

Symbolic integration in the spirit of Liouville, Abel and Lie

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Abstract

This paper provides a Liouville principle for integration in terms of elliptic integrals. Our methods are essentially those of Abel and Liouville changed to modern notation. We expose Lie theoretic aspect of Liouville's work.

1 Introduction

Our task is indefinite integration, that is given f from some class A we seek g from possibly different class B such that

$$f = g'.$$

As first step in this direction we would like to delimit form that g can take. Various results in this direction are usually called Liouville principle after famous result of Liouville.

Classically, in computation of field below a hyperbole it was observed that to integrate rational function one needs a logarithm. This lead to class of elementary functions.

In computation of arc length of ellipsis there appear integral

$$E(x, m) = \int \frac{(1-x^2)dx}{\sqrt{(1-x^2)(1-mx^2)}}$$

and question if it can be expressed "in finite terms" (in particular question if is is an elementary function). Abel made first steps and then Liouville proved that among others elliptic integral E is not elementary.

Abel and Liouville combined algebraic and analytic arguments, which causes some doubt about validity of proofs. Also, Liouville proof used repeated integration and differentiation and deeper sense of main part of his argument was

not clear. It turns out that Liouville used differential automorphisms, integration allowed passing info from generator (a derivation) to group elements, differentiation went back. We observed that this argument can be done directly inside Lie algebra of derivations. In effect, our proof is very close to original Liouville's proof. Thanks to use of Abel addition formula for elliptic integrals we can handle elliptic functions and elliptic integrals. It is curious that Liouville missed that: all ingredients were known to him and our main result is natural extension of that of Liouville.

2 Differential fields

Differential field is a field F with a derivation D , that is operation which satisfies Leibniz formula:

$$D(fg) = D(f)g + fD(g)$$

We say that $f \in F$ is a constant if and only if $D(f) = 0$.

Remark: In the sequel we consider only fields of characteristic 0, in finite characteristic several crucial results are no longer valid.

Example: Field $\mathbb{Q}(x, \exp(1/x^2))$. Function $f = ((x^2 - 2)\exp(1/x^2))/x^2$ is an element of this field.

Remark: Given function belongs to many fields, in practice we prefer small fields.

Example: Field of meromorphic functions in a complex area U with usual derivative is a differential field.

For computational purposes this field is too big, we want finitely generated fields. Field of meromorphic functions is in a sense universal example: every finitely generated differential field is isomorphic to a subfield of field of meromorphic functions in some area (Seidenberg [8]).

Assumption that we work inside a differential field introduces some limitations, for example it excludes $|x|$. In case of multivalued function sometimes we need to make choice of branches, for example

$$\sqrt{x}\sqrt{1-x} - \sqrt{x(1-x)}$$

is zero or not depending on choice of branches of square root.

In a sense this is necessary limitation, other settings quickly lead to unsolvable problems ([5], [6]).

2.1 Differential fields, representation

One way to represent differential field is via generators and relations. In characteristic 0 our field is an extension of field of rational numbers. Generators either are transcendental or defined via minimal polynomial of an algebraic extension (that is relation). We define derivative on transcendental generators in arbitrary way and extend to the whole field (in characteristic 0 there exist exactly one extension [3]).

Field $\mathbb{Q}(x, \exp(1/x^2))$ is given in this way, there are two generators (both transcendental) $\theta_1 = x$, $\theta_2 = \exp(1/x^2)$,

$$D(\theta_1) = 1,$$

$$D(\theta_2) = \frac{-2}{\theta_1^3} \theta_2.$$

Alternative point of view is that a differential field is generated by solutions of a system of algebraic differential equations.

From this point of view constants are first integrals of our system of equations, finding them can be a hard problem.

2.2 Differential fields, geometric model

Alternatively, we can treat finitely generated differential field as rational functions on an algebraic manifold M . Derivative D is a vector field on M .

Remark: saying about field of rational functions we automatically make no distinction between manifolds with the same field of rational functions (birationally equivalent).

2.3 Differential fields, towers

For us crucial role is played by extensions of transcendental degree 1, that is pairs $F \subset K$ of differential fields such that K is of transcendental degree 1 over F . Geometrically, K is a field of algebraic functions on a curve (algebraic manifold of dimension 1) over F . Assuming that K is algebraic over $F(\theta)$ we get (variant of) chain rule

$$Df = D_F f + (D\theta)\partial_\theta f$$

where D_F is derivation of K equal to D on F and such that $D_F(\theta) = 0$ and ∂_θ is derivation of K which is zero on F and such that $\partial_\theta(\theta) = 1$.

We obtain more general fields as towers, that is sequences

$$F_0 \subset F_1 \subset F_2 \cdots \subset F_n = K$$

where $F_i \subset F_{i+1}$ is an extension of transcendental degree ≤ 1 .

One can consider more general towers, but here we would like to consider more specific case.

2.4 Differential fields, extensions of degree 1

Typical examples of extensions of degree ≤ 1 , $F \subset F(\theta)$:

- algebraic extension
- extension by a primitive, $D(\theta) = f$ where $f \in F$

- extension by an exponential, $D(\theta) = D(f)\theta$ where $f \in F$
- extension by an elliptic function, $D(\theta) = D(f)q$ where $f \in F$, $q^2 = \theta^3 - a\theta - b$, a and b are constants
- extension by a Lambert W function, $D(\theta) = D(f)\frac{\theta}{f(\theta+1)}$, where $f \in F$

Intuitively in the last three cases we have $\theta = \exp(f)$, $\theta = P(a, b, f)$, $\theta = W(f)$, where P is (an alternative version of) Weierstrass P elliptic function and W is Lambert W function.

In case of elliptic functions corresponding curve is an elliptic curve, in other cases we have projective line. However, algebraic extensions can introduce arbitrary curves.

In the sequel we will need two kinds of primitives: logarithms, where $f = \frac{Dh}{h}$ for $h \in F$ and elliptic integrals.

2.5 Differential fields, elliptic integrals

To define elliptic integrals we assume that there are elements $p, q \in F$ such that $q^2 = p^3 - ap - b$ where a and b are constants.

$\theta \in K$ is an elliptic integral of the first kind if

$$D(\theta) = \frac{Dp}{q}.$$

$\theta \in K$ is an elliptic integral of the second kind if

$$D(\theta) = \frac{pDp}{q}.$$

$\theta \in K$ is an elliptic integral of the third kind if there exist constant c such that

$$D(\theta) = \frac{Dp}{(p-c)q}.$$

Our elliptic integrals are integrals of a differential form on an elliptic curve in Weierstrass form.

Traditional approach uses Legendre form curve:

$$y^2 = (1-x^2)(1-mx^2).$$

Those curves give essentially equivalent theories, however to write a curve in Legendre form we may need additional algebraic extensions. For our purposes we need control over algebraic extensions so Weierstrass form is preferable. Also, integrals on Weierstrass curve fit well with Weierstrass elliptic functions. Legendre curve naturally leads to Jacobi elliptic functions which for us are less convenient.

Elliptic integrals in Legendre form are frequently transformed to trigonometric form. For us this is complication which needlessly introduces transcendental functions.

Crucial for us Abel lemma is proved on curve in Legendre form, thanks to equivalence we will get it also for Weierstrass elliptic integrals.

2.6 Differential fields, extensions of Lie type

All extensions of degree 1 that we considered are Lie closed, that is there exists nonzero derivation X on K such that X commutes with D and X is zero on F . More generally extension $F \subset K$ of transcendental degree n is Lie closed if there are n linearly independent derivations X_i on K which commute with D and are zero on F .

Explicitly:

- For θ which is a primitive $X = \partial_\theta$
- For θ which is an exponential $X = \theta \partial_\theta$
- For θ which is an elliptic function, that is $D(\theta) = D(f)q$ where $f \in F$, $q^2 = \theta^3 - a\theta - b$, a and b are constants we take $X = q \partial_\theta$
- For θ which is a Lambert W

$$X = \frac{\theta}{\theta + 1} \partial_\theta$$

Note that $DX - XD$ is a derivation, so it is enough to check commutation on generators, that is elements of F and θ . On F our X is zero, D preserves F , so both products are zero, so D and X commute on F .

For θ which is a primitive we take $X = \partial_\theta$ so we have $\partial_\theta \theta = 1$, so $DX\theta = 0$. $D\theta \in F$ so $XD\theta = 0$ and again D and X commute.

More generally, for all our θ we have $D\theta = gh(\theta)$ where $g \in F$ and $h(\theta)$ depends only on θ (in the elliptic case h is an algebraic function, in other cases it is rational function). Now our X is

$$X = h(\theta) \partial_\theta$$

and we have

$$\begin{aligned} XD\theta &= Xgh(\theta) = gXh(\theta) = gh(\theta) \partial_\theta h(\theta), \\ DX\theta &= Dh(\theta) = gh(\theta) \partial_\theta h(\theta) \end{aligned}$$

so D and X commute.

In three cases X generates one parameter group of (differential) automorphisms:

- When θ is a primitive, then map $\theta \mapsto \theta + c$ where c is a constant extends to an automorphism of K . In other words translation by a constant gives automorphism of differential field.
- When θ is an exponential, then mapping $\theta \mapsto c\theta$ where c jest nonzero constant extends to an automorphism. So group is multiplicative group of nonzero constants.
- On elliptic curve we have multiplication, multiplication by points with constant coordinates gives automorphisms.

In other words extensions above are differential Galois (Kolchin proves that there are no other differential Galois extensions of degree 1).

The case of Lambert W is different: on algebraic level there are no group.

2.7 Differential fields, indefinite integral

We say that $g \in F$ is an indefinite integral (or a primitive) of $f \in F$ when

$$f = D(g).$$

For example

$$\int ((x^2 - 2) \exp(1/x^2))/x^2 = x \exp(1/x^2).$$

An element of a differential field whose indefinite integral is uniquely determined up to an additive constant (but an element of a differential field may be represented by multiple expressions, so we can get different formulas).

2.8 Elementary extensions

A differential field K is an elementary extension of F if there exists a tower

$$F = F_0 \subset F_1 \subset \dots \subset F_n = K$$

where each of $F_i \subset F_{i+1}$ is an algebraic extension, an extension by a logarithm or an extension by an exponential.

We say that a function f is elementary over F if it is an element of an elementary extension of F .

We say that a function f is elementary if it is elementary over the field of rational functions over constants.

Example: Let $f(x) = \sqrt{\exp(x + \log(x))}$. $f \in K = \mathbb{Q}(x)(\theta_1, \theta_2, \theta_3)$ where $\theta_1 = \log(x)$, $\theta_2 = \exp(x + \log(x))$, $\theta_3 = \sqrt{\exp(x + \log(x))}$. So f is an elementary function.

We can write the function from the previous example as $f(x) = \sqrt{x \exp(x)}$, and use $K = \mathbb{Q}(x)(\theta_1, \theta_2)$ for $\theta_1 = \exp(x)$, $\theta_2 = \sqrt{x \exp(x)}$. However, given an expression for $f(x)$ we can build an elementary extension in a natural way: each logarithm and exponential appearing in f and each irrational algebraic subexpression of f (like a root) is associated with a generator of an extension. Different expressions for f may lead to different elementary extensions.

We normally assume that extensions cannot be written in simpler form. For example, we will treat θ as an exponential or a logarithm only when it is not algebraic over a smaller field. For example, we treat $f(x) = \exp(\frac{\log(x)}{2}) = \sqrt{x}$ not as an exponential but as a solution to the algebraic equation $f^2(x) = x$. Similarly, we do not treat $x = \log(\exp(x))$ as a logarithm.

2.9 Elliptic-Lambert extensions

We will consider also a wider class of extensions "elliptic-Lambert" extensions where we allow towers in which may appear extensions by elliptic functions, elliptic integrals and Lambert W function.

2.10 Commutator formula

Lemma 1 *If X, Y are derivations on K , $p \in K$, p and $\psi(p)$ are algebraically dependent over constants, then*

$$X((Yp)\psi(p)) - Y((Xp)\psi(p)) = ([X, Y]p)\psi(p)$$

Proof: Since derivations extend uniquely to algebraic extensions we have $X\psi(p) = (Xp)\psi'(p)$ and $Y\psi(p) = (Yp)\psi'(p)$ where $'$ denotes unique derivation on $C(p, \psi(p))$ such that $p' = 1$ and C is constant field. Now,

$$X((Yp)\psi(p)) = (XYp)\psi(p) + (Yp)(Xp)\psi'(p),$$

$$Y((Xp)\psi(p)) = (YXp)\psi(p) + (Xp)(Yp)\psi'(p).$$

Subtracting the above give the result. □

3 Abel formula

We will need Abel addition formula for elliptic integrals (in Legendre form). Abel work [1] can be easily modified to modern standards. However, below we present somewhat different argument.

We consider integrals of differential forms on a curve C in Legendre form $y^2 = (1 - x^2)(1 - mx^2)$. Then

$$\Pi'(x) = \frac{1}{(1 - nx^2)y}$$

Let (x_1, y_1) i (x_2, y_2) be points on C and (x_3, y_3) be their sum. Abel gave formula

$$\Pi(x_1) + \Pi(x_2) = C + \Pi(x_3) - \frac{a}{2\Delta(a)} \log \left(\frac{a_0a + a^3 + x_1x_2x_3\Delta(a)}{a_0a + a^3 - x_1x_2x_3\Delta(a)} \right)$$

where

$$n = \frac{1}{a^2},$$

$$\Delta(a) = \sqrt{(1 - a^2)(1 - ma^2)},$$

$$a_0 = \frac{x_2^3y_1 - x_1^3y_2}{x_1y_2 - x_2y_1}.$$

In differential version

$$\frac{dx_1}{(1 - x_1^2/a^2)y_1} + \frac{dx_2}{(1 - x_2^2/a^2)y_2} = \frac{dx_3}{(1 - x_3^2/a^2)y_3} - \frac{adf}{2\Delta(a)f}$$

where

$$f = \frac{a_0a + a^3 + x_1x_2x_3\Delta(a)}{a_0a + a^3 - x_1x_2x_3\Delta(a)}.$$

In this version formula can be checked by direct calculation (it can be done on a computer), we skip details.

We will also need integrals of the first kind F and of second kind E :

$$F(x)' = \frac{1}{y},$$

$$E(X)' = \frac{1 - mx^2}{y}.$$

Lemma 2 (*Abel*)

$$\sum_{k=1}^l F(x_k)' = F(y)',$$

$$\sum_{k=1}^l E(x_k)' = E(y)' + g',$$

$$\sum_{k=1}^l \Pi(x_k)' = \Pi(y)' + f',$$

where y is sum of points on curve C , g is a rational function and f jest a sum of logarithms.

Proof: Formula for Π follows by induction from the formula above. Formulas for F and E are obtained in similar way: direct calculation for $l = 2$ and induction. Explicit formula for g when $l = 2$ is:

$$g = m \frac{-x_1x_2^3 + x_1^3x_2}{x_1y_2 - x_2y_1}$$

□

Remark: Abel obtained formula for F taking limit of formula for Π when n goes to 0 (and similarly for E). This can be justified in purely algebraic way, but in computer era direct calculation is simpler.

4 Liouville-Ostrowski theorem

Liouville-Ostrowski theorem:

Theorem 3 *If F is a differential field, $f \in F$ and f has a primitive in an elementary extension K of F , then there exists extension \bar{F} of F by algebraic constants and functions $v_i \in \bar{F}$ and constants $c_1, \dots, c_l \in \bar{F}$ such that*

$$f = D(v_0) + \sum_{i=1}^l c_i \frac{D(v_i)}{v_i} = D(v_0) + \sum_{i=1}^l c_i D(\log(v_i))$$

For modern proof see [7] (and [6] to get strong result about constants).

Liouville-Ostrowski theorem says that we can find all parts of integral of f already in F extended by algebraic constants. This is crucial property for symbolic integration algorithms.

Using his theorem Liouville proved that $\int \frac{\exp(x)}{x} dx$, $\int \exp(x^2) dx$ and elliptic integrals of the first and second kind are not elementary.

4.1 Generalization

Theorem 4 *If F is a differential field, $f \in F$ and f has a primitive in an elliptic-Lambert extension K of F , then there exist an extension \bar{F} of F by algebraic constants, functions $v_i \in \bar{F}$ and constants $c_1, \dots, c_l \in \bar{F}$, such that*

$$f = D(v_0) + \sum_{i=1}^l c_i \phi(Dv_i, v_i)$$

where $\phi(Dv_i, v_i)$ is of one of forms below

- derivative of a logarithm, that is $\phi(Dv_i, v_i) = \frac{D(v_i)}{v_i}$
- $\phi(Dv_i, v_i) = \frac{Dv_i}{q_i}$
- $\phi(Dv_i, v_i) = \frac{v_i Dv_i}{q_i}$
- $\phi(Dv_i, v_i) = \frac{Dv_i}{(v_i - c_i)q_i}$

where $q_i^2 = v_i^3 - a_i v_i - b_i$, $q_i \in \bar{F}$, a_i, b_i, c_i are constants in \bar{F} .

In other word, we can find all ingredients of the integral already in F extended by algebraic constants.

Remark: Abel proved equivalent result in case when F is algebraic over $C(x)$ (where C is constant field) and K is algebraic over F .

Proof: Proof is via induction over tower. Namely let L be elliptic-Lambert extension in which integral exists. We can write

$$\bar{F} = F_0 \subset F_1 \subset \dots \subset F_n = L$$

where each $F_j \subset F_{j+1}$ is an extension of degree ≤ 1 of form given earlier. By assumption in L we can write integral in the form given in the theorem. So it remains to prove that given expression as above with all parts in F_{j+1} we can

transform it into expression with all parts in F_j extended by constants. We will do this in a few steps.

Step 1. We will prove that:

$$X \frac{Dv}{v} = D \frac{Xv}{v}$$

where X is given earlier derivation commuting with D . Also

$$X \frac{Dp}{(p-c)q} = D \frac{Xp}{(p-c)q}$$

and similarly for remaining $\phi(Dv_i, v_i)$ terms.

Namely, all our $\phi(Dv_i, v_i)$ are of form $(Dv_i)\psi(v_i)$ where ψ is an algebraic function. Since $[X, D] = 0$ equality

$$X\phi(Dp, p) = D\phi(Xp, p).$$

follows from Lemma 1.

Step 2. Now we compute

$$\begin{aligned} 0 = Xf &= XD(v_0) + \sum_{i=1}^l c_i X\phi(Dv_i, v_i) \\ &= DXv_0 + \sum_{i=1}^l c_i D\phi(Xv_i, v_i) \\ &= D(Xv_0 + \sum_{i=1}^l c_i \phi(Xv_i, v_i)) \end{aligned}$$

So

$$c = Xv_0 + \sum_{i=1}^l c_i \phi(Xv_i, v_i)$$

is a constant.

Step 3. Now we use chain rule:

$$D = D_{F_j} + (D\theta)\partial_\theta$$

If θ is a primitive, then the above can be written as

$$D = D_{F_j} + (D\theta)X.$$

Now

$$D(v_0) + \sum_{i=1}^l c_i \phi(Dv_i, v_i) = D_{F_j}(v_0) + \sum_{i=1}^l c_i \phi(D_{F_j}v_i, v_i) +$$

$$\begin{aligned}
& (D\theta)(Xv_0 + \sum_{i=1}^l c_i \phi(Xv_i, v_i)) \\
&= D_{F_j}(v_0) + \sum_{i=1}^l c_i \phi(D_{F_j} v_i, v_i) + cD\theta
\end{aligned}$$

But $D\theta = \phi(Dp, p)$ is in the form required by the theorem, so we can add it as an additional term in the sum. $D_{F_j}(\theta) = 0$, so we have expression of required form in extended by constants.

Step 3'. In Lambert W case we have $D\theta = \frac{D(f)}{f} X\theta$ so $cD\theta$ term can be replaced by logarithmic term.

Step 3". In other cases $D\theta = D(f)X\theta$ and proceeding as before we add cf to v_0 .

It remains to consider algebraic extensions. We do this using methods of Galois theory (or more precisely method of Abel). We use trace and norm maps. For an algebraic extension $F \subset E$ we define

$$\text{Tr}(y) = \sum_{k=1}^m \iota_k(y),$$

$$\text{Norm}(y) = \prod_{k=1}^m \iota_k(y)$$

where ι_k goes over all embeddings of E over F into algebraic closure of F .

We have

$$\frac{D\text{Norm}(y)}{\text{Norm}(y)} = \text{Tr}\left(\frac{Dy}{y}\right).$$

Proof:

$$\text{Norm}(y) = \prod_{k=1}^m \iota_k(y).$$

Since derivative extends uniquely onto algebraic extension we have $D\iota_k(y) = \iota_k(Dy)$. By the logarithmic derivative formula we have

$$\frac{D\text{Norm}(y)}{\text{Norm}(y)} = \sum \frac{D\iota_k(y)}{\iota_k(y)} = \sum \frac{\iota_k(Dy)}{\iota_k(y)} = \sum \iota_k\left(\frac{D(y)}{y}\right) = \text{Tr}\left(\frac{Dy}{y}\right).$$

From Abel formula (Lemma 2)

$$\text{Tr}(D(\Pi(p))) = D\Pi(\tilde{p}) + Df$$

where \tilde{p} is sum of images on curve and f jest a sum of logarithms. By Galois (Abel) theory \tilde{p} is in E . Similar result holds for elliptic integrals of the first and second kind. Passing between Legendre and Weierstrass theory we get similar result for integrals in Weierstrass form.

Now, when $F_j \subset F_{j+1}$ is an algebraic extension and in F_{j+1} we have

$$f = D(v_0) + \sum_{i=1}^l c_i \phi(Dv_i, v_i)$$

and $f \in F_j$, then applying Tr to both sides we get similar formula with terms in F_j extended by constants, which ends the proof of algebraic case.

Our proof can introduce transcendental constants. We should prove that it is enough to use algebraic constants. This can be done in various ways, for example using Hilbert theorem about zeros (like [6]) or adapting model-theoretic proof of Hilbert theorem. We will skip details. \square

5 Further results and remarks

We can strengthen our main theorem by allowing more complicated K . Namely, we can allow towers containing Lie closed extensions such that Lie algebra spanned by X_k is spanned by commutators. Put

$$w_{X_k} = X_k v_0 + \sum_{i=1}^l c_i \phi(X_k v_i, v_i).$$

Proceeding like in step 2 of proof of Theorem we get:

$$X_k w_{X_j} - X_j w_{X_k} = [X_k, X_j] v_0 + \sum_{i=1}^l c_i \phi([X_k, X_j] v_i, v_i).$$

From step 2 we know that w_{X_k} is a constant, so

$$X_k w_{X_j} - X_j w_{X_k} = 0$$

hence

$$[X_k, X_j] v_0 + \sum_{i=1}^l c_i \phi([X_k, X_j] v_i, v_i) = 0$$

We assume that Lie algebra spanned by X_k is spanned by commutators, so also

$$w_{X_k} = 0.$$

But having this we can proceed like in step 3 using formula

$$D = D_{F_j} + \sum a_k X_k$$

where $a_k \in K$ are such that $D\theta_i = \sum a_k X_k \theta_i$ where θ_i is transcendence basis of F_{j+1} over F_j . Terms involving X_k will vanish so

$$f = Dv_0 + \sum_{i=1}^l c_i \phi(Dv_i, v_i) = D_{F_j} v_0 + \sum_{i=1}^l c_i \phi(D_{F_j} v_i, v_i).$$

It is well known (see for example [4]) that Picard-Vessiot extensions are Lie closed and that semisimple Lie algebra is generated by commutators. Many classical special functions are solutions of linear ordinary differential equations, so are elements of Picard-Vessiot extensions. Generically corresponding Lie algebra is semisimple. Let us note that there are a lot of integrals which can be expressed in terms of hypergeometric functions but not in terms of elementary functions. Our result proves that all those hypergeometric functions must correspond to degenerate cases of hypergeometric equation.

Result about elliptic integrals of first and second kind can be proved via main theorem in [2], however to handle elliptic integrals of third kind we need Abel formula.

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