. Tandem queues are often used as models of manufacturing lines, and simple computer, communication and social networks.

For a single server queue the heavy traffic problem has been studied in numerous papers going back to Kingman's classic work [12]. A recent comprehensive source is the monograph by Whitt [21]. The papers by Szczotka and Kelly [18], Szczotka [17], Boxma and Cohen [4], Szczotka and Woyczyński [20], and Czystołowski and Szczotka [5] were of direct influence in the present study. In [4], [20], and [5], the heavy tailed case has been studied.

Extensive work on Lévy-driven queues utilizing a different formalism of fluid networks and reflected processes can be found in papers by Dębicki et al. [7], Miyazawa and Rolski [15], and in the review article by Dębicki and Mandjes [8]. Nonlinear diffusion approximations to infinite tandem queues via the hydrodynamic limits for nearest neighbor exclusion interacting particle systems have been pioneered by Benassi and Fouque [1], Kipnis [13], and Srinivasan [16]. They all led to the classical nonlinear Burgers partial differential equation. More complicated regimes, including what was called the "gossiping secretaries" network, were studied in Margolius and Woyczyński [14]. They led to more general nonlinear diffusion equations and interacting particle systems with nonlocal interactions via more complex exclusion principles.

2. PRELIMINARIES

2.1. Notation. With the tandem queue described by the (introduced in Section 1) stationary sequence $\{\zeta_k := (v_k^1, \ldots, v_k^m, u_k), k \geqslant 1\}$ of the service times v_k^i at m servers in the network, and the interarrival times u_k at the first server, let $\tilde{w}_k^i$ and $\tilde{W}_k^i$ denote, respectively, the waiting time for service in the i-th queue of the k-th unit and the sojourn time of k-th unit in the first i-th queues. Also, let

$$\tilde{w}_k = (\tilde{w}_k^1, \ldots, \tilde{w}_k^m) \quad \text{and} \quad \tilde{W}_k = (\tilde{W}_k^1, \ldots, \tilde{W}_k^m)$$

be the corresponding random vectors. Clearly,

$$\tilde{W}_k^i = \sum_{j=1}^{i} (\tilde{w}_k^j + v_k^j), \quad 1 \leqslant i \leqslant m.$$

We will consider the following three system parameters:

$$\alpha = \min_{1 \leqslant i \leqslant m} (\bar{u} - \bar{v}^i), \quad \bar{u} = \mathbf{E}\, u_1, \quad \text{and} \quad \bar{v}^i = \mathbf{E}\, v_1^i,\ 1 \leqslant i \leqslant m,$$

where the expectations are assumed to be finite.

It is well known (cf. [18]) that if the sequence $\{\zeta_k, k \geqslant 1\}$ is ergodic, and $\alpha > 0$, i.e., in the case of a stable queueing system, the vectors $\tilde{w}_k$ and $\tilde{W}_k$ converge in distribution, as $k \to \infty$, i.e.,

$$\tilde{w}_k \xrightarrow{d} \tilde{\omega} = (\tilde{\omega}^1, \tilde{\omega}^2, \ldots, \tilde{\omega}^m) \quad \text{and} \quad \tilde{W}_k \xrightarrow{d} \tilde{W} = (\tilde{W}^1, \ldots, \tilde{W}^m). \tag{2.1}$$

The main result of this paper describes the asymptotic behavior of the vector $\tilde{W}$ as $\alpha \downarrow 0$, that is, in what is usually described as the heavy traffic limit, and in the situation when the distributions of interarrival times and service times have heavy tails. However, the results cover also the situation of light tails considered in [18].

2.2. Representation of the vector of sojourn times. The limit vector $\tilde{W}$ in (2.1) has a representation in terms of the original sequences of interarrival and service times that will be useful in what follows. From now onwards let us replace the stationary sequence $\{\zeta_k, k \geqslant 1\}$ by its two-sided stationary extension $\{\tilde{\zeta}_k, -\infty < k < \infty\}$, so that $\{\tilde{\zeta}_k, k \geqslant 1\} \stackrel{d}{=} \{\zeta_k, k \geqslant 1\}$. For simplicity's sake the tilde over the two-sided extension will be dropped henceforth. Below we will assume the following condition:

$$\{(v_j^1, v_j^2, \ldots, v_j^m, u_j), j \leqslant 0\} \stackrel{d}{=} \{(v_j^1, v_j^2, \ldots, v_j^m, u_j), j \geqslant 0\}. \tag{2.2}$$

The representation will also involve the interpolated processes

$$V^s(t) = \sum_{j=1}^{\lfloor t \rfloor} v_j^s \quad \text{and} \quad U(t) = \sum_{j=1}^{\lfloor t \rfloor} u_j,$$

where $\lfloor t \rfloor$ denotes the integer part of t.

PROPOSITION 2.1. *Under the assumption* (2.2) *the limit random vector* $(\tilde{W}^1, \ldots, \tilde{W}^m)$ *in* (2.1) *has the distribution identical with the distribution of the random vector* $(W^1, \ldots, W^m)$, *where*

$$W^i = \sup_{0 = n_{i+1} \leqslant n_i \leqslant n_{i-1} \leqslant \ldots \leqslant n_1 < \infty} \sum_{s=1}^{i} \Big(\sum_{j=n_{s+1}+1}^{n_s} (v_j^s - u_j) + v_{n_{s+1}}^s \Big), \tag{2.3}$$

with $n_1, \ldots, n_{i+1}$ *ranging over the set of nonnegative integers or*, *equivalently*,

$$W^i = \sup_{0 = t_{i+1} \leqslant t_i \leqslant \ldots \leqslant t_1 < \infty} \Big(\sum_{s=1}^{i} \big(V^s(t_s) - V^s(t_{s+1}) \big) - U(t_1) + \sum_{s=1}^{i} \big(V^s(t_{s+1}) - V^s(\lfloor t_{s+1} \rfloor -) \big) \Big), \tag{2.4}$$

with $t_1, \ldots, t_{i+1}$ *ranging over the set of nonnegative real numbers.*

P r o o f. The formula (3.5) from [18] implies that the i-th coordinate, $\tilde{W}^i$, of the vector $\tilde{W} = (\tilde{W}^1, \ldots, \tilde{W}^m)$ in (2.1) has the following form:

$$\tilde{W}^i = \sup_{-\infty \leqslant n_1 \leqslant \ldots \leqslant n_i \leqslant n_{i+1} = -1} \sum_{s=1}^{i} \Big(\sum_{j=n_s+1}^{n_{s+1}} (v_j^s - u_j) + v_{n_{s+1}+1}^s \Big).$$

In view of the assumption (2.2) we get $(\tilde{W}^1, \ldots, \tilde{W}^m) \stackrel{d}{=} (W^1, \ldots, W^m)$, with

$$W^i = \sup_{0 = n_{i+1} \leqslant n_i \leqslant \ldots \leqslant n_1 < \infty} \sum_{s=1}^{i} \Big(\sum_{j=n_{s+1}+1}^{n_s} (v_j^s - u_j) + v_{n_{s+1}}^s \Big).$$

Therefore,

$$\begin{aligned} W^i &= \sup_{0 = n_{i+1} \leqslant n_i \leqslant \ldots \leqslant n_1 < \infty} \Big(\sum_{s=1}^{i} \Big(\sum_{j=n_{s+1}+1}^{n_s} v_j^s + v_{n_{s+1}}^s \Big) - \sum_{j=1}^{n_1} u_j \Big) \\ &= \sup_{0 = t_{i+1} \leqslant t_i \leqslant \ldots \leqslant t_1 < \infty} \Big(\sum_{s=1}^{i} \Big(\sum_{j=\lfloor t_{s+1} \rfloor + 1}^{\lfloor t_s \rfloor} v_j^s + v_{\lfloor t_{s+1} \rfloor}^s \Big) - \sum_{j=1}^{\lfloor t_1 \rfloor} u_j \Big), \end{aligned}$$

which is the formula (2.3).

Taking into account the notation introduced before and the relation

$$v_{\lfloor t_{s+1} \rfloor}^s = V^s(t_{s+1}) - V^s(\lfloor t_{s+1} \rfloor -)$$

we arrive at the formula (2.4), thus completing the proof of the proposition. ■

REMARK 2.1. Notice that, for each $n \geqslant 1$, we have

$$\begin{aligned} W^i = \sup_{0 = t_{i+1} \leqslant t_i \leqslant \ldots \leqslant t_1 < \infty} \Big(&\sum_{s=1}^{i} \big(V^s(nt_s) - V^s(nt_{s+1}) \big) - U(nt_1) \\ &+ \sum_{s=1}^{i} \big(V^s(nt_{s+1}) - V^s(\lfloor nt_{s+1} \rfloor -) \big) \Big). \end{aligned}$$

REMARK 2.2. The assumption (2.2) is satisfied if the process $\{\zeta_k := (v_k^1, \ldots, v_k^m, u_k), -\infty < k < \infty\}$ is reversible, or if $\zeta_k := (v_k^1, \ldots, v_k^m, u_k)$, $-\infty < k < \infty$, are i.i.d. random vectors.

2.3. Continuity of the operator G. In the relation (2.4) in Proposition 2.1 we describe the action of an operator on the interpolated process V^s. It will be convenient to introduce a special notation for the operator itself,

$$\begin{aligned} G^i(x)(t) = \sup_{0 = t_{i+1} \leqslant t_i \leqslant \ldots \leqslant t_2 \leqslant t_1 = t} &\sum_{j=1}^{i} \big(x^j(t_j) - x^j(t_{j+1}) \big) \\ &+ \sum_{j=1}^{i} \big(x^j(t_{j+1}) - x^j(t_{j+1}-) \big), \quad t \geqslant 0, \end{aligned}$$

and consider it as a mapping

$$D^m[0,\infty) \ni x = (x^1, \dots, x^m) \mapsto G^i(x) \in D[0,\infty),$$

where $D^m[0,\infty)$ denotes the product of m copies of the space $D[0,\infty)$ of functions that are right-continuous and have left-hand limits at each point. $D[0,\infty)$ is being considered with the J_1 Skorokhod topology, and $D^m[0,\infty)$ with the product J_1 Skorokhod topology (see, e.g., [21], p. 83).

Notice that, for x such that $x^i(0) = 0$, we have

$$G^2(x)(t) = \sup_{0 \leqslant t_2 \leqslant t_1 = t} \big(x^1(t_1) - x^1(t_2) + x^2(t_2) + x^1(t_2) - x^1(t_2-)\big), \qquad t \geqslant 0.$$

PROPOSITION 2.2. *Mappings G^i and $G = (G^1, \dots, G^m)$ are continuous in the Skorokhod J_1 topology and the product Skorokhod J_1 topology, respectively.*

P r o o f. To prove the assertion notice that, for $x = (x^1, x^2, \dots, x^m)$ and $y = (y^1, y^2, \dots, y^m)$ belonging to $D^m[0,\infty)$, we have

$$G^i(x)(t) = G^i(x - y + y)(t) \leqslant G^i(x-y)(t) + G^i(y)(t),$$

so that

$$\sup_{0\leqslant s\leqslant t} |G^i(x)(s) - G^i(y)(s)| \leqslant \sup_{0\leqslant s\leqslant t} |G^i(x-y)(s)| + \sup_{0\leqslant s\leqslant t} |G^i(y-x)(s)|. \tag{2.5}$$

Now, let $x_n = (x_n^1, x_n^2, \dots, x_n^m)$ converge, as $n \to \infty$, to $x = (x^1, x^2, \dots, x^m)$ in $D^m[0,\infty)$ equipped with the product J_1 topology, and let t be a continuity point of x (thus it is also a continuity point of all of $x^i, i = 1, \dots, m$). Then there exist sequences $\{\lambda_n^i, n \geqslant 1\},\ 1 \leqslant i \leqslant m$, of continuous mappings of $[0,t]$ onto $[0,t]$, with strictly increasing continuous inverses, such that

$$\max_{1\leqslant j\leqslant m} \sup_{0\leqslant s\leqslant t} |\lambda_n^j(s) - s| \to 0,$$

$$\max_{1\leqslant j\leqslant m} \sup_{0\leqslant s\leqslant t} |x_n^j \circ \lambda_n^j(s) - x^j(s)| \to 0,$$

and

$$\max_{1\leqslant j\leqslant m} \sup_{0\leqslant s\leqslant t} |x_n^j \circ \lambda_n^j(s-) - x^j(s-)| \to 0,$$

as $n \to \infty$, where $\circ$ denotes the superposition of functions. In view of (2.5),

$$\begin{aligned} \sup_{0\leqslant s\leqslant t} &|G^i(x_n \circ \lambda_n)(s) - G^i(x)(s)| \\ &\leqslant \sup_{0\leqslant s\leqslant t} |G^i(x_n \circ \lambda_n - x)(s)| + \sup_{0\leqslant s\leqslant t} |G^i(x - x_n \circ \lambda_n)(s)|. \end{aligned} \tag{2.6}$$

Putting
$$y_n^j(t) = x_n^j \circ \lambda_n^j(t) - x^j(t), \quad y_n = (y_n^1, y_n^2, \ldots, y_n^m),$$
we get

(2.7)
$$\begin{aligned}\sup_{0\leqslant s\leqslant t} |G^i(x_n \circ \lambda_n - x)(s)| &= \sup_{0\leqslant s\leqslant t} |G^i(y_n)(s)| \\ &= \sup_{0\leqslant s\leqslant t} \Big| \sup_{0=t_{i+1}\leqslant t_i\leqslant\ldots\leqslant t_2\leqslant t_1=s} \sum_{j=1}^{i} \big(y_n^j(t_j) - y_n^j(t_{j+1})\big) \\ &\quad + \sum_{j=1}^{i} \big(y_n^j(t_{j+1}) - y_n^j(t_{j+1}-)\big)\Big|.\end{aligned}$$

Therefore, if
$$\sup_{0\leqslant s\leqslant t} |x_n^j \circ \lambda_n^j(s) - x^j(s)| \leqslant \varepsilon$$
and
$$\sup_{0\leqslant s\leqslant t} |x_n^j \circ \lambda_n^j(s-) - x^j(s-)| \leqslant \varepsilon,$$
then $\sup_{0\leqslant s\leqslant t} |y_n^i(s)| \leqslant \varepsilon$, and, by (2.7),

$$\sup_{0\leqslant s\leqslant t} |G^i(x_n \circ \lambda_n - x)(s)| \leqslant 2i\varepsilon. \tag{2.8}$$

In a similar way we obtain

$$\sup_{0\leqslant s\leqslant t} |G^i(x - x_n \circ \lambda_n)(s)| \leqslant 2i\varepsilon. \tag{2.9}$$

Hence, by (2.8) and (2.9),

$$\sup_{0\leqslant s\leqslant t} |G^i(x_n \circ \lambda_n)(s) - G^i(x)(s)| \leqslant 4i\varepsilon, \tag{2.10}$$

which implies the continuity of G^i, and of G as well. The proof of the proposition is now complete. ∎

The operators G^i, for different i's, are related by the following inequality:

PROPOSITION 2.3. *For each* $G^i, i = 1, \ldots, m,$ *and* $x \in D^m[0,\infty)$ *with* $x(0) = 0,$

$$\sup_{0\leqslant s\leqslant t} G^i(x) \leqslant \sup_{0\leqslant s\leqslant t} G^{i-1}(x)(s) + \sup_{0\leqslant s\leqslant t} x^i(s), \tag{2.11}$$

where, *by definition*, $G^0 \equiv 0.$

P r o o f. Indeed,

$$
\begin{aligned}
\sup_{0\leqslant s\leqslant t} G^i(x)(s) &= \sup_{0\leqslant s\leqslant t}\ \sup_{0=t_{i+1}\leqslant t_i\leqslant\ldots\leqslant t_2\leqslant t_1=s} \sum_{j=1}^{i}\big(x^j(t_j)-x^j(t_{j+1})\big)\\
&\quad+\sum_{j=1}^{i}\big(x^j(t_{j+1})-x^j(t_{j+1}-)\big)\\
&\leqslant \sup_{0\leqslant s\leqslant t}\ \sup_{0=t_{i+1}\leqslant t_i\leqslant\ldots\leqslant t_2\leqslant t_1=s} \sum_{j=1}^{i-1}\big(x^j(t_j)-x^j(t_{j+1})\big)\\
&\quad+\sum_{j=1}^{i-1}\big(x^j(t_{j+1})-x^j(t_{j+1}-)\big)\\
&\quad+\sup_{0\leqslant s\leqslant t}\ \sup_{0=t_{i+1}\leqslant t_i\leqslant\ldots\leqslant t_2\leqslant t_1=s}\big(x^i(t_i)-x^i(t_{i+1})\big)\\
&\quad+\big(x^i(t_{i+1})-x^i(t_{i+1}-)\big)\\
&=\sup_{0\leqslant s\leqslant t} G^{i-1}(x)(s)+\sup_{0\leqslant s\leqslant t} x^i(s). \quad\blacksquare
\end{aligned}
$$

2.4. The invariance principle. Let I_c and $I^c,\ c>0$, be the indicator functions of the sets $\{t<c\}$ and $\{t\geqslant c\}$, respectively. Obviously, for any $x\in D^m[0,\infty)$, we have $x=xI_c+xI^c$.

DEFINITION 2.1. We shall say that a measurable mapping $f: D^m[0,\infty)\mapsto\mathbb{R}^m$ has the *property* $(*)$ if the following four conditions are satisfied:

$$
(*)\quad \begin{cases}
0\leqslant f(x+y)\leqslant f(x)+f(y), \text{ whenever } x(0)=y(0)=0,\\
f(x+a)=f(x)+a \text{ for any } a\in\mathbb{R}^m, x\in D^m[0,\infty) \text{with } x(0)=0,\\
f\big(xI^c(\cdot)\big)=f\big(x(c+\cdot)\big),\\
f(x_nI_c)\to f(xI_c), \text{ for all } x_n\to x, \text{ in Skorokhod } J_1 \text{ product topology,}
\end{cases}
$$

where c is a continuity point of x.

DEFINITION 2.2. Given a sequence $S=\{s_k\}, s_k\uparrow\infty$, we shall say that a vector-valued process $\{X(t)=\big(X^1(t),\ldots,X^m(t)\big), t\geqslant 0\}$ in $\mathbb{R}^m$ has *S-asymptotically stationary increments* if the processes $\{X(t), t\geqslant 0\}$ and $\{X(s_k+t)-X(s_k), t\geqslant 0\}, k=1,2,\ldots$, have the same distribution.

Clearly, if $\{\xi_j, -\infty<j<\infty\}$ is a stationary sequence then the process $Y(t)=\sum_{j=1}^{\lfloor t\rfloor}\xi_j,\ t\geqslant 0$, has S-asymptotically stationary increments with $s_k=k$.

THEOREM 2.1. *Let $X=(X^1,\ldots,X^m)$ and $X_n=(X_n^1,\ldots,X_n^m)$, $n\geqslant 1$, be stochastic processes with trajectories in $D^m[0,\infty)$, and such that $X(0)=X_n(0)=(0,\ldots,0)$ a.s. Additionally, assume that $X_n,\ n\geqslant 1$, have S-asympto-*

tically stationary increments, and that $X_n \xrightarrow{d} X$ *as* $n \to \infty$, *where* X *is stochastically continuous, and* $X(t) \to (-\infty, \ldots, -\infty)$ *a.s. as* $t \to \infty$.

Then, if the mapping $f : D^m[0, \infty) \mapsto \mathbb{R}^m$ *satisfies the property* $(*)$, *and the sequence of* $M_n = (M_n^1, \ldots, M_n^m) := f(X_n)$ *is tight, then* $M_n \xrightarrow{d} M = f(X)$.

P r o o f. Notice that

$$\begin{aligned} f(X_n) &= f(X_n I_{s_k} + X_n I^{s_k}) \leqslant f(X_n I_{s_k}) + f(X_n I^{s_k}) \\ &= f(X_n I_{s_k}) + f\big(X_n(s_k + \cdot)\big) \\ &\leqslant f(X_n I_{s_k}) + f\big(X_n(s_k + \cdot) - X_n(s_k)\big) + X_n(s_k). \end{aligned}$$

Hence,

$$f(X_n) - f(X_n I_{s_k}) \leqslant f\big(X_n(s_k + \cdot) - X_n(s_k)\big) + X_n(s_k).$$

Because $\{X_n\}$ has S-asymptotically stationary increments,

$$X_n(s_k + \cdot) - X_n(s_k) \stackrel{d}{=} X_n \quad \text{and} \quad f\big(X_n(s_k + \cdot) - X_n(s_k)\big) \stackrel{d}{=} f(X_n).$$

The tightness of $\{f(X_n)\}$ implies that, for any $\varepsilon > 0$, there exists a compact set $K \in \mathbb{R}_+^m$ such that $P\big(f(X_n) \in K\big) \geqslant 1 - \varepsilon$, and $\|x\| \leqslant k_0$ for all $x \in K$. Hence, for any $\delta \in \mathbb{R}_+^m$, we have

$$\begin{aligned} &P\big(f(X_n) - f(X_n I_{s_k}) \geqslant \delta\big) \\ &\leqslant \varepsilon + P\Big(f\big(X_n(s_k+\cdot) - X_n(s_k)\big) \in K, f\big(X_n(s_k+\cdot) - X_n(s_k)\big) + X_n(s_k) \geqslant \delta\Big) \\ &\leqslant \varepsilon + P\big(k_0 + X_n(s_k) \geqslant \delta\big). \end{aligned}$$

Consequently,

$$\begin{aligned} &\limsup_n P\big(f(X_n) - f(X_n I_{s_k}) \geqslant \delta\big) \\ &\qquad \leqslant \varepsilon + \limsup_n P\big(k_0 + X_n(s_k) \geqslant \delta\big) \leqslant \varepsilon + P\big(k_0 + X_n(s_k) \geqslant \delta\big). \end{aligned}$$

But $X_n(s_k) \to (-\infty, \ldots, -\infty)$ a.s., so $P\big(k_0 + X_n(s_k) \geqslant \delta\big) \to 0$. This implies

$$\lim_k \limsup_n P\big(f(X_n) - f(X_n I_{s_k}) \geqslant \delta\big) \leqslant \varepsilon,$$

which completes the proof of the theorem. ∎

3. HEAVY TRAFFIC LIMIT

3.1. Notation. From now onwards, instead of a single tandem queue, we will consider a sequence of such systems indexed by $n \geqslant 1$. Our previous notation will now include the index indicating the number of a tandem. Hence, for the n-th tandem we have the generating sequence

$$\{\zeta_{n,k} := (v^1_{n,k}, \dots, v^m_{n,k}, u_{n,k}), -\infty \leqslant k \leqslant \infty\}.$$

Let

$$a^i_n = \bar{v}^i_n - \bar{u}_n, \quad \alpha_n = \min_{1\leqslant i\leqslant m}(-a^i_n) > 0, \quad \bar{u}_n = \mathbf{E}\, u_{n,1}, \quad \bar{v}^i_n = \mathbf{E}\, v^i_{n,1},$$

$$V^s_n(t) = \sum_{j=1}^{\lfloor nt\rfloor} v^s_{n,j}, \quad \text{and} \quad U_n(t) = \sum_{j=1}^{\lfloor nt\rfloor} u_{n,j}.$$

Then the counterpart of the vector of sojourn times $W = (W^1, W^2, \dots, W^m)$ for the n-th system is $W(n) = \big(W^1(n), W^2(n), \dots, W^m(n)\big)$, where

$$\begin{aligned} W^i(n) &= \sup_{0=t_{i+1}\leqslant t_i\leqslant\ldots\leqslant t_1<\infty} \Big(\sum_{s=1}^{i} \big(\sum_{j=\lfloor nt_{s+1}\rfloor+1}^{\lfloor nt_s\rfloor} v^s_{n,j} + v^s_{n,[nt_{s+1}]}\big) - \sum_{j=1}^{\lfloor nt_1\rfloor} u_{n,j}\Big) \\ &= \sup_{0=t_{i+1}\leqslant t_i\leqslant\ldots\leqslant t_1<\infty} \Big(\sum_{s=1}^{i}\big(V^s_n(t_s) - V^s_n(t_{s+1})\big) - U_n(t_1) \\ &\quad + \sum_{s=1}^{i}\big(V^s_n(t_{s+1}) - V^s_n(\lfloor t_{s+1}\rfloor -)\big)\Big). \end{aligned}$$

Furthermore, let us introduce the following notation:

$$\tilde{V}^i_n(t) := \frac{1}{c_n}\sum_{j=1}^{\lfloor nt\rfloor}(v^i_{n,j} - \bar{v}^i_n), \quad \tilde{U}_n(t) := \frac{1}{c_n}\sum_{j=1}^{\lfloor nt\rfloor}(u_{n,j} - \bar{u}_n),$$

$$D^i_n(t) := \tilde{V}^i_n(t) - \tilde{V}^i_n([t]-) \equiv \frac{1}{c_n}(v^i_{[nt]} - \bar{v}^i_n),$$

$$A^i_n(t) := \frac{n|a^i_n|}{c_n}\frac{[nt_i]}{n}, \quad \text{and} \quad B^i_n := \frac{1}{c_n}\sum_{s=1}^{i}\bar{v}^s_n,$$

where $\{c_n\}$ is a nondecreasing sequence of positive numbers diverging to infinity. Then

(3.1)

$$\begin{aligned} \frac{1}{c_n}W^i(n) = \sup_{0=t_{i+1}\leqslant t_i\leqslant\ldots\leqslant t_1<\infty} \Big(&\sum_{s=1}^{i}\big(\tilde{V}^s_n(t_s) - A^s_n(t_s) - \tilde{V}^s_n(t_{s+1}) + A^s_n(t_{s+1})\big) \\ &- \tilde{U}_n(t_1) + \sum_{s=1}^{i} D^s_n(t_{s+1}) + B^i_n\Big). \end{aligned}$$

3.2. Tightness of the sequence $\{W(n)/c_n\}$ in heavy traffic. To provide sufficient conditions for tightness of $\{W(n)/c_n\}$ in heavy traffic observe that its terms have the following representation:

(3.2)
$$\frac{1}{c_n}W^i(n) = \sup_{0\leqslant t<\infty} \sup_{0=t_{i+1}\leqslant t_i\leqslant\ldots\leqslant t_2\leqslant t_1=t} \sum_{s=1}^{i} \big(Z_n^s(t_s) - Z_n^s(t_{s+1}) + D_n^s(t_{s+1})\big),$$

where

$$Z_n^i(t) := \tilde{V}_n^i(t) - \tilde{U}_n^i(t) - \frac{n|a_n^i|}{c_n}\frac{\lfloor nt\rfloor}{n}.$$

Also, define

$$R_n^i := \sup_{0\leqslant t<\infty} Z_n^i(t).$$

THEOREM 3.1. *Assume that, for each $i = 1, 2, \ldots, m$,*

$$\lim_{n\to\infty} \frac{n|a_n^i|}{c_n} = d_i < \infty.$$

If, for each $i = 1, 2, \ldots, m$, the sequence $\{R_n^i\}$ is tight, then the sequence $\{W^i(n)/c_n\}$ is tight for each $i = 1, 2, \ldots, m$.

P r o o f. The assertion is proved by mathematical induction. For that purpose we need to show that $\{W^1(n)/c_n\}$ is tight. But this follows from the relation

$$R_n^1 = W^1(n)/c_n + v_{n,0}^1/c_n,$$

the convergence $v_{n,0}^1/c_n \xrightarrow{P} 0$, and the assumption that $\{R_n^1\}$ is tight.

So, assume that $\{W^i(n)/c_n\}$ is tight. To show that $\{W^{i+1}(n)/c_n\}$ is tight notice that, in view of the relation (2.11) in Proposition 2.3, we get

$$\begin{aligned}\frac{1}{c_n}W^{i+1}(n) &\leqslant \frac{1}{c_n}W^i(n) + \sup_{0\leqslant t<\infty}\sup_{0\leqslant s\leqslant t} Z_n^{i+1}(s) + v_{n,0}^{i+1}/c_n\\ &= \frac{1}{c_n}W^i(n) + R_n^{i+1} + v_{n,0}^{i+1}/c_n.\end{aligned}$$

Now, since $v_{n,0}^{i+1}/c_n \xrightarrow{P} 0$, the tightness assumption of $\{R_n^{i+1}\}$, and that induction assumption that $\{W^i(n)/c_n\}$ is tight imply that $\{W^{i+1}(n)/c_n\}$ is tight, which concludes the proof of the theorem. ■

Using Theorem 3.1 and Theorem 1 in [5] we immediately obtain the following corollary. All the Lévy processes appearing below are assumed to have finite expectations (see, e.g., [2], for background information on Lévy processes).

COROLLARY 3.1. *Assume that, for each n, $\{\zeta_{n,k}, 0 < k < \infty\}$ is a sequence of i.i.d. random vectors in $\mathbb{R}_+^{m+1}$ such that the sequence $\{(v_{n,k}^1, \ldots, v_{n,k}^m), k \geqslant 1\}$ is independent of the sequence $\{u_{n,k}, k \geqslant 1\}$. Furthermore, assume that the following three conditions are satisfied*:

(A) $(\tilde{V}_n^1, \ldots, \tilde{V}_n^m, \tilde{U}_n) \xrightarrow{d} (V^1, V^2, \ldots, V^m, U) \equiv (V, U)$, *as* $n \to \infty$, *where* V *and* U *are mutually independent stochastically continuous*, *centered Lévy processes in* $\mathbb{R}^m$ *and* $\mathbb{R}$, *respectively*;

(B) $n(\bar{u}_n - \bar{v}_n^i)/c_n \to \beta^i$, *as* $n \to \infty$, *where* $0 < \beta^i < \infty$, *for* $1 \leqslant i \leqslant m$;

(C) $n(\bar{u}_n/c_n)^2 \to c^2$ *and* $n(\bar{v}_n^i/c_n)^2 \to c^2$, $1 \leqslant i \leqslant m$, *as* $n \to \infty$, *where* $0 \leqslant c^2 < \infty$.

Then the sequence $\{W(n)/c_n\}$ *is tight.*

3.3. Asymptotic behavior of the sequence $\{W(n)/c_n\}$ in heavy traffic. Below we consider the asymptotic behavior of random vectors $W(n)$ under the heavy traffic assumption, $\alpha_n \downarrow 0$, and in the situation when the distributions of random variables $u_{n,1}, v_{n,1}^i,\ 1 \leqslant i \leqslant m$, have heavy tails but finite expectations. Let us begin by introducing the mapping

$$D^m[0,\infty) \ni x = (x^1, \ldots, x^m) \longmapsto \tilde{h}(x) = \big(h(x^1), \ldots, h(x^m)\big) \in \mathbb{R}^m,$$

where $h(x^i) = \sup_{t \geqslant 0} x^i(t)$. Observe that this mapping is not continuous in the J_1 product topology. However, the mapping $f = \tilde{h} \circ G$ satisfies the property $(*)$ of Subsection 2.4.

THEOREM 3.2. *Assume that*, *for each* $n \geqslant 1$, $\{\zeta_{n,k}, -\infty < k < \infty\}$ *is an ergodic and reversible sequence of random vectors in* $\mathbb{R}_+^{m+1}$ *such that* $\alpha_k \downarrow 0$, *and*

(A) $(\tilde{V}_n^1, \ldots, \tilde{V}_n^m, \tilde{U}_n) \xrightarrow{d} (V^1, V^2, \ldots, V^m, U) \equiv (V, U)$, *as* $n \to \infty$, *where* V *and* U *are mutually independent stochastically continuous*, *centered Lévy processes in* $\mathbb{R}^m$ *and* $\mathbb{R}$, *respectively*;

(B) $n(\bar{u}_n - \bar{v}_n^i)/c_n \to \beta^i$, *as* $n \to \infty$, *where* $0 < \beta^i < \infty$, *for* $1 \leqslant i \leqslant m$.

Moreover, assume that the sequence $\{W(n)/c_n\}$ *is tight.*

Then, *for any* $\beta = \beta^1, \ldots, \beta^m \in \mathbb{R}^m$, *as* $n \to \infty$, *the sojourn times* W_n *converge in distribution to the distribution of* $M := \tilde{h}\big(G(V - \beta) - U\big)$, *where* V *and* U *are stochastically continuous*, *centered Lévy processes in* $\mathbb{R}^m$ *and* $\mathbb{R}$, *respectively*, *and the mapping* G *is defined as in Proposition* 2.2.

P r o o f. Let us introduce the following notation:

$$Y_n^i(t) := \tilde{V}_n^i(t) - A_n^i(t), \quad Y_n := (Y_n^1, \ldots, Y_n^m),$$

$$\tilde{D}_n^i(t) := Y_n^i(t) - Y_n^i([t]-) \equiv \frac{1}{c_n}(v_{[nt]}^i - \bar{v}_n^i) - \frac{|a_n^i|}{c_n},$$

and

$$C_n^i := \frac{1}{c_n} \sum_{s=1}^{i} |a_n^s|.$$

Notice that, by (3.1), we have

$$\frac{1}{c_n} W^i(n) =$$

$$= \sup_{0 \leqslant t < \infty} \Big(\sup_{0 = t_{i+1} \leqslant t_i \leqslant \ldots \leqslant t_2 \leqslant t_1 = t} \big(\sum_{s=1}^{i} \big(Y_n^s(t_s) - Y_n^s(t_{s+1}) + \tilde{D}_n^i(t_{i+1}) \big) \big) - \tilde{U}_n(t) \Big)$$
$$+ B_n^i + C_n^i = \tilde{h}\big(G^i(Y_n) - \tilde{U}_n\big) + B_n^i + C_n^i,$$

where the last equality follows by the definition of G^i. In view of the assumptions (A) and (B),

$$(Y_n, \tilde{U}_n) \equiv \big((Y_n^1, \ldots, Y_n^m), \tilde{U}_n\big) \xrightarrow{d} \big((Y^1, \ldots, Y^m), U\big) \equiv (Y, U),$$

where

$$Y^i(t) = V^i(t) - \beta^i t.$$

Now, the continuity of mappings G^i in the product J_1 Skorokhod topology (Proposition 2.2) implies the convergence

$$\big(G(Y_n), \tilde{U}_n\big) \equiv \Big(\big(G^1(Y_n^1), \ldots, G^m(Y_n^m)\big), \tilde{U}_n\Big)$$
$$\xrightarrow{d} \Big(\big(G^1(Y^1), \ldots, G^m(Y^m)\big), U\Big) \equiv \big(G(Y), U\big).$$

Finally, an application of Theorem 2.1, with $f = \tilde{h}$ and

$$X_n = (X_n^1, \ldots, X_n^m) := \big(G^1(Y_n), \ldots, G^m(Y_n)\big) - \tilde{U}_n,$$
$$X = (X^1, \ldots, X^m) := \big(G^1(Y), \ldots, G^m(Y)\big) - U,$$

yields the assertion of the theorem. ■

COROLLARY 3.2. *Let, for each* $n \geqslant 1$, $\{\zeta_{n,k}, -\infty < k < \infty\}$ *be a sequence of i.i.d. random vectors in* $\mathbb{R}_+^{m+1}$ *such that the sequence* $\{(v_{n,k}^1, \ldots, v_{n,k}^m), -\infty < k < \infty\}$ *is independent of the sequence* $\{u_{n,k}, -\infty < k < \infty\}$, *and assume that*

(A$'$) $(\tilde{V}_n^1, \ldots, \tilde{V}_n^m) \xrightarrow{d} V$ *and* $U_n \xrightarrow{d} U$, *as* $n \to \infty$, *where* V *and* U *are mutually independent, centered Lévy processes in* $\mathbb{R}^m$ *and* $\mathbb{R}$, *respectively.*

Moreover, suppose that the conditions (B) *and* (C) *of Corollary* 3.1 *are satisfied.*

Then

$$W(n)/c_n \xrightarrow{d} M = \tilde{h}\big(\tilde{G}(V - \beta) - U\big),$$

where $\beta(t) = (\beta_1 t, \ldots, \beta_m t)$.

The situation in the special case of two identical servers in series, that is, when $m = 2$ and $v_k^1 = v_k^2$, is described in the following result:

COROLLARY 3.3. *Let, for each* $n \geqslant 1$, $\{\zeta_{n,k} = (v_{n,k}^1, v_{n,k}^1, u_{n,k}), -\infty < k < \infty\}$ *be a sequence of i.i.d. random vectors in* $\mathbb{R}_+^3$ *such that the sequences* $\{v_{n,k}^1, -\infty < k < \infty\}$ *and* $\{u_{n,k}, -\infty < k < \infty\}$ *are mutually independent, and each of them is a sequence of i.i.d. nonnegative random variables. Moreover, let*

(A′) $\tilde{V}_n^1 \xrightarrow{d} V^1$ *and* $U_n \xrightarrow{d} U$, *as* $n \to \infty$, *where* V^1 *and* U *are mutually independent, centered Lévy processes in* $\mathbb{R}$;

(B) $n(\bar{u}_n - \bar{v}_n^1)/c_n \to \beta^1$, *as* $n \to \infty$, *where* $0 < \beta^1 < \infty$;

(C) $n(\bar{u}_n/c_n)^2 \to c^2$ *and* $n(\bar{v}_n^1/c_n)^2 \to c^2$, *as* $n \to \infty$, *where* $0 \leqslant c^2 < \infty$.

Then

$$W^2(n)/c_n \xrightarrow{d} \sup_{0 \leqslant t_2 \leqslant t < \infty} \big(V^1(t) - \beta^1 t + V^1(t_2) - V^1(t_2-) - U(t)\big).$$

Note that, despite the identical service times at both servers, the above result is different than one might expect intuitively on the basis of an analysis of the single server system.

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