Elements of renewal theory, with applications: Exercise Sheet 8

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All the problems will be discussed in the class.

1. Water enters a reservoir at a random rate (phase E). When the water volume reaches 5000 litres, the water stops entering and the period of water usage starts (phase U) and continues until the whole volume has been used up. This two phases process repeats indefinitely.

   Suppose that the rates at which water enters the reservoir during the successive periods are given by independent identically distributed random variables θ_1, $\theta_2, \ldots$ with $\mathbb{E}(1/\theta_1) = a < \infty$ which are also independent of independent identically distributed random variables ρ_1, $\rho_2, \ldots$ which define the rates at which water is consumed during the successive periods of water usage. Assume that $\mathbb{E}(1/\rho_1) = b < \infty$ and that at time zero a phase E starts.

   (a) Find the asymptotics of $\mathbb{E}K(t)$ as $t \to \infty$, where $K(t)$ is the number of complete phases E within $[0, t]$.

   (b) What is the proportion of time that water is consumed?

   (c) Denote by $B(t)$ the duration of the time interval between t and the beginning of the next phase E. Find $\lim_{t\to\infty} \mathbb{P}\{B(t) > x\}$ for $x \geq 0$.

2. Let $U(t)$ be the renewal function for a zero-delayed ordinary random walk with jumps having the uniform distribution on $[0, 1]$. Using the renewal equation for $U(t)$ find $U(t)$ for $t \in [0, 2)$.

3. Let Z satisfy the renewal equation

$$Z = f + F \star Z,$$

   where F is a proper distribution function with mean $\mu < \infty$. Also, let f be a nonnegative nondecreasing function satisfying $f(+\infty) := \lim_{t\to\infty} f(t) < \infty$. Prove that $\lim_{t\to\infty} t^{-1}Z(t) = \mu^{-1} f(+\infty)$. Give two solutions: (a) use Problem 5 from Exercise sheet 6; (b) provide a direct proof.

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4. For $(N(t))_{t\geq 0}$ a renewal process, prove that

$$\mathbb{E}N(t+s)N(t) = \int_{[0,\,t]} U(t+s-y)\mathrm{d}U^*(y) + \int_{[0,\,t]} U^*(t-y)\mathrm{d}U^*(y), \quad t,s\geq 0,$$

where U is the renewal function and $U^*(t) = \mathbb{E}N(t)$, that is, $U^*(t) = U(t) - 1$ for $t \geq 0$. In particular,

$$\mathrm{Var}\, N(t) = 2\int_{[0,\,t]} U^*(t-y)\mathrm{d}U^*(y) + U^*(t) - (U^*(t))^2, \quad t\geq 0. \tag{1}$$

5. Suppose that $\sigma^2 := \mathrm{Var}\,\xi < \infty$ and that the distribution of ξ is nondegenerate.

   (a) Assuming that the underlying zero-delayed ordinary random walk is nonlattice, use formula (1) to prove that

   $$\lim_{t\to\infty} \frac{\mathrm{Var}\, N(t)}{t} = \sigma^2\mu^{-3}, \tag{2}$$

   where $\mu = \mathbb{E}\xi < \infty$ (this is an expected result in view of the central limit theorem for renewal processes).

   (b) Deduce from Problems 7 and 8 (Exercise sheet 2) that relation (2) holds irrespective of whether the underlying random walk is nonlattice or lattice.

   Hint: For part (a), use the argument of the proof of Proposition 38 in the lecture notes to check that

   $$\lim_{t\to\infty} (U^*(t) - t/\mu) = \frac{\mathbb{E}\xi^2}{\mu^2} - 1 =: c.$$

   Show that

   $$\sum_{k\geq 2}(k-1)\mathbb{P}\{S_k\leq t\} = \int_{[0,\,t]} U^*(t-y)\mathrm{d}U^*(y) = \frac{t^2}{2\mu^2} + \frac{2ct}{\mu} + o(t), \quad t\to\infty. \tag{3}$$

   For part (b), show that

   $$\lim_{t\to\infty} \frac{\mathbb{E}(N(t)-t/\mu)^2}{t} = \sigma^2\mu^{-3}$$

   and then prove that $|U^*(t) - t/\mu| \leq C$ for a positive constant $C < \infty$ (if the random walk is nonlattice, this is just Lorden's inequality; if the random walk is d-lattice, obtain a counterpart of Problem 1, Exercise sheet 3 and use a monotonicity argument to prove the desired inequality).

6. For $(N(t))_{t\geq 0}$ the renewal process which corresponds to a zero-delayed ordinary random walk $(S_n)_{n\in\mathbb{N}_0}$ with jumps of mean $\mu\in(0,\infty)$ and $\nu(t) = N(t)+1$, prove that

$$\begin{aligned}&\mathrm{Cov}\,(N(t), S_{\nu(t)} - t)\\ &= \mu\int_{[0,\,t]} U^*(t-y)\mathrm{d}U^*(y) + tU^*(t) - \mu(U^*(t))^2 - \int_0^t U^*(y)\mathrm{d}y, \quad t\geq 0,\end{aligned} \tag{4}$$

where $U^*(y) = \mathbb{E}N(y)$ for $y\geq 0$ and $\mathrm{Cov}\,(X,Y) = \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y$. Assuming that $\mathbb{E}\xi^2 < \infty$ and that $(S_n)_{n\in\mathbb{N}_0}$ is nonlattice, show that $\lim_{t\to\infty} t^{-1}\mathrm{Cov}\,(N(t), S_{\nu(t)} - t) = 0$.

Hint: To obtain (4), use the decomposition

$$\int_0^{S_{\nu(t)}} N(y)\mathrm{d}y = \int_0^t N(y)\mathrm{d}y + \int_t^{S_{\nu(t)}} N(y)\mathrm{d}y$$

having observed that $\int_t^{S_{\nu(t)}} N(y)\mathrm{d}y = N(t)(S_{\nu(t)} - t)$. Also, the first equality in (3) may prove helpful. For the convergence result, the same reasoning as in Problem 5(a) may be used.