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A STRENGTHENED ASYMPTOTIC UNIFORM DISTRIBUTION PROPERTY

BY

MICHEL J. G. WEBER (STRASBOURG)

Abstract. A sequence $X = \{X_i, i \geqslant 1\}$ of independent variables taking values in $\mathbb{Z}$, with partial sums $S_n = \sum_{k=1}^{n} X_k$, for each n is said to be asymptotically uniformly distributed, for short a.u.d., if

$$\lim_{n\to\infty} \mathbb{P}\{S_n \equiv m \,(\mathrm{mod})\, d\} = \frac{1}{d} \tag{0.1}$$

for any $d \geqslant 2$ and $m = 0, 1, \ldots, d-1$. We are concerned with estimating the rate of convergence in (0.1), a question recently discussed by Dolgopyat and Hafouta (2022). We obtain a quantitative estimate under weaker moment assumptions, by using a different approach. We deduce from a more general result that if (i) for some function $1 \leqslant \phi(t) \uparrow \infty$ as $t \to \infty$, and some constant C, we have, for all n and $\nu \in \mathbb{Z}$,

$$\left| B_n \mathbb{P}\{S_n = \nu\} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(\nu - M_n)^2}{2B_n^2}} \right| \leqslant \frac{C}{\phi(B_n)},$$

then (ii) there exists a numerical constant C_1 such that for all n with $B_n \geqslant 6$, all $h \geqslant 2$, and $\mu = 0, 1, \ldots, h-1$,

$$\left| \mathbb{P}\big\{S_n \equiv \mu \pmod h\big\} - \frac{1}{h} \right| \leqslant \frac{1}{\sqrt{2\pi} B_n} + \frac{1 + 2C/h}{\phi(B_n)^{2/3}} + C_1 e^{-(1/16)\phi(B_n)^{2/3}}.$$

Assumption (i) holds if a local limit theorem in the usual form is applicable, and (ii) yields a strengthening of Rozanov's necessary condition.

Assume in place of (i) that $\vartheta_j = \sum_{k\in\mathbb{Z}} \mathbb{P}\{X_j = k\} \wedge \mathbb{P}\{X_j = k+1\} > 0$ for each j, and $\nu_n = \sum_{j=1}^{n} \vartheta_j \uparrow \infty$. For these classes of random variables we prove strengthened forms of the asymptotic uniform distribution property, with sharp uniform rate of convergence. (iii) Let $\alpha > \alpha' > 0$, $0 < \varepsilon < 1$. Then for each n such that

$$|x| \leqslant \frac{1}{2} \left(\frac{2\alpha \log(1-\epsilon)\nu_n}{(1-\epsilon)\nu_n} \right)^{1/2} \implies \frac{\sin x}{x} \geqslant (\alpha'/\alpha)^{1/2},$$

we have

$$\sup_{u\geqslant 0}\sup_{d<\pi(\frac{(1-\epsilon)\nu_n}{2\alpha\log(1-\epsilon)\nu_n})^{1/2}}\left|\mathbb{P}\{d\,|\,S_n+u\}-\frac{1}{d}\right| \leqslant 2e^{-\frac{\epsilon^2}{2}\nu_n}+\big((1-\epsilon)\nu_n\big)^{-\alpha'}.$$

(iv) Let $0<\rho<1$ and $0<\varepsilon<1$. The sharper uniform rate of convergence $2e^{-\frac{\epsilon^2}{2}\nu_n}+e^{-((1-\epsilon)\nu_n)^{\rho}}$ is also proved (for a corresponding d-region of divisors) for each n such that

$$|x|\leqslant\frac{1}{2}\Big(\frac{2}{((1-\epsilon)\nu_n)^{1-\rho}}\Big)^{1/2}\implies\frac{\sin x}{x}\geqslant\sqrt{1-\varepsilon}.$$

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1. LOCAL LIMIT THEOREM AND ASYMPTOTIC UNIFORM DISTRIBUTION

Let $X=\{X_i,\,i\geqslant 1\}$ be a sequence of independent variables taking values in $\mathbb{Z}$, and let $S_n=\sum_{k=1}^n X_k$ for each n.

The sequence X is said to be *asymptotically uniformly distributed* with respect to lattices of span d, for short a.u.d.(d), if for $m=0,1,\dots,d-1$, we have

$$\lim_{n\to\infty}\mathbb{P}\{S_n\equiv m\ (\mathrm{mod}\ d)\}=\frac{1}{d}. \tag{1.1}$$

Equivalenty for $m=0,1,\dots,d-1$, we have

$$\lim_{n\to\infty}\mathbb{P}\{d\,|\,S_n-m\}=\frac{1}{d}. \tag{1.2}$$

The sequence X is *asymptotically uniformly distributed*, for short a.u.d., if (1.1) holds true for any $d\geqslant 2$ and $m=0,1,\dots,d-1$.

Dvoretsky and Wolfowitz [4] proved the following characterization. Assume that X consists of independent random variables taking only the values

$$0,1,\dots,h-1.$$

In order that the partial sums $\{S_n,n\geqslant 1\}$ be a.u.d.(h), it is necessary and sufficient that

$$\prod_{k=1}^{\infty}\Big(\sum_{m=0}^{h-1}\mathbb{P}\{X_k=m\}e^{\frac{2i\pi}{h}rm}\Big)=0\quad(r=1,\dots,h-1). \tag{1.3}$$

Equivalently,

$$\prod_{k=1}^{\infty}(\mathbb{E}\,e^{\frac{2i\pi}{h}rX_k})=\lim_{N\to\infty}\mathbb{E}\,e^{\frac{2i\pi}{h}rS_N}=0\quad(r=1,\dots,h-1). \tag{1.4}$$

This notion plays an important role in the study of the local limit theorem. Assume that the random variables X_k take values in a common lattice $\mathcal{L}(v_0, D)$, defined by the sequence $v_k = v_0 + Dk$, $k \in \mathbb{Z}$, v_0 and $D > 0$ being reals, and are square integrable, and let

$$M_n = \mathbb{E}\, S_n, \qquad B_n^2 = \mathrm{Var}(S_n) \to \infty. \tag{1.5}$$

We say that the *local limit theorem* (in the usual form) *is applicable* to X if

$$\sup_{N=v_0n+Dk} \left| B_n \mathbb{P}\{S_n = N\} - \frac{D}{\sqrt{2\pi}} e^{-\frac{(N-M_n)^2}{2B_n^2}} \right| = o(1), \qquad n \to \infty. \tag{1.6}$$

When the random variables X_i are identically distributed, (1.6) reduces to

$$\sup_{N=v_0n+Dk} \left| \sigma\sqrt{n}\, \mathbb{P}\{S_n = N\} - \frac{D}{\sqrt{2\pi}} e^{-\frac{(N-n\mu)^2}{2n\sigma^2}} \right| = o(1), \tag{1.7}$$

where $\mu = \mathbb{E}\, X_1$ and $\sigma^2 = \mathrm{Var}(X_1)$. By Gnedenko's Theorem [11] (see also [21, p. 187], [26, Th. 1.4], (1.7) holds if and only if the span D is maximal (there are no other real numbers v_0' and $D' > D$ for which $\mathbb{P}\{X \in \mathcal{L}(v_0', D')\} = 1$).

Note that the transformation

$$X_j' = \frac{X_j - v_0}{D} \tag{1.8}$$

allows one to reduce to the case $v_0 = 0$, $D = 1$.

REMARK 1.1. The series (in k)

$$\sum_{N=v_0n+Dk} \left(\mathbb{P}\{S_n = N\} - \frac{D}{\sqrt{2\pi} B_n} e^{-\frac{(N-M_n)^2}{2B_n^2}} \right) \tag{1.9}$$

is obviously convergent, whereas nothing can be deduced concerning its size's order from the very definition of the local limit theorem.

We have, however, the following estimate, which (up to our knowledge) has not been known:

$$\sum_{N=v_0n+Dk} \left(\mathbb{P}\{S_n = N\} - \frac{D}{\sqrt{2\pi} B_n} e^{-\frac{(N-M_n)^2}{2B_n^2}} \right) = \mathcal{O}(D/B_n). \tag{1.10}$$

The proof goes as follows. By using the Poisson summation formula, the series associated to the second summand in (1.9) satisfies

$$\sum_{N=v_0n+Dk} \frac{D}{\sqrt{2\pi} B_n} e^{-\frac{(N-M_n)^2}{2B_n^2}} = \sum_{\ell \in \mathbb{Z}} e^{2i\pi\ell\{\frac{v_0n-M_n}{D}\} - \frac{2\pi^2\ell^2 B_n^2}{D^2}}, \tag{1.11}$$

and so is $1 + \mathcal{O}(D/B_n)$, whereas the one associated to the first is 1. Therefore (1.10) follows.

When a strong local limit theorem with convergence in variation holds, we have the more informative result

$$\lim_{n\to\infty}\sum_{N=v_0n+Dk}\left|\mathbb{P}\{S_n=N\}-\frac{D}{\sqrt{2\pi}B_n}e^{-\frac{(N-M_n)^2}{2B_n^2}}\right|=0. \tag{1.12}$$

We recall that if $X=\{X_n,\ n\geqslant 1\}$ is a sequence of independent, square integrable, integer-valued random variables, and $S_{k,n}=X_{k+1}+\cdots+X_{k+n}$, $S_n=S_{0,n}$ for each $n\geqslant 1$, $k\geqslant 0$, the sequence of partial sums $\{S_n,\ n\geqslant 1\}$ is said to satisfy the *strong local limit theorem* with convergence in variation if there are constants $A_{k,n}$ and $B_{k,n}$ such that for each $k=0,1,2,\ldots,$

$$\sum_{m\in\mathbb{Z}}\left|\mathbb{P}\{S_{k,n}=m\}-\frac{1}{B_{k,n}\sqrt{2\pi}}\exp\left\{-\frac{(m-A_{k,n})^2}{2B_{k,n}^2}\right\}\right|\to 0 \tag{1.13}$$

as $n\to\infty$ and $B_{k,n}\to\infty$. See Szewczak and Weber [26, Definition 1.29].

The following result is classical.

THEOREM 1.2 (Rozanov). *Let $X=\{X_i,i\geqslant 1\}$ be a sequence of independent variables taking values in $\mathbb{Z}$, and let $S_n=\sum_{k=1}^n X_k$ for each n. The local limit theorem is applicable to X only if X has the a.u.d. property.*

REMARK 1.3. In Petrov's [21, Lemma 1, p. 194] and in Rozanov's [25, Lemma 1, p. 261], Theorem 1.2 is stated under the assumption that a local limit theorem in the strong form holds (see Section 2), which is not necessary.

REMARK 1.4. In Rozanov's [25], it was remarked that the a.u.d. property is sufficient if the X_n are iid, or if they are independent and uniformly bounded. From [25, Theorem II], it follows that this is also true if $\sup_j\|X_j\|_3<\infty$.

In this work, we are concerned with estimating the rate of convergence in (1.1). There is a recent result which presents a characterization of certain rates of convergence of S_n mod *m* by means of either decay rates of the characteristic functions at resonant points, or Edgeworth expansions: see Dolgopyat and Hafouta [3, Theorems 1.6 and 1.8]. We obtain a quantitative estimate of the rate of convergence under weaker moment assumptions.

THEOREM 1.5. *Let $X=\{X_i,i\geqslant 1\}$ be a sequence of independent variables taking values in $\mathbb{Z}$, and let $S_n=\sum_{k=1}^n X_k$, for each n. Assume that for some function $1\leqslant\phi(t)\uparrow\infty$ as $t\to\infty$, and some constant C, we have, for all n,*

$$\sup_{m\in\mathbb{Z}}\left|B_n\mathbb{P}\{S_n=m\}-\frac{1}{\sqrt{2\pi}}\,e^{-\frac{(m-M_n)^2}{2B_n^2}}\right|\leqslant\frac{C}{\phi(B_n)}. \tag{1.14}$$

Then there exists a numerical constant C_1, such that for all $0 < \varepsilon \leqslant 1$, all n such that $B_n \geqslant 6$, and all $h \geqslant 2$,

$$\sup_{\mu=0,1,\dots,h-1} \left| \mathbb{P}\big\{S_n \equiv \mu \pmod h\big\} - \frac{1}{h} \right| \leqslant \frac{1}{\sqrt{2\pi}\, B_n} + \frac{2C}{h\sqrt{\varepsilon}\,\phi(B_n)} + \mathbb{P}\left\{ \frac{|S_n - M_n|}{B_n} > \frac{1}{\sqrt{\varepsilon}} \right\} + C_1 e^{-1/(16\varepsilon)}.$$

Theorem 1.5 provides an explicit link between the local limit theorem and the a.u.d. property, through the quantitative estimate of the difference $\mathbb{P}\{S_n \equiv m \,(\mathrm{mod}\, h)\} - 1/h$. Further, Theorem 1.5 contains Theorem 1.2, since by definition such a function ϕ exists if the local limit theorem is applicable to X.

REMARK 1.6. It follows from the proof that $C_1 = 2e\sqrt{\pi}$ is suitable.

Choosing $\varepsilon = \phi(B_n)^{-2/3}$ and using Chebyshev's inequality, we get

COROLLARY 1.7. *For all n such that $B_n \geqslant 6$, and all $h \geqslant 2$, we have*

$$\sup_{\mu=0,1,\dots,h-1} \left| \mathbb{P}\{S_n \equiv \mu \pmod h\} - \frac{1}{h} \right| \leqslant H_n \tag{1.15}$$

with

$$H_n = \frac{1}{\sqrt{2\pi}\, B_n} + \frac{1 + 2C/h}{\phi(B_n)^{2/3}} + C_1\, e^{-(1/16)\phi(B_n)^{2/3}}. \tag{1.16}$$

REMARK 1.8. Examples of LLTs with speed of convergence are given in Appendix.

Proof of Theorem 1.5. By assumption,

$$\left| B_n \mathbb{P}\{S_n = m\} - \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(m-M_n)^2}{2B_n^2}} \right| \leqslant \frac{C}{\phi(B_n)}$$

for all m and n. Let $\varepsilon > 0$. We have

$$\left| \mathbb{P}\{S_n \equiv m \,(\mathrm{mod}\, h)\} - \sum_{\substack{|k-M_n| \leqslant B_n/\sqrt{\varepsilon} \\ k \equiv m\,(h)}} \mathbb{P}\{S_n = k\} \right| \leqslant \mathbb{P}\left\{ \frac{|S_n - M_n|}{B_n} > \frac{1}{\sqrt{\varepsilon}} \right\},$$

$$\left| \sum_{\substack{|k-M_n| \leqslant B_n/\sqrt{\varepsilon} \\ k \equiv m\,(h)}} \mathbb{P}\{S_n = k\} - \frac{1}{\sqrt{2\pi} B_n} \sum_{\substack{|k-M_n| \leqslant B_n/\sqrt{\varepsilon} \\ k \equiv m\,(h)}} e^{-\frac{(k-M_n)^2}{2B_n^2}} \right| \leqslant \frac{C}{B_n \phi(B_n)} \sum_{\substack{|k-M_n| \leqslant B_n/\sqrt{\varepsilon} \\ k \equiv m\,(h)}} 1 \leqslant \frac{2C}{h\sqrt{\varepsilon}\,\phi(B_n)}.$$

Letting $z_n = \{M_n\}$, where $\{u\}$ stands for the fractional part of a real u, we have

$$\sum_{\substack{k\in\mathbb{Z}\\ |k-M_n|>B_n/\sqrt{\varepsilon}}} e^{-\frac{(k-M_n)^2}{2B_n^2}} \leqslant \sum_{\substack{Z\in\mathbb{Z}\\ |Z-z_n|>B_n/\sqrt{\varepsilon}}} e^{-\frac{(Z-z_n)^2}{2B_n^2}}.$$

Now using the elementary inequality $(a+b)^2 \leqslant 2(a^2+b^2)$ for reals a, b, we have $|Z-z_n|^2 \geqslant |Z|^2/2 - z_n^2$, and obviously $|Z-z_n| \leqslant |Z| + |z_n|$. We can thus continue as follows:

$$\leqslant \sum_{\substack{Z\in\mathbb{Z}\\ |Z|+|z_n|>B_n/\sqrt{\varepsilon}}} e^{-\frac{(Z-z_n)^2}{2B_n^2}} \leqslant e^{\frac{1}{2B_n^2}} \sum_{\substack{Z\in\mathbb{Z}\\ |Z|>(B_n/\sqrt{\varepsilon})-1}} e^{-\frac{Z^2}{4B_n^2}}.$$

Assume that $B_n \geqslant \max(1/\sqrt{2}, 4\sqrt{\varepsilon})$; then $\frac{B_n}{\sqrt{\varepsilon}} - 2 \geqslant \frac{B_n}{2\sqrt{\varepsilon}}$. In particular, $|Z| \geqslant 1$ in the previous series, and so we have the estimates

$$\begin{aligned}
\leqslant 2e^{\frac{1}{2B_n^2}} \sum_{Z>B_n/(2\sqrt{\varepsilon})+1} e^{-\frac{Z^2}{4B_n^2}} &\leqslant 2e \sum_{Z>B_n/(2\sqrt{\varepsilon})+1} \int_{Z-1}^{Z} e^{-\frac{t^2}{4B_n^2}}\,\mathrm{d}t \\
&\leqslant 2e \int_{B_n/(2\sqrt{\varepsilon})}^{\infty} e^{-\frac{t^2}{4B_n^2}}\,\mathrm{d}t \\
(t=\sqrt{2}B_n u) \qquad &= 2\sqrt{2}eB_n \int_{1/(2\sqrt{2\varepsilon})}^{\infty} e^{-u^2/2}\,\mathrm{d}u \\
&\leqslant 2\sqrt{2}eB_n\sqrt{\pi/2}e^{-1/(16\varepsilon)} \\
&= 2e\sqrt{\pi}B_n e^{-1/(16\varepsilon)},
\end{aligned}$$

since $e^{x^2/2}\int_x^\infty e^{-t^2/2}\,\mathrm{d}t \leqslant \sqrt{\pi/2}$ for any $x \geqslant 0$.

Therefore

$$\begin{aligned}
(1.17) \qquad &\left|\mathbb{P}\{S_n \equiv m \ (\mathrm{mod}\ h)\} - \frac{1}{\sqrt{2\pi}B_n}\sum_{k\equiv m\,(h)} e^{-\frac{(k-M_n)^2}{2B_n^2}}\right| \\
&\qquad\leqslant \mathbb{P}\left\{\frac{|S_n-M_n|}{B_n} > \frac{1}{\sqrt{\varepsilon}}\right\} + \frac{2C}{h\sqrt{\varepsilon}\,\phi(B_n)} + C_1 e^{-1/(16\varepsilon)}
\end{aligned}$$

with $C_1 = 2e\sqrt{\pi}$.

Recall the Poisson summation formula: for $x \in \mathbb{R},\ 0 \leqslant \delta \leqslant 1$,

$$(1.18) \qquad \sum_{\ell\in\mathbb{Z}} e^{-(\ell+\delta)^2\pi x^{-1}} = x^{1/2}\sum_{\ell\in\mathbb{Z}} e^{2i\pi\ell\delta-\ell^2\pi x}.$$

Write $k = m + lh$, $M'_n = M_n - m$,

$$\frac{(k-M_n)^2}{2B_n^2} = \frac{(lh-M'_n)^2}{2B_n^2} = \frac{(l-\lceil M'_n/h\rceil + \{M'_n/h\})^2}{2B_n^2/h^2} = \frac{(\ell+\{M'_n/h\})^2}{2B_n^2/h^2}, \tag{1.19}$$

letting $\ell = l - \lceil M'_n/h\rceil$.

By applying it with $x = 2B_n^2\pi/h^2$, $\delta = \{M'_n/h\}$, we get

$$\sum_{k\equiv m\,(h)} e^{-\frac{(k-M_n)^2}{2B_n^2}} = \sum_{\ell\in\mathbb{Z}} e^{-\frac{(\ell-\{M'_n/h\})^2}{2B_n^2/h^2}} = \frac{\sqrt{2\pi}B_n}{h}\sum_{\ell\in\mathbb{Z}} e^{-2i\pi\ell\{M'_n/h\}-2\pi^2B_n^2\ell^2/h^2}. \tag{1.20}$$

Hence

$$\left|\frac{h}{\sqrt{2\pi}B_n}\sum_{k\equiv m\,(h)} e^{-\frac{(k-M_n)^2}{2B_n^2}} - 1\right| \leqslant \sum_{|\ell|\geqslant 1} e^{-2\pi^2B_n^2\ell^2/h^2}. \tag{1.21}$$

But for any positive real a,

$$\sum_{H=1}^{\infty} e^{-aH^2} \leqslant \frac{\sqrt{\pi}}{2}\min\left(\frac{1}{\sqrt{a}}, \frac{1}{a}\right). \tag{1.22}$$

Therefore with $a = 2\pi^2B_n^2/h^2$,

$$\left|\frac{h}{\sqrt{2\pi}B_n}\sum_{k\equiv m\,(h)} e^{-\frac{(k-M_n)^2}{2B_n^2}} - 1\right| \leqslant \sqrt{\pi}\min\left(\frac{h}{\sqrt{2}\pi B_n}, \frac{h^2}{2\pi^2B_n^2}\right) \leqslant \frac{h}{\sqrt{2\pi}\,B_n}.$$

We have thus obtained the explicit bound

$$\left|\frac{1}{\sqrt{2\pi}B_n}\sum_{k\equiv m\,(h)} e^{-\frac{(k-M_n)^2}{2B_n^2}} - \frac{1}{h}\right| \leqslant \frac{1}{\sqrt{2\pi}B_n}. \tag{1.23}$$

By carrying it back to (1.17), we find that for any $\varepsilon > 0$, all n such that $B_n \geqslant \max(1/\sqrt{2}, 4\sqrt{\varepsilon})$, and all $h \geqslant 2$,

$$\sup_{\mu=0,1,\dots,h-1}\left|\mathbb{P}\{S_n \equiv \mu \pmod h\} - \frac{1}{h}\right| \leqslant \frac{1}{\sqrt{2\pi}B_n} + \frac{2C}{h\sqrt{\varepsilon}\phi(B_n)} + \mathbb{P}\left\{|S_n - M_n| > \frac{B_n}{\sqrt{\varepsilon}}\right\} + C_1e^{-1/(16\varepsilon)}. \tag{1.24}$$

This is fulfilled if we choose $0 < \varepsilon \leqslant 1$ and n such that $B_n \geqslant 6$, whence the claimed estimate. ∎

2. LOCAL LIMIT THEOREM IN THE STRONG FORM

There are easy examples of sequences X for which the fulfilment of the local limit theorem depends on the behavior of the first members of X. Hence it is reasonable to introduce the following definition due to Prokhorov [22]. A local limit theorem in the *strong form* (or *in a strengthened form*) is said to be applicable to X if a local limit theorem in the usual form is applicable to any subsequence extracted from X which differs from X only in a finite number of members. This definition can be made a bit more convenient (see Gamkrelidze [8]). Let

$$S_{k,n} = X_{k+1} + \cdots + X_{k+n}, \quad A_{k,n} = \mathbb{E}\, S_{k,n}, \quad B^2_{k,n} = \mathrm{Var}(S_{k,n}). \tag{2.1}$$

The local limit theorem in the strong form holds if and only if

$$\mathbb{P}\{S_{k,n} = m\} = \frac{D}{B_{k,n}\sqrt{2\pi}} e^{-\frac{(m-A_{k,n})^2}{2B^2_{k,n}}} + o\Big(\frac{1}{B_{k,n}}\Big), \tag{2.2}$$

uniformly in m and every finite k, $k = 0, 1, 2, \ldots$, as $n \to \infty$ and $B_{k,n} \to \infty$.

Rozanov's necessary condition can be stated as follows.

THEOREM 2.1 ([25, Th. I]). *Let $X = \{X_j, j \geqslant 1\}$ be a sequence of independent, square integrable random variables taking values in $\mathbb{Z}$. Let $b_k^2 = \mathrm{Var}(X_k)$ and $B_n^2 = b_1^2 + \cdots + b_n^2$. Assume that*

$$B_n \to \infty \quad \text{as } n \to \infty. \tag{2.3}$$

The following condition is necessary for the applicability of a local limit theorem in the strong form to the sequence X:

$$\prod_{k=1}^{\infty} \Big[\max_{0\leqslant m<h} \mathbb{P}\{X_k \equiv m \,(\mathrm{mod}\, h)\}\Big] = 0 \quad \text{for any } h \geqslant 2. \tag{2.4}$$

REMARK 2.2. No additional regularity condition or stronger uniform integrability condition is required. See [26, Theorem 1.33, Lemma 1.34 and proofs]. Prokhorov characterized the local limit theorem in the strong form for independent, uniformly bounded random variables in [22]; see also [26, Theorem 1.25] and related results of Rvačeva, Maejima and Stone.

Condition (2.4) is also sufficient in some important examples, in particular if the X_i have stable limit distribution, namely when each X_i has distribution function $F_i(x)$ defined by

$$F_i(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ 1 - [c_i + \alpha_i(x)]\frac{1}{x^\alpha} & \text{for } x > 0, \end{cases} \tag{2.5}$$

where $|\alpha_i(x)| \leqslant \alpha(x)$, $\alpha(x) \to 0$ as $x \to \infty$, $0 < c' < c_i < c'' < \infty$, $i = 1, 2, \dots$, and $0 < \alpha < 1$.

Let also

$$B_n = \Big(\sum_{i=1}^{n} c_i\Big)^{1/\alpha}, \qquad S_n = \sum_{i=1}^{n} X_i, \qquad p_{kj} = \mathbb{P}\{X_k = j\}.$$

Let $G(x)$ denote the stable distribution function for which $M(x) = 0$, $N(x) = -1/x^{\alpha}$, $\sigma^2 = 0$ and $\gamma(\tau) = \alpha\tau^{1-\alpha}/(1-\alpha)$, in the Lévy–Khinchin formula. Rogozin [24] proved that, as $n \to \infty$,

$$\mathbb{P}\left\{\frac{S_n}{B_n} < x\right\} \to G(x). \tag{2.6}$$

A local limit theorem holds for the sequence $\{X_j,\ j \geqslant 1\}$ if

$$B_n\mathbb{P}\{S_n = m\} - g\left(\frac{m}{B_n}\right) \to 0 \tag{2.7}$$

as $n \to \infty$, uniformly in m, $-\infty < m < \infty$, where

$$g\left(\frac{m}{B_n}\right) = G'\left(\frac{m}{B_n}\right).$$

The concept of local limit theorem in the strong form is extended similarly.

Mitalauskas [16] proved that in order that the sequence $\{X_j,\ j \geqslant 1\}$ satisfies a local limit theorem in the strong form, it is necessary and sufficient that Rozanov's condition be fulfilled. See Szewczak and Weber [26, p. 31 and Theorem 1.40 with its proof].

We briefly indicate how Theorem 2.1 is proved. If the local limit theorem in the strong form is applicable to the sequence X, then

$$\sum_{k=1}^{\infty} \mathbb{P}\{X_k \not\equiv 0 \pmod h\} = \infty \qquad \text{for any } h \geqslant 2. \tag{2.8}$$

Indeed, otherwise given $h \geqslant 2$, by the Borel–Cantelli lemma, on a set of measure greater than $3/4$, $X_k \equiv 0 \pmod h$ for all $k \geqslant k_0$, say. The new sequence X' defined by $X'_k = 0$ if $k < k_0$, $X'_k = X_k$ otherwise, with partial sums S'_n, satisfies $\mathbb{P}\{S'_n \equiv 0 \pmod h\} > 3/4$ for all n large enough, and this can be used to deduce a contradiction with the fact that $\mathbb{P}\{S'_n \equiv 0 \pmod h\}$ should converge to $1/h$.

REMARK 2.3. After reduction to the case when m_k is the most likely residue of X_k mod h, replacing X_k by $X_k - m_k$ (in the above proof) shows that (2.8) implies (2.4).

The arithmetical quantity

$$\max_{0\leqslant m<h} \mathbb{P}\{X_k \equiv m \ (\mathrm{mod}\ h)\}$$

also appears in the study of local limit theorems with arithmetical sufficient conditions. The approaches used (Freiman, Moskvin and Yudin [6], Mitalauskas [17], Raudelyunas [23] and later Fomin [5], for instance) require the random variables not to overly concentrate in a particular residue class m (mod h) of $\mathbb{Z}$, and impose arithmetical conditions of the type: for all $h \geqslant 2$,

$$\max_{0\leqslant m<h} \mathbb{P}\{X_k \equiv m \ (\mathrm{mod}\ h)\} \leqslant 1-\alpha_k \tag{2.9}$$

for all k, where α_k is some specific sequence of reals decreasing to 0. In addition, one generally has $\sum_k \alpha_k = \infty$. Although the simple form of the local limit theorem is here considered, for obvious reasons, condition (2.4) brings nothing more in this context, as the requirements made are stronger.

As a consequence of the quantitative formulation of the a.u.d. property obtained in Theorem 1.5, we have the following result.

THEOREM 2.4. *Under the assumptions of Theorem* 2.1*, assume further that the local limit theorem is applicable to a sequence* X.

(i) *We have*

$$\limsup_{h\to\infty} \prod_{k=1}^{\infty} \max_{0\leqslant m<h} \mathbb{P}\{X_k \equiv m \ (\mathrm{mod}\ h)\} = 0.$$

(ii) *There exists a function* $1 \leqslant \phi(t) \uparrow \infty$ *as* $t \to \infty$ *such that*

$$\sum_{k=1}^{n} \frac{\max_{0\leqslant m<h} \mathbb{P}\{X_k \equiv m \ (\mathrm{mod}\ h)\}}{1-\max_{0\leqslant m<h} \mathbb{P}\{X_k \equiv m \ (\mathrm{mod}\ h)\}} \geqslant -\log\Big(\frac{1}{h} + H_n\Big),$$

where

$$H_n = \frac{1}{\sqrt{2\pi} B_n} + \frac{1+2C/h}{\phi(B_n)^{2/3}} + C_1 e^{-(1/16)\phi(B_n)^{2/3}},$$

and C, C_1 *are absolute constants.*

Proof. Consider a sequence Y where $Y_k = X_k - m_k$, m_k are integers, for all $k \geqslant 1$. Let $h \geqslant 2$ be fixed. Choose m_k so that

$$\max_{0\leqslant m<h} \mathbb{P}\{X_k \equiv m \ (\mathrm{mod}\ h)\} = \mathbb{P}\{X_k \equiv m_k \ (\mathrm{mod}\ h)\} = \mathbb{P}\{Y_k \equiv 0 \ (\mathrm{mod}\ h)\},$$

and let $\mu_n = \sum_{k=1}^n m_k$. Note that $\sum_{k=1}^n Y_k = S_n - \mu_n$ and $\mathrm{Var}(\sum_{k=1}^n Y_k) = \mathrm{Var}(S_n) = B_n^2$.

As the local limit theorem is applicable to the sequence X, condition (1.14) is satisfied for some function $1 \leqslant \phi(t) \uparrow \infty$ as $t \to \infty$, namely for all n,

$$\sup_{\nu \in \mathbb{Z}} \left| B_n \mathbb{P}\{S_n = \nu\} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(\nu - M_n)^2}{2B_n^2}} \right| \leqslant \frac{C}{\phi(B_n)}.$$

Given n, letting $\nu = m + \mu_n$ and observing that $\mathbb{P}\{\sum_{k=1}^n Y_k = m\} = \mathbb{P}\{S_n - \mu_n = m\}$, we get for $m \in \mathbb{Z}$, $n \geqslant 1$,

$$\left| B_n \mathbb{P}\Big\{ \sum_{k=1}^n Y_k = m \Big\} - \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(m + \mu_n - M_n)^2}{2B_n^2}} \right| \leqslant \frac{C}{\phi(B_n)}.$$

Thus Y satisfies condition (1.14) with the same function $\phi(n)$.

Applying Corollary 1.7 to the sequence Y, it follows that

$$\begin{aligned} \text{(2.10)} \quad \prod_{k=1}^n \max_{0 \leqslant m < h} \mathbb{P}\{X_k \equiv m \ (\mathrm{mod}\ h)\} &= \prod_{k=1}^n \mathbb{P}\{Y_k \equiv 0 \ (\mathrm{mod}\ h)\} \\ &\leqslant \mathbb{P}\Big\{ \sum_{k=1}^n Y_k \equiv 0 \ (\mathrm{mod}\ h) \Big\} \leqslant \frac{1}{h} + H_n, \end{aligned}$$

where H_n has the form given in the statement, and $H_n \to 0$ as $n \to \infty$.

Letting $n \to \infty$ in (2.10) implies

$$\text{(2.11)} \qquad \prod_{k=1}^\infty \max_{0 \leqslant m < h} \mathbb{P}\{X_k \equiv m \ (\mathrm{mod}\ h)\} \leqslant \frac{1}{h}.$$

This being true for each $h \geqslant 2$, letting now $h \to \infty$ in (2.11) yields

$$\text{(2.12)} \qquad \limsup_{h \to \infty} \prod_{k=1}^\infty \max_{0 \leqslant m < h} \mathbb{P}\{X_k \equiv m \ (\mathrm{mod}\ h)\} = 0.$$

By using the elementary inequality $\log(1 - x) \geqslant -x/(1 - x)$, $0 \leqslant x < 1$, we also have

$$\begin{aligned} \prod_{k=1}^n \mathbb{P}\{Y_k \equiv 0 \ (\mathrm{mod}\ h)\} &= \prod_{k=1}^n \big(1 - \mathbb{P}\{Y_k \not\equiv 0 \ (\mathrm{mod}\ h)\}\big) \\ &= e^{\sum_{k=1}^n \log(1 - \mathbb{P}\{Y_k \not\equiv 0 \ (\mathrm{mod}\ h)\})} \\ &\geqslant e^{-\sum_{k=1}^n \mathbb{P}\{Y_k \not\equiv 0 \ (\mathrm{mod}\ h)\}/(1 - \mathbb{P}\{Y_k \not\equiv 0 \ (\mathrm{mod}\ h)\})}. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{k=1}^n \frac{\max_{0 \leqslant m < h} \mathbb{P}\{X_k \equiv m \ (\mathrm{mod}\ h)\}}{1 - \max_{0 \leqslant m < h} \mathbb{P}\{X_k \equiv m \ (\mathrm{mod}\ h)\}} &= \sum_{k=1}^n \frac{\mathbb{P}\{Y_k \not\equiv 0 \ (\mathrm{mod}\ h)\}}{1 - \mathbb{P}\{Y_k \not\equiv 0 \ (\mathrm{mod}\ h)\}} \\ &\geqslant -\log\Big(\frac{1}{h} + H_n \Big). \quad \blacksquare \end{aligned}$$

REMARK 2.5. (i) Note that the bound used in (2.10) is very weak since by independence of X_k, thus of Y_k,

$$\prod_{k=1}^{n} \mathbb{P}\{Y_k \equiv m \pmod h\} = \mathbb{P}\Big\{\forall J \subset [1,n],\ \sum_{k\in J} Y_k \equiv m \pmod h\Big\}.$$

One can consider, in place of individuals Y_k, sums over blocks according to any partition of $\{1,\dots,n\}$.

(ii) Sets of multiples serve as good test sets for the applicability of the local limit theorem because addition is a closed operation. What can be derived when testing the applicability of the local limit theorem with other remarkable sets of integers (squarefree numbers, prime numbers, power numbers, geometric sequences, ...) is unknown. Concerning the squarefree integers, that is, having no squared prime factors, we note the bound

$$\left|2^{-n}\sum_{j\ \text{squarefree}}\binom{n}{j} - \frac{6}{\pi^2}\right| \leqslant C_1 e^{-C_2(\log n^{3/5})/(\log\log n)^{1/5}}. \tag{2.13}$$

We refer to [2].

3. RANDOM SEQUENCES SATISFYING THE A.U.D. PROPERTY

It has some interest to relate the a.u.d. property for Bernoulli sums to the one of sets having Euler density, in this particular case here, arithmetic progressions. A subset A of $\mathbb{N}$ is said to have *Euler density* λ with parameter ϱ (for short E_ϱ density λ) if

$$\lim_{n\to\infty}\sum_{j\in A}\binom{n}{j}\varrho^j(1-\varrho)^{n-j} = \lambda.$$

By a result due to Diaconis and Stein, we have the following characterization.

THEOREM 3.1 ([2, Th. 1]). *For any $A \subset \mathbb{N}$, and $\varrho \in \,]0,1[$ the following assertions are equivalent:*

(i) *A has E_ϱ density λ,*

(ii) $\displaystyle\lim_{t\to\infty} e^{-t}\sum_{j\in A}\frac{t^j}{j!} = \lambda$,

(iii) *for all $\varepsilon > 0$,*

$$\lim_{n\to\infty}\frac{\#\{j\in A : n\leqslant j < n+\varepsilon\sqrt{n}\}}{\varepsilon\sqrt{n}} = \lambda.$$

Applying (iii) with $\rho = \frac{1}{2}$ to

$$A = \{u + kd : k \geqslant 1\} \tag{3.1}$$

straightforwardly implies

LEMMA 3.2. *Let $\mathcal{B}_n = \beta_1 + \cdots + \beta_n$, where β_i are i.i.d. Bernoulli random variables. Then $\{\mathcal{B}_n,\ n \geqslant 1\}$ is a.u.d.(d) for any $d \geqslant 2$.*

Now consider the independent case and introduce the following characteristic. Let Y be a random variable with values in $\mathbb{Z}$. Put

$$\vartheta_Y = \sum_{k\in\mathbb{Z}} \mathbb{P}\{Y = k\} \wedge \mathbb{P}\{Y = k+1\}, \tag{3.2}$$

where $a \wedge b = \min(a, b)$. Note that $0 \leqslant \vartheta_Y < 1$.

THEOREM 3.3. *Let $X = \{X_j,\ j \geqslant 1\}$ be a sequence of independent random variables taking values in $\mathbb{Z}$. Assume that $\vartheta_{X_j} > 0$ for each j. Further assume that the series $\sum_{j=1}^{\infty} \vartheta_{X_j}$ diverges. Then X is a.u.d. The conclusion holds in particular if the X_j are i.i.d. and $\vartheta_{X_1} > 0$.*

Note that no integrability condition is required, whereas square integrability is required in order that the local limit theorem be applicable. We prove in the next section that if the series $\sum_{j=1}^{\infty} \vartheta_{X_j}$ diverges, much more is in fact true. Under the assumption that $\vartheta_{X_j} > 0$ for all j, each X_j admits a Bernoulli component. This is the principle of a coupling method, *the Bernoulli part extraction*, introduced by McDonald [15] and Davis and McDonald [1] in the study of the local limit theorem. See Weber [30] for an application of this method to the almost sure local limit theorem, and Giuliano and Weber [10] where this method is used to obtain approximate local limit theorems with effective rate. See also Szewczak and Weber's recent survey [26], where this method is studied in detail.

Before passing to the proof, we briefly recall some facts and state an auxiliary lemma. Let $\mathcal{L}(v_0, D)$ be a lattice defined by the sequence $v_k = v_0 + Dk$, $k \in \mathbb{Z}$, v_0 and $D > 0$ being real numbers. Let X be a random variable such that $\mathbb{P}\{X \in \mathcal{L}(v_0, D)\} = 1$, and assume that $\vartheta_X > 0$ (see (3.2)). Let $f(k) = \mathbb{P}\{X = v_k\}$, $k \in \mathbb{Z}$. Let also $0 < \vartheta \leqslant \vartheta_X$. Associate to ϑ and X a sequence $\{\tau_k,\ k \in \mathbb{Z}\}$ of non-negative reals such that

$$\tau_{k-1} + \tau_k \leqslant 2f(k), \qquad \sum_{k\in\mathbb{Z}} \tau_k = \vartheta. \tag{3.3}$$

For instance $\tau_k = \frac{\vartheta}{\nu_X}(f(k) \wedge f(k+1))$ is suitable. Next define a pair of random variables (V, ε) as follows:

$$\begin{cases} \mathbb{P}\{(V,\varepsilon) = (v_k, 1)\} = \tau_k \\ \mathbb{P}\{(V,\varepsilon) = (v_k, 0)\} = f(k) - \frac{\tau_{k-1}+\tau_k}{2} \end{cases} \qquad (\forall k \in \mathbb{Z}). \tag{3.4}$$

LEMMA 3.4. *Let L be a Bernoulli random variable which is independent of (V, ε), and let $Z = V + \varepsilon D L$. Then $Z \stackrel{\mathcal{D}}{=} X$.*

Proof of Theorem 3.3. We apply Lemma 3.4 with $D = 1$ to each X_j, with $\vartheta = \vartheta_{X_j}$. One can associate to them a sequence of independent vectors $(V_j, \varepsilon_j, L_j)$, $j = 1, \ldots, n$, such that

$$\{V_j + \varepsilon_j L_j,\, j = 1, \ldots, n\} \stackrel{\mathcal{D}}{=} \{X_j,\, j = 1, \ldots, n\}. \tag{3.5}$$

Further, the sequences $\{(V_j, \varepsilon_j),\, j = 1, \ldots, n\}$ and $\{L_j,\, j = 1, \ldots, n\}$ are independent. For each $j = 1, \ldots, n$, the law of (V_j, ε_j) is defined by (3.4). And $\{L_j,\, j = 1, \ldots, n\}$ is a sequence of independent Bernoulli random variables. Set

$$W_n = \sum_{j=1}^{n} V_j, \quad M_n = \sum_{j=1}^{n} \varepsilon_j L_j, \quad B_n = \sum_{j=1}^{n} \varepsilon_j. \tag{3.6}$$

Denoting again $X_j = V_j + \varepsilon_j L_j$, $j \geqslant 1$, we have

$$\mathbb{P}\{d \,|\, S_n + u\} = \mathbb{E}_{(V,\varepsilon)}\, \mathbb{P}_L\Big\{d \,|\, \Big(\sum_{j=1}^{n} \varepsilon_j L_j + W_n\Big) + u\Big\}. \tag{3.7}$$

As $\sum_{j=1}^{n} \varepsilon_j L_j \stackrel{\mathcal{D}}{=} \sum_{j=1}^{B_n} L_j$, we have

$$\mathbb{P}_L\Big\{d \,|\, \Big(\sum_{j=1}^{n} \varepsilon_j L_j + W_n\Big) + u\Big\} = \mathbb{P}_L\Big\{d \,|\, \sum_{j=1}^{B_n} L_j + (W_n + u)\Big\}.$$

In view of the dominated convergence theorem, it suffices to prove that for each $d \geqslant 2$,

$$\mathbb{P}_L\Big\{d \,|\, \sum_{j=1}^{B_n} L_j + (W_n + u)\Big\} \to \frac{1}{d}$$

as $n \to \infty$, $\mathbb{P}_{(V,\varepsilon)}$ almost surely. But the set (compare with (3.1))

$$A = \{(W_n + u) + kd,\ k \geqslant 1\}$$

now depends on W_n, thus on n, which complicates things. However, since

$$\chi\Big(d \,|\, \sum_{j=1}^{B_n} L_j + (W_n + u)\Big) = \frac{1}{d} \sum_{j=0}^{d-1} e^{2i\pi \frac{j}{d}(W_n+u)} e^{2i\pi \frac{j}{d} \sum_{j=1}^{B_n} L_j},$$

by integrating with respect to $\mathbb{P}_L$ we get

$$\mathbb{P}_L\Big\{d \,|\, \sum_{j=1}^{B_n} L_j + (W_n + u)\Big\} = \frac{1}{d} + \frac{1}{d} \sum_{j=1}^{d-1} e^{2i\pi \frac{j}{d}(W_n+u)} \Big(\cos \frac{\pi j}{d}\Big)^{B_n}.$$

By assumption, B_n tends to infinity $\mathbb{P}_{(V,\varepsilon)}$ almost surely (see [31, (8.3.5)] for instance). Thus the latter sum tends to 0 as $n \to \infty$, $\mathbb{P}_{(V,\varepsilon)}$ almost surely. Therefore by the convergence argument invoked before, $\mathbb{P}\{d \,|\, S_n + u\}$ tends to $1/d$ as $n \to \infty$, for any $d \geqslant 2$ and $u \in \mathbb{N}$. Hence the sequence $\{S_n,\, n \geqslant 1\}$ is a.u.d. ∎

4. RANDOM SEQUENCES SATISFYING A STRENGTHENED A.U.D. PROPERTY

For Bernoulli sums, the a.u.d. property is only a rough aspect of the value distribution of divisors of $\mathcal{B}_n + u$, $u \geqslant 0$ integer. Much more is known.

THEOREM 4.1 ([27, Th. 2.1]). *We have the uniform estimate*

$$\sup_{u\geqslant 0}\sup_{2\leqslant d\leqslant n}\left|\mathbb{P}\{d \mid \mathcal{B}_n + u\} - \frac{1}{d}\sum_{0\leqslant |j|<d} e^{i\pi(2u+n)\frac{j}{d}} e^{-n\frac{\pi^2 j^2}{2d^2}}\right| = \mathcal{O}((\log n)^{5/2} n^{-3/2}).$$

The special case $u = 0$ was proved in [32, Th. II]. We introduce the elliptic Theta function

$$\Theta_u(d,n) = \sum_{\ell\in\mathbb{Z}} e^{i\pi(2u+n)\frac{\ell}{d}}\, e^{-n\frac{\pi^2\ell^2}{2d^2}}. \tag{4.1}$$

By the Poisson summation formula,

$$\Theta_u(d,n) = \left(d\sqrt{\frac{2}{\pi n}}\right)\sum_{\ell\in\mathbb{Z}} e^{-(\ell+\{\frac{u+n/2}{d}\})^2\frac{2d^2}{n}}. \tag{4.2}$$

As a consequence of Theorem 4.1, we get

COROLLARY 4.2. *We have the uniform estimate*

$$\sup_{u\geqslant 0}\sup_{2\leqslant d\leqslant n}\left|\mathbb{P}\{d \mid \mathcal{B}_n + u\} - \frac{\Theta_u(d,n)}{d}\right| \leqslant C(\log n)^{5/2} n^{-3/2}.$$

Apart from this important but specific case, it seems that the speed of convergence in the limit (1.1) has not been investigated, in particular when d and n vary simultaneously.

Consider the independent case and assume, as in Theorem 3.3, that $\nu_n = \sum_{j=1}^n \vartheta_{X_j} \uparrow \infty$. The speed of uniform convergence over regions (in d and n) presents a singularity when d is close to $\sqrt{\nu_n}$. That quantity already appears in Davis and McDonald [1]. On the other hand, when d is not close to $\sqrt{\nu_n}$, in a sense that we shall make precise, we show that an explicit speed of convergence can be assigned, under the *sole* assumption of divergence of the series $\sum_{j=1}^\infty \vartheta_{X_j}$. So, for this important class of independent sequences, the well-known a.u.d. necessary condition turns up to be a particularly weak requirement. Further, one can show by using the Poisson summation formula that in the Bernoulli case, the local limit theorem implies a weaker speed of convergence than the one obtained in Theorem 4.1.

The speed of uniform convergence problem for *all* d and n, $n \geqslant d \geqslant 2$, $n \to \infty$, is more complicated and one must restrict to the i.i.d. case. In place of the limiting term $1/d$ appears the more complicated elliptic Theta function $\Theta_u(d,n)$.

For the independent case, the approach used becomes inoperative, due to appearance of integral products with interlaced integrands. In fact, what will make it possible to handle the independent case is not just that d and $\sqrt{\nu_n}$ are not too close, but also that in the background, symmetry properties of the Bernoulli model permitted to effect the necessary calculations in the first quadrant and *not* in the half-circle. This point is crucial for getting the uniform speed of convergence in Theorem 4.1. This is explained in [27, Lemma 3.3 and after the proof]. In short, when the Bernoulli extraction part applies, these symmetry properties allow one to get a speed of convergence.

The proof for the Bernoulli case can be transposed to other systems of random variables when such symmetries exist. This is not the case for the Hwang and Tsai model of the Dickman function [12, 9], nor for the Cramér model of primes [29]. We prove the following result.

THEOREM 4.3. *Assume that $D = 1$, $\vartheta_{X_j} > 0$ for each j, and the series $\sum_{j=1}^{\infty} \vartheta_{X_j}$ diverges. Let $\alpha > \alpha' > 0$ and $0 < \varepsilon < 1$. Then for each n such that*

$$|x| \leqslant \frac{1}{2}\sqrt{\frac{2\alpha \log(1-\epsilon)\nu_n}{(1-\epsilon)\nu_n}} \implies \frac{\sin x}{x} \geqslant (\alpha'/\alpha)^{1/2},$$

recalling that $\nu_n = \sum_{j=1}^{n} \vartheta_j$, we have

$$\sup_{u\geqslant 0} \sup_{d<\pi\sqrt{\frac{(1-\epsilon)\nu_n}{2\alpha\log(1-\epsilon)\nu_n}}} \left|\mathbb{P}\{d \mid S_n + u\} - \frac{1}{d}\right| \leqslant 2e^{-\frac{\epsilon^2}{2}\nu_n} + ((1-\epsilon)\nu_n)^{-\alpha'}.$$

For the proof we use the following lemma.

LEMMA 4.4 ([14, Theorem 2.3]). *Let $X_1, \dots, X_n$ be independent random variables with $0 \leqslant X_k \leqslant 1$ for each k. Let $S_n = \sum_{k=1}^{n} X_k$ and $\mu = \mathbb{E}\, S_n$. Then for any $\epsilon > 0$,*

$$\text{(a) } \mathbb{P}\{S_n \geqslant (1+\epsilon)\mu\} \leqslant e^{-\frac{\epsilon^2\mu}{2(1+\epsilon/3)}},$$

$$\text{(b) } \mathbb{P}\{S_n \leqslant (1-\epsilon)\mu\} \leqslant e^{-\frac{\epsilon^2\mu}{2}}.$$

We also need the following result.

PROPOSITION 4.5 ([27, Corollary 3.6]).

(i) *For each $\alpha > \alpha' > 0$ and n such that $\tau_n \geqslant (\alpha'/\alpha)^{1/2}$, where*

$$\tau_n = \frac{\sin \varphi_n/2}{\varphi_n/2}, \qquad \varphi_n = \left(\frac{2\alpha \log n}{n}\right)^{1/2},$$

we have

$$\sup_{u\geqslant 0} \sup_{d<\pi\sqrt{\frac{n}{2\alpha\log n}}} \left|\mathbb{P}\{d \mid \mathcal{B}_n + u\} - \frac{1}{d}\right| \leqslant n^{-\alpha'}.$$

(ii) *Let* $0 < \rho < 1$ *and* $0 < \eta < 1$, *and suppose that* n *is so large that* $\widetilde{\tau}_n \geqslant \sqrt{1-\eta}$, *where*

$$\widetilde{\tau}_n = \frac{\sin \psi_n/2}{\psi_n/2}, \qquad \psi_n = \left(\frac{2n^\rho}{n}\right)^{1/2}.$$

Then

$$\sup_{u\geqslant 0} \sup_{d<(\pi/\sqrt{2})n^{(1-\rho)/2}} \left|\mathbb{P}\{d \,|\, \mathcal{B}_n + u\} - \frac{1}{d}\right| \leqslant e^{-(1-\eta)\, n^\rho}.$$

REMARK 4.6. Corollary 3.6 in [27] presents simple cases deduced from Theorem 3.1 in the same paper. It is transparent from the proof that estimates in Theorem 3.1 and Corollary 3.6 are valid for integers $u \leqslant 0$. A reason for this is that the value of u does not play any role in the successive estimates appearing in the proof: this value appears in the complex exponential factor, and is just transported along the proof until the final estimate is established.

Proof of Theorem 4.3. We use the Bernoulli part extraction displayed in Lemma 3.4, (3.5), (3.6) as well as the notation introduced. Let

$$A_n = \{B_n \leqslant (1-\varepsilon)\nu_n\}. \tag{4.3}$$

We deduce from Lemma 4.4 that $\mathbb{P}\{A_n\} \leqslant e^{-\epsilon^2\nu_n/2}$ for all positive n. We write

$$\begin{aligned}\mathbb{P}\{d \,|\, S_n\} - \frac{1}{d} \\ = \mathbb{E}_{(V,\varepsilon)}(\chi(A_n) + \chi(A_n^c))\Big(\mathbb{P}_L\Big\{d \,|\, \Big(\sum_{j=1}^n \varepsilon_j L_j + W_n\Big)\Big\} - \frac{1}{d}\Big).\end{aligned} \tag{4.4}$$

We have

$$\begin{aligned}\mathbb{E}_{(V,\varepsilon)}\,\chi(A_n)\left|\mathbb{P}_L\Big\{d \,|\, \Big(\sum_{j=1}^n \varepsilon_j L_j + W_n\Big)\Big\} - \frac{1}{d}\right| &\leqslant 2\mathbb{P}\{A_n\} \\ &\leqslant 2e^{-\frac{\epsilon^2}{2}\nu_n},\end{aligned} \tag{4.5}$$

so that

$$\begin{aligned}\left|\mathbb{P}\{d \,|\, S_n\} - \frac{1}{d}\right| \\ \leqslant 2e^{-\frac{\epsilon^2}{2}\nu_n} + \mathbb{E}_{(V,\varepsilon)}\,\chi(A_n^c)\cdot\left|\mathbb{P}_L\Big\{d \,|\, \Big(\sum_{j=1}^n \varepsilon_j L_j + W_n\Big)\Big\} - \frac{1}{d}\right|.\end{aligned} \tag{4.6}$$

Now on A_n^c, $B_n \geqslant (1-\epsilon)\nu_n$, and since $\sqrt{x/\log x}$ is increasing on $[e,\infty)$, we have

$$\sqrt{\frac{(1-\epsilon)\nu_n}{2\alpha\log((1-\epsilon)\nu_n)}} \leqslant \sqrt{\frac{B_n}{2\alpha\log B_n}}. \tag{4.7}$$

Also

(4.8)

$$\varphi_n = \sqrt{\frac{2\alpha\log B_n}{B_n}} \leqslant \sqrt{\frac{2\alpha\log((1-\epsilon)\nu_n)}{(1-\epsilon)\nu_n}} \quad \text{and thus} \quad \frac{\sin(\varphi_n/2)}{\varphi_n/2} \geqslant (\alpha'/\alpha)^{1/2},$$

by assumption.

By applying Proposition 4.5 we have, $\mathbb{P}_{(V,\varepsilon)}$ almost surely,

$$\sup_{u\geqslant 0}\ \sup_{d<\pi\sqrt{\frac{B_n}{2\alpha\log B_n}}} \left|\mathbb{P}_L\Big\{d\,|\,\Big(\sum_{j=1}^{B_n} L_j + W_n + u\Big)\Big\} - \frac{1}{d}\right| \leqslant B_n^{-\alpha'}.$$

Hence on A_n^c,

$$\begin{aligned}
\sup_{u\geqslant 0}\ \sup_{d<\pi\sqrt{\frac{(1-\epsilon)\nu_n}{2\alpha\log((1-\epsilon)\nu_n)}}} &\left|\mathbb{P}_L\Big\{d\,|\,\Big(\sum_{j=1}^{B_n} L_j + W_n + u\Big)\Big\} - \frac{1}{d}\right| \\
&\leqslant \sup_{u\geqslant 0}\ \sup_{d<\pi\sqrt{\frac{B_n}{2\alpha\log B_n}}} \left|\mathbb{P}_L\Big\{d\,|\,\Big(\sum_{j=1}^{B_n} L_j + W_n + u\Big)\Big\} - \frac{1}{d}\right| \\
&\leqslant B_n^{-\alpha'} \leqslant ((1-\epsilon)\nu_n)^{-\alpha'}.
\end{aligned} \tag{4.9}$$

In view of (4.6) and (4.9), for all $u \geqslant 0$ and $d < \pi\sqrt{\frac{(1-\epsilon)\nu_n}{2\alpha\log((1-\epsilon)\nu_n)}}$ we get

$$\begin{aligned}
\left|\mathbb{P}\{d\,|\,S_n+u\} - \frac{1}{d}\right| &\leqslant 2e^{-\frac{\epsilon^2}{2}\nu_n} + \ ((1-\epsilon)\nu_n)^{-\alpha'}\,\mathbb{E}_{(V,\varepsilon)}\,\chi(A_n^c) \\
&\leqslant 2e^{-\frac{\epsilon^2}{2}\nu_n} + ((1-\epsilon)\nu_n)^{-\alpha'}. \quad \blacksquare
\end{aligned} \tag{4.10}$$

The next result shows a considerable variation of the speed of convergence when d is less close to $\sqrt{\nu_n}$.

THEOREM 4.7. *Let* $0<\rho<1$ *and* $0<\varepsilon<1$. *Then for each* n *such that*

$$|x| \leqslant \frac{1}{2}\sqrt{\frac{2}{((1-\epsilon)\nu_n)^{1-\rho}}} \implies \frac{\sin x}{x} \geqslant \sqrt{1-\varepsilon}$$

we have

$$\sup_{u\geqslant 0}\ \sup_{d<(\pi/\sqrt{2})((1-\varepsilon)\nu_n)^{(1-\rho)/2}} \left|\mathbb{P}\{d\,|\,S_n+u\} - \frac{1}{d}\right| \leqslant 2e^{-\frac{\epsilon^2}{2}\nu_n} + e^{-((1-\epsilon)\nu_n)^{\rho}}.$$

Proof. The proof is similar. We operate with the same set A_n as in (4.3), and use the decomposition (3.7). Let $0 < \rho < 1$ and $0 < \varepsilon < 1$.

By applying Proposition 4.5 with $\eta = \varepsilon$, for n such that $\widetilde{\tau}_n \geqslant \sqrt{1-\varepsilon}$, where

$$\widetilde{\tau}_n = \frac{\sin(\psi_n/2)}{\psi_n/2} \quad \text{with} \quad \psi_n = \left(\frac{2B_n^\rho}{B_n}\right)^{1/2},$$

we have, $\mathbb{P}_{(V,\varepsilon)}$ almost surely,

$$\sup_{u \geqslant 0} \sup_{d < (\pi/\sqrt{2}) B_n^{(1-\rho)/2}} \left| \mathbb{P}_L \Big\{ d \,|\, \Big(\sum_{j=1}^{B_n} L_j + W_n + u \Big) \Big\} - \frac{1}{d} \right| \leqslant e^{-(1-\varepsilon) B_n^\rho}.$$

By using estimates corresponding to (4.7), (4.8), namely that on A_n^c,

$$\psi_n = \left(\frac{2}{B_n^{1-\rho}}\right)^{1/2} \leqslant \left(\frac{2}{((1-\varepsilon)\nu_n)^{1-\rho}}\right)^{1/2},$$

so that $\widetilde{\tau}_n \geqslant \sqrt{1-\varepsilon}$, we deduce that on A_n^c,

$$\begin{aligned} &\sup_{u \geqslant 0} \sup_{d < (\pi/\sqrt{2})((1-\varepsilon)\nu_n)^{(1-\rho)/2}} \left| \mathbb{P}_L \Big\{ d \,|\, \Big(\sum_{j=1}^{B_n} L_j + W_n + u \Big) \Big\} - \frac{1}{d} \right| \\ &\qquad \leqslant \sup_{u \geqslant 0} \sup_{d < (\pi/\sqrt{2}) B_n^{(1-\rho)/2}} \left| \mathbb{P}_L \Big\{ d \,|\, \Big(\sum_{j=1}^{B_n} L_j + W_n + u \Big) \Big\} - \frac{1}{d} \right| \leqslant e^{-(1-\varepsilon) B_n^\rho}. \end{aligned}$$

Therefore

$$\begin{aligned} &\sup_{u \geqslant 0} \sup_{d < (\pi/\sqrt{2})((1-\varepsilon)\nu_n)^{(1-\rho)/2}} \left| \mathbb{P}\{ d \,|\, S_n + u \} - \frac{1}{d} \right| \\ &\qquad \leqslant 2e^{-\frac{\varepsilon^2}{2}\nu_n} + \mathbb{E}_{(V,\varepsilon)}\, \chi(A_n^c) e^{-(1-\varepsilon) B_n^\rho} \leqslant 2e^{-\frac{\varepsilon^2}{2}\nu_n} + e^{-(1-\varepsilon)^{1+\rho}\nu_n^\rho}. \quad \blacksquare \end{aligned}$$

REMARK 4.8. We have only considered necessary conditions for the validity of the local limit theorem, which are formulated in terms of the a.u.d. property, as well as strengthenings of this property yielding effective speed of convergence bounds. It is important to mention in this context that in 1984, Mukhin found a remarkable necessary and sufficient condition for the validity of the local limit theorem. Let $\{S_n,\, n \geqslant 1\}$ be a sequence of $\mathbb{Z}$-valued random variables such that an integral limit theorem holds: there exist $a_n \in \mathbb{R}$ and real $b_n \to \infty$ such that the sequence of distributions of $(S_n - a_n)/b_n$ converges weakly to an absolutely continuous distribution G with density $g(x)$, which is uniformly continuous in $\mathbb{R}$. The local limit theorem is valid if

$$\mathbb{P}\{S_n = m\} = B_n^{-1} g\left(\frac{m - A_n}{B_n}\right) + o(B_n^{-1}), \tag{4.11}$$

uniformly in $m \in \mathbb{Z}$. Muhkin showed that the validity of the local limit theorem is equivalent to the existence of a sequence of integers $v_n = o(b_n)$ such that

$$\sup_m |\mathbb{P}\{S_n = m + v_n\} - \mathbb{P}\{S_n = m\}| = o\Big(\frac{1}{b_n}\Big). \tag{4.12}$$

Revisiting the succinct proof given in [20], we however could only prove in [28] rigorously a strictly weaker necessary and sufficient condition, with a significantly different formulation, namely that a necessary and sufficient condition for the local limit theorem in the usual form to hold is

$$\sup_{\substack{m,k\in\mathbb{Z} \\ |m-k|\leqslant \max\{1,[\sqrt{\varepsilon}_n b_n]\}}} |\mathbb{P}\{S_n = m\} - \mathbb{P}\{S_n = k\}| = o\Big(\frac{1}{b_n}\Big), \tag{4.13}$$

where

$$\varepsilon_n := \sup_{x\in\mathbb{R}} \Big|\mathbb{P}\Big\{\frac{S_n - a_n}{b_n} < x\Big\} - G(x)\Big| \ \to\ 0, \tag{4.14}$$

by the integral limit theorem. General relations of type (4.12) are also considered in [28]. Mukhin wrote in this regard in [20]: "... getting from here more general sufficient conditions turns out to be difficult in view of the lack of good criteria. Working with asymptotic equidistribution properties is more convenient in this respect".

APPENDIX A. LLTS WITH SPEED OF CONVERGENCE

Let $S_n = X_1 + \cdots + X_n$, $n \geqslant 1$, where X_j are independent random variables such that $\mathbb{P}\{X_j \in \mathcal{L}(v_0, D)\} = 1$.

Assume first that the random variables X_j are identically distributed. Then we have the following characterization result.

THEOREM A.1. *Let F denote the distribution function of X_1.*

(i) ([13, Theorem 4.5.3]) *In order that*

$$\sup_{N=an+Dk} \Big|\frac{\sigma\sqrt{n}}{D}\mathbb{P}\{S_n = N\} - \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(N-n\mu)^2}{2n\sigma^2}}\Big| = \mathcal{O}(n^{-\alpha/2}), \tag{A.1}$$

where $0 < \alpha < 1$, it is necessary and sufficient that the following conditions be satisfied:

$$(1)\ D \text{ is maximal}, \qquad (2) \int_{|x|\geqslant u} x^2 F(dx) = \mathcal{O}(u^{-\alpha}) \quad \text{as } u \to \infty.$$

(ii) ([21, Theorem 6, p. 197]) *If $\mathbb{E}\,|X_1|^3 < \infty$, then* (A.1) *holds with $\alpha = 1/2$.*

Now consider the non-identically-distributed case. Assume that (see (3.2))

(A.2) $$\vartheta_{X_j} > 0, \quad j = 1, \dots, n.$$

Let $\nu_n = \sum_{j=1}^n \vartheta_j$. Let $\psi : \mathbb{R} \to \mathbb{R}^+$ be even, convex and such that $\frac{\psi(x)}{x^2}$ and $\frac{x^3}{\psi(x)}$ are non-decreasing on $\mathbb{R}^+$. We further assume that

(A.3) $$\mathbb{E}\,\psi(X_j) < \infty.$$

Put

$$L_n = \frac{\sum_{j=1}^n \mathbb{E}\,\psi(X_j)}{\psi(\sqrt{\mathrm{Var}(S_n)})}.$$

The following result is Corollary 1.7 in Giuliano–Weber [10].

THEOREM A.2. *Assume that* $\frac{\log \nu_n}{\nu_n} \leqslant 1/14$*. Then, for all* $\kappa \in \mathcal{L}(v_0 n, D)$ *such that*

$$\frac{(\kappa - \mathbb{E}\,S_n)^2}{\mathrm{Var}(S_n)} \leqslant \sqrt{\frac{7\log \nu_n}{2\nu_n}},$$

we have

$$\left| \mathbb{P}\{S_n = \kappa\} - \frac{De^{-\frac{(\kappa-\mathbb{E}\,S_n)^2}{2\mathrm{Var}(S_n)}}}{\sqrt{2\pi \mathrm{Var}(S_n)}} \right| \leqslant C_3 \left\{ D\left(\frac{\log \nu_n}{\mathrm{Var}(S_n)\nu_n} \right)^{1/2} + \frac{L_n + \nu_n^{-1}}{\sqrt{\nu_n}} \right\}.$$

And $C_3 = \max(C_2, 2^{3/2} C_{\mathrm{E}})$*,* C_{E} *being an absolute constant arising from Berry–Esseen's inequality.*

We pass to another speed of convergence result due to Mukhin. Consider the structural characteristic of a random variable X, introduced and studied by Mukhin in [18, 19] for instance,

$$H(X, d) = \mathbb{E}\langle X^* d\rangle^2,$$

where $\langle \alpha \rangle$ denotes the distance from α to the nearest integer, and X^* is a symmetrization of X. Let φ_X be the characteristic function X. The two-sided inequality

(A.4) $$1 - 2\pi^2 H\left(X, \frac{t}{2\pi}\right) \leqslant |\varphi_X(t)| \leqslant 1 - 4H\left(X, \frac{t}{2\pi}\right),$$

is established in the above references. See also Szewczak and Weber [26] for more.

The following is the one-dimensional version of [19, Theorem 5], see also [26] and is stated without proof, however.

THEOREM A.3 (Mukhin). *Let $X_1, \dots, X_n$ have zero mean and finite third moments. Let*

$$B_n^2 = \sum_{j=1}^n \mathbb{E}\,|X_j|^2, \qquad H_n = \inf_{1/4 \leqslant d \leqslant 1/2} \sum_{j=1}^n H(X_j, d), \qquad L_n = \frac{\sum_{j=1}^n \mathbb{E}\,|X_j|^3}{(B_n)^{3/2}}.$$

Then

$$\sup_{N = v_0 n + Dk} \left| B_n \mathbb{P}\{S_n = N\} - \frac{D}{\sqrt{2\pi}} e^{-\frac{(N-M_n)^2}{2B_n^2}} \right| \leqslant C L_n\,(B_n/H_n). \tag{A.5}$$

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IRMA, UMR 7501
Université Louis-Pasteur et C.N.R.S.
Strasbourg, France
E-mail: michel.weber@math.unistra.fr

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