PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 43, Fasc. 2 (2023), pp. 241–246
Published online 19.2.2024
doi:10.37190/0208-4147.00146

AN OPERATOR-VALUED FREE POINCARÉ INEQUALITY

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Abstract. The purpose of this short note is to give an operator-valued free Poincaré inequality, which provides a simple proof to (an improvement of) a lemma of Voiculescu (2000) asserting that the kernel of the free difference quotient is exactly the coefficients.

2020 Mathematics Subject Classification: Primary 46L54; Secondary 46L52, 47B47.

Key words and phrases: Sobolev space, free difference quotient, projective tensor product, conditional expectation, von Neumann algebra.

1. INTRODUCTION

Let M be a von Neumann algebra with a faithful normal tracial state τ, and B be a unital von Neumann subalgebra of M with a (unique) τ-preserving conditional expectation E from M onto B. Let X be a self-adjoint element of M, which is assumed to be algebraically free from B. Let $B\langle X\rangle$ denote the family of all B-valued non-commutative polynomials, i.e., the linear span of all monomials $b_0Xb_1X\ldots Xb_n$, $b_i \in B$, and μ denotes the usual multiplication on $B\langle X\rangle$. The *free difference quotient*

$$\partial_{X:B} : B\langle X\rangle \to B\langle X\rangle^{\otimes 2}$$

is a unique $B\langle X\rangle^{\otimes 2}$-valued derivation on $B\langle X\rangle$ that satisfies $\partial_{X:B}[X] = 1\otimes 1$ and $\partial_{X:B}[b] = 0$ for any $b \in B$. Let $L^2(M,\tau) = L^2(M)$ denote the completion of M with respect to the (tracial) L^2-norm defined by $|a|_2 = \tau(a^*a)^{1/2}$ for every $a \in M$. Set $B\langle t\rangle := B * \mathbb{C}\langle t\rangle$ (algebraic free product) with indeterminate t. Note that any element of $B\langle t\rangle$ is a linear combination of monomials $b_0tb_1t\cdots tb_n$ ($b_i \in B$). For any $R>0$, let $B_R\{t\}$ be the completion of $B\langle t\rangle$ with respect to the norm $|||\cdot|||_R$ defined by

$$|||p(t)|||_R = \inf\left\{ \sum_{k=1}^{n} \|b_{k,0}\|\cdot\|b_{k,1}\|\cdots\|b_{k,m(k)}\|R^{m(k)} \;\middle|\; p(t) = \sum_{k=1}^{n} b_{k,0}tb_{k,1}\cdots tb_{k,m(k)} \right\}$$

for every $p(t) \in B\langle t\rangle$.

The purpose of this short note is to give an operator-valued free Poincaré inequality, which is almost of the same form as what Voiculescu conjectured (see [7]) but we choose the norm of $\partial_{X:B}[p(X)]$ here to be the projective tensor norm instead of the L^2-norm. Hence, our inequality may be called a free Poincaré inequality. Nevertheless, it gives a rather simple proof to (an improvement of) [6, Lemma 3.4], an important fact asserting that the kernel of $\partial_{X:B}$ is exactly the algebra B in the analytic setup. Actually, the inequality is a byproduct of our investigation on [6], which became the groundwork for [2, 3]. (Compare the discussion here to Voiculescu's.) We remark that a scalar-valued free Poincaré inequality has been established by Voiculescu in his unpublished note, and its proof can also be found in e.g. [4, Section 8.1].

2. RESULTS

In this section, $C^*(B\langle X\rangle)^{\overline{\otimes}2}$ and $C^*(B\langle X\rangle)^{\widehat{\otimes}2}$ denote the minimal tensor product and the projective tensor product, respectively, that is, they are the completions of the algebraic tensor product $C^*(B\langle X\rangle)^{\otimes 2}$ with respect to the C^*-norm $\|\cdot\|$ and the Banach $*$-norm $\|\cdot\|_\pi$, respectively, defined as follows:

$$\|\xi\| = \|(\rho_1 \otimes \rho_2)(\xi)\|_{B(H\otimes K)}, \quad \xi \in C^*(B\langle X\rangle)^{\otimes 2},$$

with some faithful $*$-representations ρ_1 and ρ_2 of $C^*(B\langle X\rangle)$ on some Hilbert spaces H_1 and H_2, respectively, and

$$\|\xi\|_\pi = \inf\left\{\sum_{k=1}^N \|\xi_{k,1}\|\,\|\xi_{k,2}\| \;\middle|\; \xi = \sum_{k=1}^N \xi_{k,1}\otimes\xi_{k,2},\, \xi_{k,j}\in C^*(B\langle X\rangle),\, N\in\mathbb{N}\right\}$$

for any $\xi \in C^*(B\langle X\rangle)^{\otimes 2}$. Note that the minimal C^*-tensor norm $\|\cdot\|$ does not depend on the choice of the faithful $*$-representations (ρ_1, H_1) and (ρ_2, H_2).

Assume that $\partial_{X:B}$ from $(C^*(B\langle X\rangle), \|\cdot\|)$ to $(C^*(B\langle X\rangle)^{\overline{\otimes}2}, \|\cdot\|)$ is closable (this follows from the existence of conjugate variable in $L^2(M)$, see [5, Corollary 4.2] and [6, Section 3.2]). We denote by $\overline{\partial}_{X:B}$ the closure of $\partial_{X:B}$ with respect to $\|\cdot\|$ on both sides. Note that the natural map from the tensor product $C^*(B\langle X\rangle)^{\widehat{\otimes}2} \subset M^{\widehat{\otimes}2}$ to $C^*(B\langle X\rangle)^{\overline{\otimes}2} \subset M^{\overline{\otimes}2}$ is injective. This indeed follows from Haagerup's famous work [1, Proposition 2.2]. Hence, $\partial_{X:B}$ from $(C^*(B\langle X\rangle), \|\cdot\|)$ to $(C^*(B\langle X\rangle)^{\widehat{\otimes}2}, \|\cdot\|_\pi)$ is closable if it is so from $(C^*(B\langle X\rangle), \|\cdot\|)$ to $(C^*(B\langle X\rangle)^{\overline{\otimes}2}, \|\cdot\|)$. Let $\widehat{\partial}_{X:B}$ denote the closure of $\partial_{X:B}$ with respect to $\|\cdot\|$ and $\|\cdot\|_\pi$.

Voiculescu introduced a certain smooth subalgebra of $C^*(B\langle X\rangle)$, which is a kind of Sobolev space (see [5, Section 4]). Let $B^{(1)}(X)$ be the completion of $B\langle X\rangle$ with respect to the norm $|||\cdot|||_{(1)}$ defined by

$$|||p(X)|||_{(1)} := \|p(X)\| + \|\partial_{X:B}[p(X)]\|_\pi$$

for any $p(X) \in B\langle X\rangle$. The resulting space becomes a Banach $*$-algebra. Here, we can show two lemmas.

LEMMA 2.1. *We have the following facts:*

(1) *For any $\eta \in B^{(1)}(X)$ there exist a unique $\eta_\pi \in C^*(B\langle X\rangle)^{\widehat{\otimes}2}$, a unique $\eta_\infty \in C^*(B\langle X\rangle)$ and a net $\{p_\lambda\}$ of $B\langle X\rangle$ such that $|||\eta|||_{(1)} = \|\eta_\infty\| + \|\eta_\pi\|_\pi$ and*

$$\begin{aligned} p_\lambda &\to \eta && \text{in } B^{(1)}(X),\\ p_\lambda &\to \eta_\infty && \text{in } C^*(B\langle X\rangle),\\ \partial_{X:B}[p_\lambda] &\to \eta_\pi && \text{in } C^*(B\langle X\rangle)^{\widehat{\otimes}2}. \end{aligned}$$

(2) *The correspondence $\iota : B^{(1)}(X) \to C^*(B\langle X\rangle)$ given by $\iota[\eta] := \eta_\infty$ for every $\eta \in B^{(1)}(X)$ defines a contractive algebra homomorphism with $\iota|_{B\langle X\rangle} = \mathrm{id}_{B\langle X\rangle}$. With this map, we regard $B^{(1)}(X)$ as a $*$-subalgebra of $C^*(B\langle X\rangle)$.*

(3) *The correspondence $\widetilde{\partial}_{X:B} : B^{(1)}(X) \to C^*(B\langle X\rangle)^{\widehat{\otimes}2}$ given by $\widetilde{\partial}_{X:B}[\eta] := \eta_\pi$ for every $\eta \in B^{(1)}(X)$ defines a contractive derivation. Moreover, $\widetilde{\partial}_{X:B} = \widehat{\partial}_{X:B} \circ \iota$ and hence $\widetilde{\partial}_{X:B}|_{B\langle X\rangle} = \partial_{X:B}$.*

(4) *The non-commutative functional calculus map $f(t) \mapsto f(X)$ from $B_R\{t\}$ to $C^*(B\langle X\rangle)$ sending t to X is well defined as long as $\|X\| < R$, and its range becomes a $*$-subalgebra of $B^{(1)}(X)$.*

Proof. We give only a sketch of proof.
(1) This follows from the definition of $(B^{(1)}(X), |||\cdot|||_{(1)})$.
(2) The well-definedness of ι follows from the fact that η_∞ is unique.
(3) The well-definedness of $\widetilde{\partial}_{X:B}$ follows similarly to (2). By the construction of $\widetilde{\partial}_{X:B}$ and the closability of $\widehat{\partial}_{X:B}$, we have $\widetilde{\partial}_{X:B} = \widehat{\partial}_{X:B} \circ \iota$. That $\widetilde{\partial}_{X:B}$ is a derivation follows from the first part of [6, Lemma 3.1], which is valid in the present setting.
(4) Use the following inequalities (see [5, Section 4]):

$$\|p(X)\| \leqslant |||p(t)|||_R, \quad \|\partial_{X:B}[p(X)]\| \leqslant \|\partial_{X:B}[p(X)]\|_\pi \leqslant C|||p(t)|||_R$$

for any $p(t) \in B\langle t\rangle$, where $C = \sup_{n\in\mathbb{N}} n\|X\|^{n-1}/R^n$. ∎

LEMMA 2.2. *The map $\iota : B^{(1)}(X) \to C^*(B\langle X\rangle)$ is injective. Moreover, the range of ι is exactly* $\mathrm{dom}(\widehat{\partial}_{X:B})$.

Proof. The first part is clear from Lemma 2.1. Next, we show the second part. By Lemma 2.1(3), it follows that $\mathrm{ran}(\iota) \subset \mathrm{dom}(\widehat{\partial}_{X:B})$. Conversely, for any

$f(X) \in \mathrm{dom}(\widehat{\partial}_{X:B})$ there exists a sequence $\{p_n(X)\}_{n=1}^{\infty} \subset B\langle X\rangle$ such that $p_n(X) \xrightarrow{n\to\infty} f(X)$ in $\|\cdot\|$ and $\partial_{X:B}[p_n(X)] \xrightarrow{n\to\infty} \widehat{\partial}_{X:B}[f(X)]$ in $\|\cdot\|_{\pi}$. Then

$$\begin{aligned}|||p_n(X)-p_m(X)|||_{(1)} &= \|p_n(X)-p_m(X)\| + \|\partial_{X:B}[p_n(X)] - \partial_{X:B}[p_m(X)]\|_\pi \\ &\xrightarrow{n\to\infty} \|f(X)-f(X)\| + \|\widehat{\partial}_{X:B}[f(X)] - \widehat{\partial}_{X:B}[f(X)]\|_\pi = 0.\end{aligned}$$

Therefore, there exists an $\eta \in B^{(1)}(X)$ such that $p_n(X) \xrightarrow{n\to\infty} \eta$ in $|||\cdot|||_{(1)}$ and we have $f(X) = \iota[\eta]$. Thus, $\mathrm{dom}(\widehat{\partial}_{X:B}) \subset \mathrm{ran}(\iota)$. ∎

We are now in a position to give the desired inequality.

THEOREM 2.1 (An operator-valued free Poincaré inequality). *For an arbitrary element $f(X) \in \mathrm{dom}(\widehat{\partial}_{X:B})$,*

$$|f(X) - E[f(X)]|_2 \leqslant 2|X|_2 \|\widehat{\partial}_{X:B}[f(X)]\|_\pi;$$

equivalently, by Lemma 2.2, for any $f(X) \in B^{(1)}(X)$, the same inequality also holds with $\widetilde{\partial}_{X:B}[f(X)]$ in place of $\widehat{\partial}_{X:B}[f(X)]$, where $\|\cdot\|_\pi$ is the projective tensor norm on $C^*(B\langle X\rangle)^{\widehat{\otimes}2}$.

Proof. By the continuity of E and of the norm, it suffices to show the inequality for any non-commutative polynomial $p(X) \in B\langle X\rangle$ (in this case, we have $\partial_{X:B}[p(X)] = \widehat{\partial}_{X:B}[p(X)] = \widetilde{\partial}_{X:B}[p(X)]$). We denote by μ the multiplication map from $B\langle X\rangle^{\otimes 2}$ to $B\langle X\rangle$. Let $\sharp$ be a bilinear map on $B\langle X\rangle^{\otimes 2}$ such that $(a_1 \otimes a_2) \sharp (a_3 \otimes a_4) = (a_1 a_3) \otimes (a_4 a_2)$ for every $a_i \in B\langle X\rangle$. For any $p(X) \in B\langle X\rangle$ and any expression $\partial_{X:B}[p(X)] = \sum_{i=1}^N q_{i,1}(X) \otimes q_{i,2}(X) \in B\langle X\rangle^{\otimes 2}$ with monomials $q_{i,j}(X)$, we have

$$\begin{aligned}&(\mu \circ (\mathrm{id} \otimes E))\big(\partial_{X:B}[p(X)] \sharp (X \otimes 1 - 1 \otimes X)\big) \\ &\qquad = \sum_{i=1}^N \big(q_{i,1}(X) X E[q_{i,2}(X)] - q_{i,1}(X) E[X q_{i,2}(X)]\big).\end{aligned}$$

On the other hand, for any monomial $q(X) = b_0 X b_1 \cdots X b_n \in B\langle X\rangle$, we have

$$\begin{aligned}&\partial_{X:B}[q(X)] \sharp (X \otimes 1 - 1 \otimes X) \\ &\quad = \Big(\sum_{i=1}^n b_0 X b_1 \cdots b_{i-1} \otimes b_i X \cdots X b_n\Big) \sharp (X \otimes 1 - 1 \otimes X) \\ &\quad = b_0 X \otimes b_1 \cdots X b_n - b_0 \otimes X b_1 \cdots X b_n \\ &\qquad + b_0 X b_1 X \otimes b_2 \cdots X b_n - b_0 X b_1 \otimes X b_2 \cdots X b_n \\ &\qquad + b_0 X b_1 X b_2 X \otimes b_3 \cdots X b_n - b_0 X b_1 X b_2 \otimes X b_3 \cdots X b_n \\ &\qquad\quad \vdots \\ &\qquad + b_0 X b_1 X \cdots b_{n-1} X \otimes b_n - b_0 X b_1 X \cdots b_{n-1} \otimes X b_n.\end{aligned}$$

Since E is a B-bimodule map, it follows that

$$\begin{aligned}
&(\mu \circ (\mathrm{id} \otimes E))(\partial_{X:B}[q(X)] \sharp (X \otimes 1 - 1 \otimes X)) \\
&\qquad = b_0 X E[b_1 \cdots X b_n] - b_0 E[X b_1 \cdots X b_n] \\
&\qquad\quad + b_0 X b_1 X E[b_2 \cdots X b_n] - b_0 X b_1 E[X b_2 \cdots X b_n] \\
&\qquad\quad + b_0 X b_1 X b_2 X E[b_3 \cdots X b_n] - b_0 X b_1 X b_2 E[X b_3 \cdots X b_n] \\
&\qquad\qquad \vdots \\
&\qquad\quad + b_0 X b_1 X \cdots b_{n-1} X E[b_n] - b_0 X b_1 X \cdots b_{n-1} E[X b_n] \\
&\qquad = b_0 X E[b_1 \cdots X b_n] - E[q(X)] \\
&\qquad\quad + b_0 X b_1 X E[b_2 \cdots X b_n] - b_0 X E[b_1 X b_2 \cdots X b_n] \\
&\qquad\quad + b_0 X b_1 X b_2 X E[b_3 \cdots X b_n] - b_0 X b_1 X E[b_2 X b_3 \cdots X b_n] \\
&\qquad\qquad \vdots \\
&\qquad\quad + q(X) - b_0 X b_1 X \cdots X E[b_{n-1} X b_n] \\
&\qquad = q(X) - E[q(X)].
\end{aligned}$$

By linearity, we obtain

$$(\mu \circ (\mathrm{id} \otimes E))(\partial_{X:B}[p(X)] \sharp (X \otimes 1 - 1 \otimes X)) = p(X) - E[p(X)]$$

for any $p(X) \in B\langle X \rangle$. Therefore,

$$\begin{aligned}
|p(X) - E[p(X)]|_2 &= |(\mu \circ (\mathrm{id} \otimes E))(\partial_{X:B}[p] \sharp (X \otimes 1 - 1 \otimes X))|_2 \\
&= \Big| \sum_{i=1}^{N} \big(q_{i,1}(X) X E[q_{i,2}(X)] - q_{i,1}(X) E[X q_{i,2}(X)] \big) \Big|_2 \\
&\leqslant \sum_{i=1}^{N} \big(|q_{i,1}(X) X E[q_{i,2}(X)]|_2 + |q_{i,1}(X) E[X q_{i,2}(X)]|_2 \big) \\
&\leqslant 2|X|_2 \sum_{i=1}^{N} \|q_{i,1}(X)\| \cdot \|q_{i,2}(X)\|
\end{aligned}$$

since τ is tracial and E is contractive. It follows that

$$|p(X) - E[p(X)]|_2 \leqslant 2|X|_2 \|\partial_{X:B}[p(X)]\|_\pi$$

by the definition of the projective tensor norm. ■

The inequality still holds even if the L^2-norm is replaced with the operator norm. The proof is completely identical.

COROLLARY 2.1. *Both* $\ker \widehat{\partial}_{X:B}$ *and* $\ker \widetilde{\partial}_{X:B}$ *are exactly* B.

From $\|\partial_{X:B}[p(X)]\| \leqslant \|\partial_{X:B}[p(X)]\|_\pi$ for every $p(X) \in B\langle X\rangle$, and Lemmas 2.1(4) and 2.2, we have

$$\{f(X) \mid f(t) \in B_R\{t\}\} \subset B^{(1)}(X) = \mathrm{dom}(\widehat{\partial}_{X:B}) \subset \mathrm{dom}(\overline{\partial}_{X:B})$$

when $\|X\| < R$ and $\overline{\partial}_{X:B}$ is an extension of $\widehat{\partial}_{X:B}$ (via the natural injection from $M^{\widehat{\otimes}2}$ to $M^{\overline{\otimes}2}$ due to [1, Proposition 2.2]). Therefore, Corollary 2.1 yields the following corollary:

COROLLARY 2.2. $\ker \overline{\partial}_{X:B} \cap B^{(1)}(X) = B$.

This statement is an improvement of [6, Lemma 3.4]; giving a concise proof of it was our original purpose.

Acknowledgments. The author thanks his supervisor Prof. Yoshimichi Ueda for conversations, comments and editorial support of this note. The author also expresses his sincere gratitude to Prof. Dan Voiculescu who kindly gave him and his supervisor an additional explanation to [6]. Furthermore, the author thanks the referee for several suggestions.

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Received 17.10.2023;
accepted 9.2.2024