PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 9.1 (1988), p. 115-123

PROBABILITIES OF MODERATE DEVIATIONS FOR RANDOMLY INDEXED SUMS OF RANDOM VARIABLES WITH MULTIDIMENSIONAL INDICES

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Abstract. Probabilities of moderate deviations for randomly indexed sums of independent and identically distributed random variables with multidimensional indices are studied. The results presented extend some theorems of Ahmad [1] and Gut [11].

1. Introduction. Let Z^d, where $d \geqslant 1$ is an integer, denote the positive integer d-dimensional lattice points. The points in Z^d will be denoted by $\boldsymbol{m}$, $\boldsymbol{n}$, etc. or, sometimes, when necessary, more explicitly by $(m_1, \ldots, m_d)$, $(n_1, \ldots, n_d)$, etc. The notation $\boldsymbol{m} \leqslant \boldsymbol{n}$ means that $m_i \leqslant n_i$ for each i, $1 \leqslant i \leqslant d$. We write $\mathbf{1}$ for the point $(1, \ldots, 1) \in Z^d$. Also, for all $\boldsymbol{n}$, we define $n = \prod n_i$ ($i = 1, 2, \ldots, d$), and $\boldsymbol{n} \to \infty$ is interpreted as $|\boldsymbol{n}| \to \infty$.

Let $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in Z^d\}$ be a collection of random variables defined on a probability space $(\Omega, \mathscr{A}, \mathrm{P})$. Throughout the paper we assume that $X_{\boldsymbol{n}}$, $\boldsymbol{n} \in Z^d$, are i.i.d. random variables. For $\boldsymbol{n} \geqslant \mathbf{1}$ define the partial sum $S_{\boldsymbol{n}} = \sum X_{\boldsymbol{k}}$, $\boldsymbol{k} \leqslant \boldsymbol{n}$.

Let $\{N_{\boldsymbol{n}}, \boldsymbol{n} \in Z^d\}$ be a set of Z^d-valued random variables defined on $(\Omega, \mathscr{A}, P)$, i.e., for every $\boldsymbol{n} \in Z^d$, $N_{\boldsymbol{n}} = (N_{\boldsymbol{n}}^{(1)}, \ldots, N_{\boldsymbol{n}}^{(d)})$, where $N_{\boldsymbol{n}}^{(i)}$, $1 \leqslant i \leqslant d$, are positive integer-valued random variables. Let

$$S_{N_{\boldsymbol{n}}} = \sum_{\boldsymbol{k} \leqslant N_{\boldsymbol{n}}} X_{\boldsymbol{k}}, \qquad \boldsymbol{n} \in Z^d.$$

If $d = 1$, the complete convergence result of Hsu and Robbins [12] and Erdős [5, 6] has been extended by Szynal [18], Csörgő and Rychlik [4], Csörgő and Révész [2], Theorem 7.1.1, p. 252. Ahmad [1] gives the order of magnitude of the series

$$\sum_{n=1}^{\infty} \mathrm{P}(|S_{N_n}| > \varepsilon n^{\alpha+1/2}) \quad \text{as } \varepsilon \to 0^+,$$

where $\{N_n\}$ is a sequence of positive integer-valued random variables, not necessarily independent of X_n's, such that, for some constant $\tau > 0$, $N_n/n \to \tau$ a.s. as $n \to \infty$. Recently Gut [11] has proved the following result:

THEOREM 1. (a) *Let* $\alpha r > 1$ *and* $\alpha > 1/2$. *Suppose that* $\mathrm{E}|X_1|^r < \infty$ *and that* $\mathrm{E}X_1 = 0$ *if* $r \geqslant 1$. *If, for some* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha r-2}\,\mathrm{P}(|N_n - \lambda n| > n\varepsilon) < \infty, \tag{1}$$

where λ *is a positive random variable such that* $P(\lambda \geqslant a) = 1$ *for some* $a > \varepsilon > 0$, *then*

$$\sum_{n=1}^{\infty} n^{\alpha r-2}\,\mathrm{P}(|S_{N_n}| > N_n^{\alpha}\varepsilon) < \infty. \tag{2}$$

(b) *Let* $\alpha r = 1$ *and* $\alpha > 1/2$. *Suppose that* $\mathrm{E}|X_1|^r \log_+ |X_1| < \infty$ *and that* $\mathrm{E}X_1 = 0$ *if* $1 \leqslant r$ (< 2). *If* (1) *holds with* $\mathrm{P}(a \leqslant \lambda \leqslant b) = 1$, *where* $0 < \varepsilon < a \leqslant b < \infty$, *then* $\mathrm{E}|X_1|^r < \infty$ *for* $r \geqslant 1$ *and* $\mathrm{E}X_1 = 0$ *imply* (2).

(c) *Let* $\alpha r \geqslant 1$ *and* $\alpha > 1/2$. *Suppose that* $\mathrm{E}|X_1|^r < \infty$ *and that* $\mathrm{E}X_1 = 0$ *if* $r \geqslant 1$. *If* (2) *holds with* $\mathrm{P}(\lambda \leqslant b) = 1$ *for some* $b > 0$, *then*

$$\sum_{n=1}^{\infty} n^{\alpha r-2}\,\mathrm{P}(|S_{N_n}| > n^{\alpha}\varepsilon) < \infty. \tag{3}$$

The main aim of this paper is to extend the results mentioned above to the d-dimensional random fields $\{X_{\boldsymbol{n}},\ \boldsymbol{n} \in Z^d\}$. Moreover, we give the exact order of magnitude of the series

$$\sum_{\boldsymbol{n} \geqslant 1} f(|\boldsymbol{n}|)\,\mathrm{P}(|S_{N_{\boldsymbol{n}}}| \geqslant t\,|N_{\boldsymbol{n}}|^{1/2} g(|N_{\boldsymbol{n}}|)), \quad \text{as } t \to 0^+,$$

for some functions f and g. From the results presented we also get extensions, for $d \geqslant 2$, of some theorems given by Gut [9, 10], Klesov [13], and Łagodowski and Rychlik [14].

2. Preliminaries. In this section we collect some general facts and a lemma which will be needed later on.

Let $d(x) = \mathrm{Card}\,\{\boldsymbol{n} \in Z^d\colon |\boldsymbol{n}| = [x]\}$ and $M_d(x) = \mathrm{Card}\,\{\boldsymbol{n} \in Z^d\colon |\boldsymbol{n}| < [x]\}$, where $[x]$ denotes the integral part of x. We have (cf. [17])

$$M_d(x) = \frac{x(\log_+ x)^{d-1}}{(d-1)!} - M_{d-1}(x), \quad d \geqslant 2, \tag{4}$$

so that

$$M(x) = O\big(x(\log_+ x)^{d-1}\big) \quad \text{as } x \to \infty. \tag{5}$$

Furthermore, for every $\delta > 0$, $d(x) = o(x^{\delta})$ as $x \to \infty$.

LEMMA. *Let* $\{X_n,\ n\in Z^d\}$ *be a sequence of independent standard normal random variables. Then, for every* $z>0$ *and* $\alpha>0$,

(6)
$$\lim_{t\to 0^+}\left(\ln\frac{1}{t}\right)^{-s-d}\sum_{n\geq 1}|n|^{-1}(\log_+|n|)^s\,\mathrm{P}(|S_n|\geq tz^\alpha|n|^{\alpha+1/2})=\frac{1}{\alpha^{s+d}(s+d)(d-1)!},$$

(7)
$$\lim_{t\to 0^+}\left(\ln\frac{1}{t}\right)^{1-s-d}\sum_{n\geq 1}|n|^r(\log_+|n|)^s\,\mathrm{P}(|S_n|\geq tz^\alpha|n|^{\alpha+1/2})=\frac{C_{(r+1)/\alpha}}{\alpha^{s+d-1}(r+1)(d-1)!\,z^{r+1}},$$

where $C_\alpha=\pi^{-1/2}2^{1/2\alpha}\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)$, $r,\ s=0,1,2,\ldots$, *and*

(8)
$$\lim_{t\to 0^+}t^{(s+d)/u}\sum_{n\geq 1}|n|^{-1}(\log_+|n|)^s\,\mathrm{P}(|S_n|\geq tz^\alpha|n|^{\alpha+1/2}(\log z|n|)^u)=\frac{C_{u/(s+d)}}{(s+d)(d-1)!}$$

for $u=1,2,\ldots$,

Observe that (6) and (7) immediately follow from the Lemma given in [15]. We can prove (8) by the same way as relation (17) of [15] if we observe that, for $z>0$,

$$\sum_{j=1}^{k}jd(j)(\log_+ zj)^s\approx\frac{(\log zk)^{s+d}}{(s+d)(d-1)!}\qquad\text{as } k\to\infty.$$

3. Main results. Let $\{X_n,\ n\in Z^d\}$ be i.i.d. random variables with mean 0 and variance 1 and let $\{N_n,\ n\in Z^d\}$ be a sequence of Z^d-valued random variables. All random variables are defined on the same probability space $(\Omega,\mathcal{A},\mathrm{P})$.

THEOREM 2. *Let*

$$\Delta_n(x)=|\mathrm{P}(S_{N_n}<x|N_n|^{1/2})-\Phi(x)|$$

and assume

$$\Delta_n=\sup_x\Delta_n(x)\to 0\qquad \text{as } n\to\infty.$$

If

(9)
$$\mathrm{E}X_l^2(\log_+|X_1|)^{s+d-1-2u}<\infty,$$

then, for every $s+d>2u$,

(10)
$$\underline{\lim}_{t\to 0^+}t^{(s+d)/u}\Big\{\sum_{n\geq 1}|n|^{-1}(\log_+|n|)^s\times \times\mathrm{P}\big(|S_{N_n}|\geq t|N_n|^{1/2}(\log_+|N_n|)^u\big)+H(-1,s)\Big\}\geq\frac{C_{u/(s+d)}}{(s+d)(d-1)!}$$

and

$$(11)\quad \overline{\lim_{t\to 0^+}}\, t^{(s+d)/u}\Big\{\sum_{n\geqslant 1} |\boldsymbol{n}|^{-1}(\log_+|\boldsymbol{n}|)^s \times$$

$$\times \mathrm{P}\big(|S_{N_n}| \geqslant t\,|N_n|^{1/2}(\log_+|N_n|)^u\big) - H(-1, s)\Big\} \leqslant \frac{C_{u/(s+d)}}{(s+d)(d-1)!}.$$

If

$$(12)\quad \lim_{K\to\infty}\overline{\lim_{t\to 0^+}}\left(\ln\frac{1}{t}\right)^{-s-d} t^{-2/(1+2\alpha)}\,\mathrm{E}\,|X_1|^{2/(1+2\alpha)} \times$$

$$\times(\log_+|X_1|)^{s+d-1}\,\mathrm{I}\,[|X_1| > t^{-1/2\alpha}K^{\alpha+1/2}] = 0,$$

then

$$(13)\quad \underline{\lim_{t\to 0^+}}\left(\ln\frac{1}{t}\right)^{-s-d}\Big\{\sum_{n\geqslant 1}|\boldsymbol{n}|^{-1}(\log_+|\boldsymbol{n}|)^s \times$$

$$\times \mathrm{P}(|S_{N_n}| \geqslant t\,|N_n|^{\alpha+1/2}) + H(-1, s) \geqslant \frac{1}{\alpha^{s+d}(s+d)(d-1)!}$$

and

$$(14)\quad \overline{\lim_{t\to 0^+}}\left(\ln\frac{1}{t}\right)^{-s-d}\Big\{\sum_{n\geqslant 1}|\boldsymbol{n}|^{-1}(\log_+|\boldsymbol{n}|)^s \times$$

$$\times \mathrm{P}(|S_{N_n}| \geqslant t\,|N_n|^{\alpha+1/2}) - H(-1, s) \leqslant \frac{1}{\alpha^{s+d}(s+d)(d-1)}.$$

If

$$(15)\qquad \mathrm{E}\,\{|X_1|^{2(r+2)/(1+2\alpha)}(\log_+|X_1|)^{s+d-1} < \infty,$$

then, for every $(r+1)/2 > \alpha > 0$,

$$(16)\quad \underline{\lim_{t\to 0^+}}\, t^{(r+1)/\alpha}\left(\ln\frac{1}{t}\right)^{1-s-d}\Big\{\sum_{n\geqslant 1}|\boldsymbol{n}|^{r}(\log_+|\boldsymbol{n}|)^s \times$$

$$\times \mathrm{P}\big(|S_{N_n}| \geqslant t\,|N_n|^{\alpha+1/2}\big) + H(r, s)\Big\} \geqslant \frac{C_{(r+1)/\alpha}}{\alpha^{s+d-1}\,b^{r+1}(d-1)!}$$

and

$$(17)\quad \overline{\lim_{t\to 0^+}}\, t^{(r+1)/\alpha}\left(\ln\frac{1}{t}\right)^{1-s-d}\Big\{\sum_{n\geqslant 1}|\boldsymbol{n}|^{r}(\log_+|\boldsymbol{n}|)^s \times$$

$$\times \mathrm{P}(|S_{N_n}| \geqslant t\,|N_n|^{\alpha+1/2}) - H(r, s)\Big\} \leqslant \frac{C_{(r+1)/\alpha}}{\alpha^{s+d-1}\,a^{r+1}(d-1)!},$$

where

$$H(r, s) = \sum_{n\geqslant 1}|\boldsymbol{n}|^r(\log_+|\boldsymbol{n}|)^s\,\mathrm{P}\big(\big||N_n| - \lambda\,|\boldsymbol{n}|\big| > t\,|\boldsymbol{n}|\big),$$

and λ *is a positive random variable such that* $\mathrm{P}(a \leqslant \lambda \leqslant b) = 1$ *for some* $0 < a \leqslant b < \infty$.

Proof. First we prove (10) and (11). Let

$$a_n(t) = (a-t)|n|, \quad b_n(t) = (b+t)|n|,$$
$$I_n(t) = \big[\big||N_n| - \lambda|n|\big| \leqslant t|n|\big].$$

It is easy to see that

$$(18)\quad \sum_{n\geqslant 1} |n|^{-1}(\log_+|n|)^s \mathrm{P}\big(|S_{N_n}| \geqslant t|N_n|^{1/2}(\log_+ b_n(t))^u, I_n(t)\big)$$
$$\leqslant \sum_{n\geqslant 1} (|n|)^{-1}(\log_+|n|)^s \mathrm{P}\big(|S_{N_n}| \geqslant t|N_n|^{1/2}(\log_+|N_n|)^u\big)$$
$$\leqslant \sum_{n\geqslant 1} |n|^{-1}(\log_+|n|)^s \times$$
$$\times \mathrm{P}\big(|S_{N_n}| \geqslant t|N_n|^{1/2}(\log_+ a_n(t))^u, I_n(t)\big) + H(-1, s).$$

Let us put, for every $t > 0$,

$$A_t = \{n \in Z^d\colon |n| \leqslant n_0(t)\},$$

and for every positive number K define

$$B_{tK} = \{n \in Z^d\colon |n| \leqslant \exp(Kt^{-1/u})\},$$

where $n_0(t)$ is a positive integer, $n_0(t) \to \infty$, and

$$(19)\qquad t^{(s+d)/u}\big(\log_+ n_0(t)\big)^{s+d} \to 0 \quad \text{as } t \to 0^+.$$

Write

$$F_a(r, s, \alpha, u) = 2\sum_{n\geqslant 1} |n|^r(\log_+|n|)^s \Phi\big(-t|n|^{1/2}(a|n|)^\alpha(\log_+ a|n|)^u\big),$$

where $\Phi(x)$ is the distribution function of a standard normal random variable. Thus, by (18), we have

$$(20)\quad t^{(s+d)/u}\Big\{\sum_{n\geqslant 1} |n|^{-1}(\log_+|n|)^s \mathrm{P}\big(|S_{N_n}| \geqslant t|N_n|^{1/2}(\log_+|N_n|)^u\big) - H(-1, s)\Big\}$$
$$\leqslant t^{(s+d)/u} \sum_{n\in B_{tK}} |n|^{-1}(\log_+|n|)^s \times$$
$$\times \big|\mathrm{P}\big(|S_{N_n}| \geqslant t|N_n|^{1/2}(\log_+ a_n(t))^u\big) - 2\Phi\big(-t(\log_+ a_n(t))^u\big)\big| +$$
$$+ t^{(s+d)/u} F_a(-1, s, 0, u) + t^{(s+d)/u} \sum_{n\in B_{tK}^c} |n|^{-1}(\log_+|n|)^s \times$$
$$\times \mathrm{P}\big(|S_{N_n}| \geqslant t a_n^{1/2}(t)(\log_+|n|)^u, I_n(t)\big),$$

where $B_{tK}^c = Z^d - B_{tK}$. Furthermore, by Abel's transform and (4) we get the following asymptotic expansion:

(21) $$\sum_{k=1}^{n} d(k)k^{-1}(\log_+ k)^s \approx \frac{(\log_+ n)^{s+d}}{(s+d)(d-1)!} \quad \text{as } n\to\infty.$$

Hence, by (19), (21) and the assumption $\Delta_n \to 0$ as $n\to\infty$, we have

(22) $$\lim_{K\to\infty}\overline{\lim_{t\to 0^+}}\, t^{(s+d)/u}\sum_{n\in B_{tK}} |n|^{-1}(\log_+|n|)^s \times$$
$$\times\left|\mathrm{P}\left(|S_{N_n}|\geqslant t|N_n|^{1/2}(\log_+ a_n(t))^u\right)-2\Phi\left(-t(\log_+ a_n(t))^u\right)\right|$$
$$\leqslant 2\,\overline{\lim_{t\to 0^+}}\, t^{(s+d)/u}\Big\{\sum_{n\in A_t}|n|^{-1}(\log_+|n|)^s\Delta_n+\sum_{n\in B_{tK}}|n|^{-1}(\log_+|n|)^s\Delta_n\Big\}$$
$$\leqslant 2\lim_{K\to\infty}\overline{\lim_{t\to 0^+}}\left\{\frac{2t^{(s+d)/u}\left(\log_+ n_0(t)\right)^{s+d}}{(s+d)(d-1)!}+\max_{n\in B_{tK}-A_t}\frac{\Delta_n}{(s+d)(d-1)!}\right\}=0.$$

On the other hand, by (18), we have

(23) $$t^{(s+d)/u}\Big\{\sum_{n>1}|n|^{-1}(\log_+|n|)^s\,\mathrm{P}\left(|S_{N_n}|\geqslant t|N_n|^{1/2}(\log_+|N_n|)^u\right)+H(-1,s)\Big\}$$
$$\geqslant\Big\{-t^{(s+d)/u}\sum_{n\in B_{tK}}|n|^{-1}(\log_+|n|)^s\times\left|\mathrm{P}\left(|S_{N_n}|\geqslant t|N_n|^{1/2}(\log_+ b_n(t))^u\right)-\right.$$
$$\left.-2\Phi\left(-t(\log_+ b_n(t))^u\right)\right|\Big\}+$$
$$+t^{(s+d)/u}F_b(-1,s,0,u)+2\Big\{-t^{(s+d)/u}\sum_{n\in B_{tK}^c}\Phi\left(-t(\log_+ b_n(t))^u\right)\Big\}+$$
$$+t^{(s+d)/u}\sum_{n\in B_{tK}^c}|n|^{-1}(\log_+|n|)^s\,\mathrm{P}\left(|S_{N_n}|\geqslant tb_n^{1/2}(t)(\log_+ b_n(t))^u,\, I_n(t)\right).$$

By our assumptions we have

(24) $$\mathrm{P}\left(|S_{N_n}|\geqslant ta_n^{1/2}(t)(\log_+ a_n(t))^u,\, I_n(t)\right)\leqslant \mathrm{P}\left(\max_{1\leqslant k\leqslant\beta_n}|S_k|\geqslant ta_n^{1/2}(t)(\log_+ a_n(t))^u\right),$$

where $\beta_n=([b_n(t)],1,\ldots,1)$. But, by the result of Fuk [8] (Corollary 3 with $t=2$, $y_1=\ldots=y_n=y=\frac{1}{2}tca_n^{1/2}(t)(\log_+ a_n(t))^u$, c being a positive constant such that $c<2u/(s+d)$), we get

(25) $$\mathrm{P}\left(\max_{1\leqslant k\leqslant\beta_n}|S_k|\geqslant ta_n^{1/2}(t)(\log_+ a_n(t))^u\right)$$
$$\leqslant|\beta_n|\,\mathrm{P}\left(|X_1|\geqslant tca_n^{1/2}(t)(\log_+ a_n(t)^u/2\right)<2^{1+2/c}\left(4+ct^2a_n(t)(\log_+ a_n(t))^{2u}/|\beta_n|\right)^{-1/c}$$
$$+2\exp\left\{\frac{-t^2a_n(t)(\log_+ a_n(t))^{2u}}{8e^2|\beta_n|}\right\}.$$

Now, using Abel's transform and the definitions of β_n, $a_n(t)$, and (4) we get

$$\lim_{K\to\infty}\overline{\lim_{t\to 0+}}\, t^{(s+d)/u-2/c}\times\sum_{n\in B_{tK}^c}|n|^{-1}(\log_+|n|)^s\left(|\beta_n|/a_n(t)\left(\log_+ a_n(t)\right)^{2u}\right)^{1/c} \tag{26}$$

$$\leqslant C\lim_{K\to\infty}\overline{\lim_{t\to 0+}}\, t^{(s+d)/u-2/c}\sum_{k\geqslant\exp(Kt^{-1/u})} d(k)k^{-1}(\log_+ k)^{s-2u/c}$$

$$= C\lim_{K\to\infty}\overline{\lim_{t\to 0+}}\, t^{(s+d)/u-2/c}\sum_{k\geqslant\exp(Kt^{-1/u})}\frac{k^{-1}(\log_+ k)^{s+d-2u/c-1}}{(d-1)!}=0,$$

$$\lim_{K\to\infty}\overline{\lim_{t\to 0+}}\, t^{(s+d)/u}\times\sum_{n\in B_{tK}^c}|n|^{-1}(\log_+|n|)^s\exp\left\{\frac{-t^2 a_n(t)\left(\log_+ a_n(t)\right)^{2u}}{8e^2|\beta_n|}\right\}=0. \tag{27}$$

Indeed, since

$$x^{1/c}\exp(-\gamma x)\leqslant(c\gamma e)^{-1/c},\qquad c,\gamma,x>0, \tag{28}$$

we infer that, for sufficiently small t,

$$\exp\left\{\frac{-t^2 a_n(t)\left(\log_+ a_n(t)\right)^{2u}}{8e^2|\beta_n|}\right\}\leqslant Ct^{-2/c}(\log_+|n|)^{-2u/c}, \tag{29}$$

where C is a positive constant which depends only on a, b, c and u.

Now, by (29), (27) can be shown similarly as (26). By our assumptions, Abel's transform and the definition of $a_n(t)$, we have

$$\lim_{K\to\infty}\overline{\lim_{t\to 0+}}\, t^{(s+d)/u}\sum_{n\in B_{tK}^c}(\log_+|n|)^s\,\mathrm{P}\left(|X_1|\geqslant t|n|^{1/2}(\log_+ a_n(t))^u\right) \tag{30}$$

$$\leqslant C\lim_{K\to\infty}\overline{\lim_{t\to 0+}}\, t^{(s+d)/u}\sum_{k\geqslant\exp(Kt^{-1/u})} d(k)\log_+^s k\,\mathrm{P}\left(|X_1|\geqslant tk^{1/2}(\log_+ k)^u\right)$$

$$\leqslant C\lim_{K\to\infty}\overline{\lim_{t\to 0+}}\, t^{(s+d)/u-2}\,\mathrm{E}\{X_1^2(\log_+|X_1|/t)^{s+d-1-2u}\times I[|X_1|$$

$$\geqslant K^u\exp(Kt^{-1/u}/2)]\}=0$$

since $(s+d)>2u$ and $EX_1^2(\log_+|X_1|)^{s+d-1-2u}<\infty$, where C is a positive constant which depends only on a. Thus the proof of (11) is completed.

To see (10) let us observe that, by Lemma 2 ([7], p. 166),

$$\Phi(-x)\leqslant(2\pi)^{-1/2}x^{-1}\exp(-x^2/2)\qquad\text{as } x\to\infty, \tag{31}$$

so that

$$\Phi\left(-t\left(\log_+ b_n(t)\right)^u\right) \leqslant (2\pi)^{-1/2} t^{-1}\left(\log_+ b_n(t)\right)^{-u} \exp\left(-t^2\left(\log_+ b_n(t)\right)^{2u}/2\right).$$

Hence, taking into account inequality (28), the last term is dominated by $Ct^{-1-2/c}(\log_+ |n|)^{-u-2u/c}$. Thus, similarly as in (26), we get

$$\lim_{K\to\infty} \overline{\lim_{t\to 0^+}}\, t^{(s+d)/u} \sum_{n\in B_{tK}^c} \Phi\left(-t\left(\log_+ b_n(t)\right)^u\right) = 0,$$

which together with (22), (26), (29) and (30) completes the proof of (10).

The proofs of (13), (14), (16) and (17) are some modifications of the proofs of (10) and (11). Namely, we use the sets $C_{tK} = \{n\in Z^d\colon |n| \leqslant Kt^{-1/\alpha}\}$ and $C_{tK}^c = Z^d - C_{tK}$ instead of the sets B_{tK} and B_{tK}^c, respectively, and apply Abel's transform as well as Fuk's inequality with adequate functions. Thus the details are omitted.

Remark 1. By the similar way we can obtain results, analogous to the given above, replacing $H(r, s)$ by

$$H_1(r, s) = \sum_{n\geqslant 1} |n|^r (\log_+ |n|)^s \, \mathrm{P}\big(\max_{1\leqslant i\leqslant d} |N_n^{(i)}/n_i - \lambda_i| > t\big),$$

where $n = (n_1, \ldots, n_d)$ and λ_i, $1 \leqslant i \leqslant d$, are positive random variables such that

$$\mathrm{P}(a \leqslant \min_{1\leqslant i\leqslant d} \lambda_i \leqslant \max_{1\leqslant i\leqslant d} \lambda_i \leqslant b) = 1$$

for some $0 < a \leqslant b < \infty$. Both cases are of course equivalent if $d = 1$.

Remark 2. Let $\{X_n, n\in Z^d\}$ be i.i.d. random variables with the mean 0 and variance 1, $\{N_n, n\in Z^d\}$ be a sequence of Z^d-valued random variables such that $|N_n|/|n| \xrightarrow{\mathrm{P}} \lambda$, where λ is a positive random variable. Then (cf. [3] and [16])

$$\Delta_n = \sup_x |\mathrm{P}(S_{N_n} < x|N_n|^{1/2}) - \Phi(x)| \to 0 \quad \text{as } n\to\infty.$$

Now, let us observe that the assumptions concerning the sequence $\{N_n, n\in Z^d\}$, given in Theorem 2, are weaker than those given in Theorem 1 by Gut [11].

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Received on 25. 3. 1986