Elements of renewal theory, with applications: Exercise Sheet 3

Alexander Iksanov*

All the problems are to be considered in the class.

1. Prove the following: if $\mathbb{P}\{\xi = 0\} = 0$, $\mathbb{E}\xi^2 < \infty$, and the distribution of ξ is 1-lattice, then
$$U(n) \le \frac{n+1}{\mu} + \frac{\mathbb{E}\xi^2}{\mu^2}, \quad n \in \mathbb{N},$$
where U is the renewal function for an ordinary random walk with generic jump ξ and $\mu = \mathbb{E}\xi < \infty$.

2. Let $\nu(t) = \inf\{k \in \mathbb{N} : S_k > t\}$ for $t \ge 0$. Prove that, for each $p > 0$, the family $\big((\nu(t)/t)^p\big)_{t\ge 1}$ is uniformly integrable, that is, $\lim\limits_{a\to\infty} \sup_{t\ge1} \mathbb{E}(\nu(t)/t)^p \mathbb{1}_{\{(\nu(t)/t)^p > a\}} = 0$. Check that $\lim\limits_{t\to\infty} \mathbb{E}(\nu(t)/t)^p = 1/(\mathbb{E}\xi)^p$ for all $p > 0$, where the right-hand side is interpreted as 0 if $\mathbb{E}\xi = \infty$.

   Hint. Prove that $\sup_{t\ge1} \mathbb{E}(\nu(t)/t)^{p+\varepsilon} < \infty$ for some $\varepsilon > 0$ is sufficient for uniform integrability of $\big((\nu(t)/t)^p\big)_{t\ge1}$. Use distributional subadditivity of $\nu(t)$ to prove the last inequality and thereupon uniform integrability. The strong law of large numbers for $\nu(t)$ in combination with the uniform integrability guarantee convergence of moments, see Theorem 5.5.2 on p. 259 in R. Durrett, *Probability: theory and examples*, 4th edition. Cambridge University Press, 2010.

3. Assuming that $\mathbb{E}\xi^{1+\delta} < \infty$ for some $\delta > 0$, prove that $\lim\limits_{t\to\infty} t^\delta \mathbb{P}\{S_{\nu(t)} - t > at\} = 0$ for every $a > 0$, where $\nu(t)$ is as in Exercise 2.

4. Check that a concave function $f : [0,\infty) \to [0,\infty)$ is subadditive on $[0,\infty)$. Prove that a convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ is *superadditive* on $[0,\infty)$, that is, $g(x+y) \ge g(x) + g(y)$ for $x, y \ge 0$.

5. Let $(\widehat{N}(t))_{t\ge0}$ be a stationary renewal process which corresponds to a nonlattice ordinary random walk with jumps of finite mean $\mu \in (0,\infty)$. Prove that $\mathbb{E}\widehat{N}(t) = t/\mu$ for $t \ge 0$, not using Laplace transforms.

*E-mail: iksan@univ.kiev.ua Homepage: do.unicyb.kiev.ua/iksan

Hint. Use stationarity of increments of $(\widehat{N}(t))$ to check the equality $\mathbb{E}\widehat{N}(t) = t\mathbb{E}\widehat{N}(1)$ for positive integer and then for positive rational t. Use the fact that paths of the process are a.s. nondecreasing and right-continuous in combination with Lebesgue's dominated convergence theorem to prove the aforementioned equality for positive irrational t. Use the elementary renewal theorem to find $\mathbb{E}\widehat{N}(1)$.

6. Let $h : [0, \infty) \to \mathbb{R}$ be a continuous decreasing non-integrable function for which $\int_0^\infty h^2(t)\mathrm{d}t < \infty$. Prove that the limit

$$\lim_{t\to\infty}\left(\int_{[0,\,t]} h(y)\mathrm{d}\widehat{N}(y) - \mu^{-1}\int_0^t h(y)\mathrm{d}y\right)$$

exists in L_2, where $(\widehat{N}(t))_{t\geq 0}$ is a stationary renewal process which corresponds to ξ with $\mathbb{E}\xi^2 < \infty$ and a nonlattice distribution, and $\mu = \mathbb{E}\xi < \infty$.

Hint. In order to show that the limit $\lim\limits_{t\to\infty} X_t$ exists in L_2, it suffices to check that $\lim_{s\to\infty}\sup_{t>s}\mathbb{E}(X_t - X_s)^2 = 0$. Also, it is useful to note that $\mu^{-1}\int_0^t h(y)\mathrm{d}y = \mathbb{E}\int_{[0,\,t]} h(y)\mathrm{d}\widehat{N}(y)$.