Referee Report to the Ph D Thesis
Odwzorowania gęstych podzbiorów płaszczyzny i kostki Hilberta
(= Mappings of dense subsets of the plane and of the Hilobert cube)

The thesis of Pietroń is devoted to certain stronger forms of topological homogneity of the Euclidean plane and of the Hilbert cube. The homogeneity of the Euclidean space is a trivial fact, which however has a fundamental importance for understanding the physical properties (namely, isotropy) of the physical space in which all of us live. In contrast, the homogeneity of the Hilbert cube is rather a surprising and inexpected phenomenon, which was first discovered by Keller in 1931.

Studying properties of the linearly ordered set of rationals, Georg Cantor noticed that any two countable dense subsets of the real line are ambiently homeomorphic. This property of the real line has been called the *countable dense homgeneity.* Afterwards it was noticed that many homogeneous spaces are in fact countable dense homogeneous. In particular, in 1913 Brouwer proved that Euclidean spaces $\mathbb{R}^n$ are countable dense homogeneous; the countable dense homogeneity of the Hilbert cube was proved by Fort in 1962. According to MathReviews, the term "countable dense homogeneus space" was coined by Ben Fitzpatrick in 1972 who proved that a countable dense homogeneous connected locally compact metric space must be locally connected. Since then many brilliant mathematicians (R.Bennet, T.Dobrowolski, K.Kuperberg, W.Kuperberg, J.Kennedy, S.Balwin, S.Watson, P.Simon, R.Heath, J. van Mill. M.Hrusak, I.Farah, J. Dijkstra, A.Medini, A.Arhangelski, L.Zdomskyy, K.Kunen, P.Szeptycki) made their contribution to studying countable dense homogeneous spaces. MathReviews shows 48 papers with the keyword "countable dense homogeneous space".

Returning back to the Ph D Thesis of Pietroń, let us remark that besides the Introduction and Acknowledgements, the thesis consists of two chapters (2 and 3) corresponding to two separate papers, unified by the topic of countable dense homogeneity. The main result of Chapter 2 is Theorem 2.4.7 establishing the countable dense homogeneity of the Hilbert cube $\mathbb{I}^{\mathbb{N}}$ via measure preserving homeomorphisms of $\mathbb{I}^{\mathbb{N}}$. This theorem claims that for any countable dense subsets $A, B \subset \mathbb{I}^{\mathbb{N}}$ there exists a measure-preserving homeomorphism h of $\mathbb{I}^{\mathbb{N}}$ such that $h(A) = B$. To prove the theorem the author develops the measure-preserving modification of the classical Anderson's proof of the homogeneity of the Hilbert cube. The overall proof of Theorem 2.4.7 is technically quite complicated and takes 10 pages of the Thesis.

Theorem 2.4.7 is a nice and important result of "measure-preserving" infinite-dimensional topology, which deserves to be included to future textbooks in infinite-dimensional topology. It should be mentioned that the measure-preserving and analytic countable dense homogeneity of the Euclidean spaces was proved by Michał Morayne in 1987 and independently by J.P.Rosey and W.Rudin in 1988.

The main result of Chapter 3 is also technically quite complicated and concerns the stronger version of countable dense homogeneity of the plane. This main result is Theorem 3.3.1 saying that two countable families $\mathcal{C}$ an $\mathcal{K}$ of subsets of the plane are ambiently homeomorphic, if they consist of pairwise disjoint and pairwise ambiently +-homeomorphic continua whose diameters tends to zero and unions are dense in $\mathbb{R}^2$. This theorem implies the well-known countable dense homogeneity of the plane. For cell-like subsets of the plane theorem 3.3.1 was proved by Banakh and Repovš in 2013. However, the Banakh-Repovš results remains true in all dimensions, whereas Theorem 3.3.1 of Petroń essentially uses some peculiar properties of the Euclidean plane (and does not directly generalize to higher dimensions). The proof of Theorem 3.3.1 follows the idea of the proof of the topological equivalence of all Sepriński carpets on the plane, due to Whyburn (1958) and uses some nontrivial results of Kuratowski, Moore, Oversteegen and Tymchatyn of partitions and isotopies of the plane.

The overall impression of the Thesis is very good. It is accurately written and red out (many times both by dissertant and his scientific supervisor), so it was very difficult to find even a small error. Nonetheless, after some attempts I have found several inessential misprints:

p.3_{10} in 10th line from below on page 3, after "family" it would be helpful to insert $\mathcal{A}$.

p.4. In the 4th line of the subsection 2.1 "prove" should be "proof".

p.7. In the 2-nd line of section 2.4 "the paper" should be changed to "the chapter".

p.15. In the second line of Theorem 3.3.4 the first $cl(D_i') \cap cl(D_j')$ should be $cl(D_i) \cap cl(D_j)$ (i.e., without primes).

p.16. A bit more serious problem concerns the proof of Lemma 3.3.6, namely the continuity of the map Φ which is not that obvious. Fortunately, the proof of this lemma can be rewritten using the well-known fact that the mapping class group of the circle has cardinality 2.

p.17. In the last line it would be helpful to remark that the distance $\rho(A, B)$ is well-known in mathematics as the Hausdorff metric.

p.21. In the 4th line one closing parenthesis should be deleted.

Summing up, I would say that the Ph D Thesis of Maciej Pietroń is very well written, it contains two important and difficult theorems (with long, technical and complicated proofs), witnessing that the dissertant, Maciej Pietroń deserves (without any doubts) awarding the Ph D degree.

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