Generalised solutions to linear and non–linear Schrödinger–type equations with singularities

Ivana Vojnović

(joint work with Nevena Dugandžija, Alessandro Michelangeli)

We consider Schrödinger equations of Hartree and cubic type in three spatial dimensions and its approximations of singular, point-like perturbations.

As approximants to the Hartree equation, we analyze the equation of the form

$$i\partial_t u_\varepsilon \;=\; -\Delta u_\varepsilon + V_\varepsilon u_\varepsilon + (w * |u_\varepsilon|^2)u_\varepsilon\,, \tag{1}$$

for $\varepsilon \in (0,1]$. Here V_ε is a real-valued potential and is meant to represent a singular, delta-like profile centered at $x=0$.

We assume that

$$V_\varepsilon(x) \;:=\; \frac{1}{\varepsilon^\sigma} V\Big(\frac{x}{\varepsilon}\Big)\,, \tag{2}$$

for a given measurable function $V:\mathbb{R}^3 \to \mathbb{R}$ and a given $\sigma \geq 0$.

The corresponding nets of approximate solutions represent generalised solutions for the singular-perturbed Schrödinger equation. The behaviour of such nets is investigated for $\sigma \in [0,3]$.

We also study a generalised solution in the Colombeau algebra $\mathcal{G}_{C^1,H^2}$ for cubic and Hartree equation with (and without) delta potential, which corresponds to the case $\sigma = 3$ in (2). In the case of the Hartree equation with delta potential compatibility between the Colombeau solution and the solution of the classical Hartree equation is established. More precisely, we prove that

$$\lim_{\varepsilon\downarrow 0} \|u_\varepsilon - u\|_{L^\infty([0,\mathrm{T}],L^2(\mathbb{R}^3))} \;=\; 0\,, \tag{3}$$

where Colombeau solution is represented by net (u_ε) and u is the unique solution in $C(\mathbb{R}, L^2(\mathbb{R}^3))$ to the Cauchy problem

$$i\partial_t u \;=\; -\Delta u + (w * |u|^2)u$$

with initial datum $a \in L^2(\mathbb{R}^3)$.

References

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable models in quantum mechanics*, AMS Chelsea Publishing, Providence, RI, (2005)

[2] N. Dugandžija, A. Michelangeli and I. Vojnović, *Generalised solutions to linear and non-linear Schrödinger-type equations with point defect: Colombeau and non-Colombeau regimes*, arXiv:2209.07207

[3] N. Dugandžija and M. Nedeljkov, *Generalized solution to multidimensional cubic Schrödinger equation with delta potential*, Monatsh. Math., 190 (2019), 481–499.

[4] N. Dugandžija and I. Vojnović, *Nonlinear Schrödinger equation with singularities*, *Qualitative Properties of Dispersive PDEs*, V. Georgiev, A. Michelangeli, and R. Scandone editors, Springer-INdAM Series, (2022)

[5] N. Dugandžija and I. Vojnović, *Singular solution of the Hartree equation with a delta potential*, Monatsh Math (2022), https://doi.org/10.1007/s00605-022-01804-z

[6] A. Michelangeli, A. Olgiati, and R. Scandone, *Singular Hartree equation in fractional perturbed Sobolev spaces*, J. Nonlinear Math. Phys., 25 (2018), 558–588.