Elements of renewal theory, with applications: Exercise Sheet 4

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All the problems are to be considered in the class.

1. Prove that for a lattice ordinary random walk with $\mathbb{E}\xi = \infty$ we have $\lim_{t\to\infty}(U(t+h) - U(t)) = 0$ for any fixed $h > 0$, where U is the corresponding renewal function.

2. Let $E_1, \ldots, E_n$ be independent random variables having the exponential distribution with unit mean which are independent of a zero-delayed ordinary random walk $(S_k)_{k\in\mathbb{N}_0}$ with jumps of infinite mean. Show that $M_n - \nu(\log n) \overset{\mathbb{P}}{\to} 0$ as $n \to \infty$, where $\nu(t) = \inf\{k \in \mathbb{N} : S_k > t\}$ for $t \geq 0$, and M_n is the index of the right-most interval $[S_k, S_{k+1})$ which contains at least one exponential point (out of $E_1, \ldots, E_n$).

3. Suppose that $\mathbb{E}\xi = \infty$ and let $f : [0, \infty) \to [0, \infty)$ be a measurable and locally bounded function such that $\lim_{t\to\infty}(f(t)/\mathbb{P}\{\xi > t\}) = c \in [0, \infty]$. Prove that

$$\lim_{t\to\infty} \int_{[0,\,t]} f(t-y)\mathrm{d}U(y) = c.$$

   Hint. Using Problem 1 in the lattice case and Blackwell's theorem in the nonlattice case prove that $t - S_{\nu(t)-1} \overset{\mathbb{P}}{\to} \infty$ as $t \to \infty$. To proceed, observe that

$$\int_{[0,\,t]} f(t-y)\mathrm{d}U(y) = \mathbb{E}g(t - S_{\nu(t)-1}), \quad t \geq 0,$$

   where $g(t) := f(t)/\mathbb{P}\{\xi > t\}$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying

$$\lim_{x\to\infty}\big(f(x+h) - f(x)\big) = ch$$

   for all $h \in \mathbb{R}$ and some nonnegative constant c. Prove that the convergence in this relation is locally uniform in h, that is, for any finite $a < b$, the convergence is uniform in $h \in [a, b]$.

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Hint. Show that $g(x) := f(x) - cx$ satisfies $\lim_{x\to\infty}\big(g(x+h) - g(x)\big) = 0$ for all $h \in \mathbb{R}$. Check that the function $\ell(x) := \exp\big(g(\log x)\big)$ is slowly varying at ∞, that is,

$$\lim_{x\to\infty} \frac{\ell(xh)}{\ell(x)} = 1$$

for all $h > 0$. The original problem is equivalent to showing that the convergence in the last limit relation is locally uniform on $(0, \infty)$. This is true by Theorem 1.2.1 in N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular variation.* Cambridge University Press, 1987. In the class we shall discuss the 5th proof of this theorem which is based on Egorov's theorem.