Elements of renewal theory, with applications: Exercise Sheet 9

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All the problems will be discussed in the class.

1. Let η_1 and η_2 be independent random variables. Show that $\eta_1 + \eta_2$ has a nonlattice distribution whenever η_1 has a distribution which is not concentrated on $(a + nh)_{n\in\mathbb{Z}}$ for $a, h > 0$. Is the statement true if $a = 0$?

   Hint. Prove the following: (a) if a distribution is δ-lattice and has characteristic function ψ, then $\psi(2\pi/\delta) = 1$; (b) if a characteristic function φ satisfies $|\varphi(t_0)| = 1$ for some $t_0 \neq 0$, $t_0 \in \mathbb{R}$, then the corresponding distribution is concentrated on $(a + k\delta)_{k\in\mathbb{Z}}$ for some $a \geq 0$ and $\delta > 0$ and $\varphi(2\pi/\delta) = \exp(\mathrm{i}(2\pi a)/\delta)$.

2. Let $(\xi_k)_{k\in\mathbb{N}}$ be independent identically distributed positive random variables with finite mean and $(\mathcal{G}_k)_{k\in\mathbb{N}_0}$ is a filtration defined by $\mathcal{G}_0 := \{\emptyset, \Omega\}$ and $\mathcal{G}_k = \sigma(\xi_1, \ldots, \xi_k)$ for $k \in \mathbb{N}$. Show that the distribution of $\xi_1 + \ldots + \xi_\tau$ is nonlattice, where $\tau \geq 1$ is a stopping time with respect to $(\mathcal{G}_k)_{k\in\mathbb{N}_0}$ such that $\mathbb{E}\tau < \infty$, if, and only if, the distribution of ξ_1 is nonlattice.

   Hint: Assuming that the distributions of ξ_1 and $\xi_1 + \ldots + \xi_\tau$ are nonlattice and lattice, respectively, obtain a contradiction as follows. Construct a zero-delayed ordinary random walk with jumps distributed like $\xi_1 + \ldots + \xi_\tau$ which is embedded into the original random walk with jumps ξ_k. Use Blackwell's theorem to conclude.

3. In the setting of Example 105 (shuttle buses at an airport) from the lecture notes find the Laplace transform $\int_0^\infty e^{-st}\mathbb{P}\{X(t) = j\}\mathrm{d}t$ for $j = 0, \ldots, K-1$ and $s > 0$ under the assumption that η has an exponential distribution.

4. Let $(\xi_k, \eta_k)_{k\in\mathbb{N}}$ be independent copies of an $(0,\infty)\times(0,\infty)$-valued random vector (ξ, η) with arbitrary dependence of components. Suppose that $\sigma^2 = \mathrm{Var}\,\xi < \infty$ and $\mathbb{E}\eta = \infty$. Prove a central limit theorem for $R(t) := \sum_{k\geq 1} \mathbb{1}_{\{\xi_1+\ldots+\xi_{k-1}+\eta_k \leq t\}}$:

$$\frac{R(t) - \mu^{-1}t + \mu^{-1}\int_0^t \mathbb{P}\{\eta > y\}\mathrm{d}y}{\sqrt{\sigma^2\mu^{-3}t}} \xrightarrow{d} \text{normal}\,(0,1), \quad t \to \infty,$$

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where $\mu = \mathbb{E}\xi$.

Hint: Prove that (a) $\mathbb{E}C(t) = \int_0^t (1 - G(t))\mathrm{d}y + O(1)$ as $t \to \infty$, where

$$C(t) = \sum_{k\geq 1}(1 - G(t - S_{k-1}))\mathbb{1}_{\{S_{k-1}\leq t\}}$$

and $G(y) = \mathbb{P}\{\eta \leq y\}$; (b) $\operatorname{Var} C(t) = o(t)$ as $t \to \infty$. Use a decomposition

$$\frac{R(t) - t/\mu + \mathbb{E}C(t)}{\sqrt{\sigma^2\mu^{-3}t}} + \frac{C(t) - \mathbb{E}C(t)}{\sqrt{\sigma^2\mu^{-3}t}} + \frac{T(t) - C(t)}{\sqrt{\sigma^2\mu^{-3}t}} = \frac{\nu(t) - t/\mu}{\sqrt{\sigma^2\mu^{-3}t}}, \tag{1}$$

where

$$T(t) = \sum_{k\geq 1}\mathbb{1}_{\{\xi_1+\ldots+\xi_{k-1}\leq t<\xi_1+\ldots+\xi_{k-1}+\eta_k\}}$$

and $\nu(t)$ is the level t first passage time of a zero-delayed ordinary random walk with jumps ξ_k. The right-hand side in (1) converges in distribution to a standard normal variable by the central limit theorem for the renewal processes. The second and third terms on the left-hand side converge to zero in probability by (b) and formula (5.32) from the lecture notes, respectively, in combination with Chebyshev's inequality. Finally, one can replace $\mathbb{E}C(t)$ in the first term on the left-hand side of (1) with $\int_0^t(1 - G(t))\mathrm{d}y$ by (a).

5. Let $(\xi_k, \eta_k)_{k\in\mathbb{N}}$ be independent copies of an $\mathbb{R}^2$-valued random vector (ξ, η) with arbitrary dependence of components. Assuming that $\mathbb{E}\xi \in (-\infty, 0)$ and $\mathbb{E}\eta^+ = \infty$, prove that the series (perpetuity)

$$e^{\eta_1} + e^{\xi_1\eta_2} + e^{\xi_1\xi_2\eta_3} + \ldots$$

diverges almost surely.

6. Let $(W_j)_{j\in\mathbb{N}}$ be independent identically distributed random variables taking values in $(0, 1)$ which are independent of a Poisson process $\Pi := (\Pi(t))_{t\geq 0}$ of unit intensity. Balls are thrown in the boxes $1, 2, \ldots$ at the epochs of the Poisson process Π with (random) probability $W_1 \cdot \ldots \cdot W_{k-1}(1 - W_k)$ of hitting the box k. Denote by $K(t)$ the number of boxes containing at least one ball at time t. Show that, as $t \to \infty$, $K(t) - \mathbb{E}(K(t)|(W_j)) \xrightarrow{\mathrm{P}} 0$ if $\mathbb{E}|\log W| = \infty$, whereas $(K(t) - \mathbb{E}(K(t)|(W_j)))/a(t) \xrightarrow{\mathrm{P}} 0$ for any function $a(t)$ diverging to ∞ if $\mathbb{E}|\log W| < \infty$.

Hint: For $k \in \mathbb{N}$, denote by $\Pi_k(t)$ the number of balls which fall into the box k within $[0, t]$ and observe that $K(t) = \sum_{k\geq 1}\mathbb{1}_{\{\Pi_k(t)\geq 1\}}$. By the thinning property of Poisson processes, given $(W_j)_{j\in\mathbb{N}}$, (a) for each k the process $(\Pi_k(t))_{t\geq 0}$ is a Poisson process of intensity $W_1 \cdot \ldots \cdot W_{k-1}(1 - W_k)$; (b) for different k these processes are independent. Use an extension of Problem 4 from Exercise sheet 5 to the integrals over $[0, \infty)$ rather than $[0, t]$ when the distribution of $|\log W|$ is lattice and Problem 5 (Ex. sheet 5) when the distribution of $|\log W|$ is nonlattice in combination with Problem 2 (Ex. sheet 5) for f_1 to show that, as $t \to \infty$, $\mathbb{E}\,\mathrm{Var}\,[K(t)|(W_j)]$ vanishes in the first case and remains bounded in the second.