Expansions of vectors spaces with a generic submodule

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Outline

Geometric Theories

Generic predicates

Examples

Generic substructures

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Examples

Special cases

Preservation Theorems for vector space expansions

Stability

NTP2

Thanks

Outline

- Geometric Theories.
- Some expansions and preservation theorems.
- Our work: the vector space with a generic submodule.

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Geometric Theories

We start with T a complete geometric theory:
-In every model M of T the algebraic closure satisfies the exchange property. That is, if $M \models T$, $A \subseteq M$, $b, c \in M$ and $b \in \mathrm{acl}(Ac) \setminus A$ then $c \in \mathrm{acl}(Ab)$.

It induces a good notion of independence:
For $M \models T$, $\vec{c} \in M^n$, $A \subseteq B \subseteq M$,
$\vec{c} \mathop{\smile\!\!\!\!\!\!|}_A B$ if $\dim_{acl}(\vec{c}/A) = \dim_{acl}(\vec{c}/B)$.

-It eliminates the quantifier $\exists^\infty$.

The dimension of a definable set is a definable condition in terms of the parameters. If $M \models T$ is suff. saturated, $|\vec{x}| = n$ there are formulas $\psi_0(\vec{y}), \ldots, \psi_n(\vec{y})$ such that

$$\dim(\varphi(M^{\vec{x}}, \vec{a})) = k \text{ if and only if } M \models \psi_k(\vec{a})$$

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Geometric Theories-examples

Example

T strongly minimal, $\dim(\varphi(\vec{x}, \vec{a})) = MR(\varphi(\vec{x}, \vec{a}))$.

Example

T of SU-rank one, $\dim_{acl}(\vec{c}/\vec{a}) = SU(\vec{c}/\vec{a})$.

Example

T dense *o-minimal*, $\dim(\varphi(\vec{x}, \vec{a})) = geom.\ dim(\varphi(\vec{x}, \vec{a}))$.

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Expansions by random predicates. Chatzidakis-Pillay

Let T be a complete theory with Q.E. that eliminates the quantifier $\exists^\infty$, let P be a new unary predicate and let $L_P = L \cup \{P\}$. Then there is a model companion T^* of T in the language L_P and it satisfies:

1. Every formula is equiv. to an existential formula.
2. T^* eliminates $\exists^\infty$ (Dolich-Miller-Steinhorn).
3. $\mathrm{acl}^* = \mathrm{acl}$.

In particular if T is geometric then T^* is geometric.
If T is o-minimal, then T^* has open core (DMS).

Examples of expansions by random predicates

Example

$T = DLO$

Model of T^**:* $(\mathbb{R}, <, \mathbb{Q})$*. The theory* T^* *is strongly dependent of dp-rank* 1*.*

Example

Let $V = (V, +, 0, \{\lambda_f\}_{f \in \mathbb{F}}) \models T$ *be a vector space,* P *a random predicate and define* aRb *if* $P(a - b)$ *holds, then* R *behaves like a random graph.*

Theorem

(Pillay-Chatzidakis) If T *is simple, then* T^* *is simple and* $\perp\!\!\!\!\smile^* = \perp\!\!\!\!\smile$*.*

Theorem

(Chernikov) If T *is* NTP_2*, then* T^* *is also* NTP_2*.*

Dense Pairs: B., Vassiliev

Start with T a geometric theory with Q.E. and $M \models T$.

Definition

We say that (M, P) is a dense-codense pair if

1. *(Density property) For every $\mathcal{L}$-formula $\phi(x, \vec{y})$ and $\vec{b} \in M$, if $\exists^{\infty} x \phi(x, \vec{b})$, there is $a \in P(M)$ such that $a \models \phi(x, \vec{b})$.*
2. *(Codensity property) For any -formulas $\phi(x, \vec{y})$ and $\psi(x, \vec{y}, \vec{z})$, $n \geq 1$ and $\vec{b} \in M$, if $(\exists^{\infty} x \phi(x, \vec{b}) \wedge \forall \vec{z} \exists^{\leq n} x \psi(x, \vec{y}, \vec{b}))$ then there is $a \in M$ such that $a \models \phi(x, \vec{b}) \wedge \forall \vec{z}(P(\vec{z}) \rightarrow \neg \psi(x, \vec{y}, \vec{b})))$.*

H-structures and lovely pairs

T a geometric theory with Q.E. and $M \models T$.

Definition

Let (M, P) be a dense-codense pair.

1. *(Lovely pair) The pair (M, P) is a lovely pair if $P(M)$ is an elementary substructure of M.*
2. *(H-structures) The pair (M, P) is an H-structure if $P(M)$ is algebraically independent.*

T_P (resp. T^{ind}) common theory of lovely pairs (resp. H structures).
In T_P and T^{ind} every formula is equiv. to a boolean combination of *bounded* existential formulas. If T is stable (respectively NIP), so are T_P and T^{ind}.

If (M, P) is an H-structure, then $(M, \mathrm{acl}(P))$ is a lovely pair.

H-structures and lovely pairs

Example ($T = DLO$.)

Model of T^P *and* T^{ind}: $(\mathbb{R}, <, \mathbb{Q})$. *The theory* T^* *is strongly dependent of dp-rank* 1.

Example (Vector spaces)

Let $V = (V, +, 0, \{\lambda_f\}_{f \in \mathbb{F}})$ *be a vector space over a finite field and* W *a subspace with* $\dim(V/W) = \dim(W) = \aleph_0$. *Then* (V, W) *is a lovely pair, has MR= 2, is* 1*-based.*

Let $(e_i : i \in \mathbb{N})$ *be a basis for* V *and let* $H = (e_{2i} : i \in \mathbb{N})$. *Then* $(V, H) \models T^{ind}$, *has Morley rank* ω *and it is CM-trivial.*

Let $\bar{a} = \mathrm{dcl}(a)$ *and work in* $V^* = \{\bar{a} : a \in V \setminus \{0\}\}$. *Define* $\bar{a}R\bar{b}$ *if there is* $h \in H$ *with* $a \in \mathrm{dcl}(bh) \setminus \mathrm{dcl}(b)$. *Then* V^* *looks like an infinite branching tree.*

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H-structures and lovely pairs

Example

T = a complete theory of SU-rank 1.
Then both T^P and T^{ind} are supersimple of SU-rank $\leq \omega$.

Example

T = a complete NTP_2 geometric theory (for example $(\mathbb{Q}, +, <, R)$). Then both T^P and T^{ind} are NTP_2.

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Work of Block Gorman-Hieronymi-Kaplan

Given two languages $L_0 \subseteq L_1$ say an expansion (M, P) (when M is an L_1-structure and P is an L_0-structure) satisfies the **Mordell-Lang property** if every algebraic L_1-condition of elements in $P(M)$ can be witnessed inside $P(M)$ as an L_0-structure.

Example

(van den Dries, Gunaydin)
Let $T = ACF$ or $T = RCF$ (both geometric) in $\mathcal{L}_{rings}$. Let $F \models T$ and choose $G \leq F^{\times}$. We say that G satisfy the Mordell-Lang property *if for every algebraic set $V \subset F^n$, $V \cap G^n$ is definable in the pure group language. It can be described in first order: for $a_1, \ldots, a_n$ in the prime field of F, the equation*

$$a_1 x_1 + \cdots + a_n x_n = 1$$

only has finitely many non-degenerate solutions.

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Work of Block Gorman-Hieronymi-Kaplan

Theorem

(Block Gorman-Hieronymi-Kaplan)
We are given two languages $L_0 \subseteq L_1$ and an expansion (M, P). Assume T_1 geometric and that Mordell-Lang is first order in a dense-codense expansion. Then

1. *Every formula is a boolean combination of formulas of the form $\exists \vec{x} \in P(\varphi_0(\vec{x}) \wedge \varphi_1(\vec{x}, \vec{y}))$, φ_i an L_i-formula.*
2. *Stability and NIP in T_0 and T_1 transfer to $Th(M, P)$.*

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Expansions of vector spaces: B.-D'Elbée-Vassiliev

Let $V = (V, +, 0, \{\lambda_f\}_{f \in \mathbb{F}}, \dots)$ be an expansion of a vector space and assume that

1. $Th(V)$ eliminates $\exists^\infty$.
2. $\mathrm{acl} = \mathrm{dcl} = span_{\mathbb{F}}$.

Then $Th(V)$ is geometric and dcl is a linear pregeometry.

Let R be a subring of $\mathbb{F}$, let $L_0 = \{+, 0, \{\lambda_f\}_{f \in R}\}$ and let $\hat{R} = Frac(R) =$ smallest subfield extending R.

Expansions of vector spaces: B.-D'Elbée-Vassiliev

We consider expansion in $\mathcal{L}_G = \mathcal{L} \cup \{G\}$ satisfying:

1. Mordell-Lang property
   If $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ are $\hat{R}$-linearly independent, then for all $g_1, \ldots, g_n \in G$
   $$\lambda_1 g_1 + \cdots + \lambda_n g_n = 0 \implies \bigwedge_i g_i = 0.$$
2. (Density Property) for all $r \in R \setminus \{0\}$, rG is dense in V.
3. (Codensity property) V is codense w.r.t. G.

We call T^G the new theory.

Expansions of vector spaces: B-D'Elbée-Vassiliev

Easy first case $\hat{R} = \mathbb{F}$.
The theory of R-modules need not have quantifier elimination, but after adding predicates for all pp-R-formulas to get Q.E., T^G is the *model companion* of the theory of expansion of V by an R-module. Furthermore T^G has Q.E. and for $\vec{a}, \vec{b} \in V^n$, $\mathrm{tp}_{\mathcal{L}_G}(\vec{a}) = \mathrm{tp}_{\mathcal{L}_G}(\vec{b})$ iff

1. $\mathrm{tp}_{\mathcal{L}}(\vec{b}) = \mathrm{tp}_{\mathcal{L}}(\vec{b})$
2. $\mathrm{tp}_{R-mod}(G(\mathrm{span}_{\mathbb{F}}\, \vec{a})) = \mathrm{tp}_{R-mod}(G(\mathrm{span}_{\mathbb{F}}\, \vec{b}))$

Examples

Natural model of T^G:
$(V, +, 0, \dots, \langle H \rangle_R)$, where $(V, +, 0, \dots, H)$ is an H-structure.

Example

T = pure vector space over $\mathbb{F} = \mathbb{Q}$ and take $R = \mathbb{F}$. Then if $(V, W_0) \models T^V = T_P$, it has Morley Rank 2.

Example

Let T = be the theory of a pure vector space over $\mathbb{Q}$ and let $V \models T$ with basis $\{e_i : i \leq \omega\}$. Assume $R = \mathbb{Z}$. Let $W_0 = \mathrm{span}_{\mathbb{Z}}(e_{2i} : i \in \omega)$. Then $(V, W_0) \models T_G$. $Th(V, W_0)$ is strictly stable: $\{2^n W_0 : n \geq 0\}$ is a strictly descending chain of definable subgroups.

Examples

Example

Let $L = \{+, 0, <, \{\lambda_f\}_{f \in \mathbb{F}}\}$ *and let* $T =$ *ordered vector spaces over* $\mathbb{F}$*. Let* $V = \mathbb{R}$*. Take* $R = \mathbb{Q}$*, and let* $(e_i : i \in \omega)$ *be independent dense and* $W_0 = \mathrm{span}_{\mathbb{Z}}(e_i : i \in \omega)$ *then* $(V, W_0) \models T_G = T_P$*. It is strongly dependent of dp-rank* 2*.*

Example

Let $T =$ *be the theory of an ordered vector space over* $\mathbb{Q}$ *and let* $(\mathbb{R}, +, 0, <) \models T$*. Assume* $R = \mathbb{Z}$ *and let* $(e_i : i \in \omega)$ *be independent dense and* $G = \mathrm{span}_{\mathbb{Z}}(e_i : i \in \omega)$*. Then* $(V, G) \models T_G$*. It is NIP, but not strongly dependent, use the family* $\{pW_0 : p \text{ prime}\}$*.*

Examples

Example

Let $\mathbb{F} = \mathbb{Q}$*, let* $L = \{+, 0, <, \{\lambda_f\}_{f \in \mathbb{F}}, S\}$ *and let* $T =$ *ordered vector space over* $\mathbb{F}$ *and* S *a random subset. Take* $R = \mathbb{F}$*, then* $(V, W_0) \models T_G = T_P$ *NTP*$_2$ *(my guess: of inp-rank 2).*

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Expansions of vector spaces: B-D'Elbée-Vassiliev

Second harder case $\hat{R} \subsetneq \mathbb{F}$.
In this case T^V is near model-complete and for $\vec{a}, \vec{b} \in V^n$, both *G-independent*, $\text{tp}_{\mathcal{L}_G}(\vec{a}) = \text{tp}_{\mathcal{L}_G}(\vec{b})$ iff

1. $\text{tp}_{\mathcal{L}}(\vec{b}, G(\vec{a})) = \text{tp}_{\mathcal{L}}(\vec{b}, G(\vec{b}))$
2. $\text{tp}_{R-mod}(G(\vec{a}))) = \text{tp}_{R-mod}(G(\vec{b}))$

In particular, if $\varphi(\vec{x}, \vec{y})$ is an $\mathcal{L}_G$-formula and $\vec{a} \in G$ is G-independent, there are a $\mathcal{L}$-formula $\psi(\vec{x}, \vec{y})$ and an R-module formula $\theta(\vec{x}, \vec{z})$ such that

$$\varphi(\vec{x}, \vec{a}) \wedge G(\vec{x}) \quad \leftrightarrow \quad \psi(\vec{x}, \vec{a}) \wedge \theta(\vec{x}, G(\vec{a})) \wedge G(\vec{x})$$

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Stability, NIP, simplicity, NTP_2

Theorem (B. D'Elbée, Vasiliev)

Let $T = Th(V, +, 0, \{\lambda_f\}_{f\in\mathbb{F}}, \dots)$. Then T^G is near model complete, furthermore:

1. *If T is stable, then T^G is stable.*
2. *If T is NIP, then T^G is NIP.*
3. *If T is NTP_1, then T^G is NTP_1.*
4. *If T is NTP_2, then T^G is NTP_2.*
5. *If T is $NSOP_1$, then T^G is $NSOP_1$.*
6. *(Ahn, Kim, Lee, Lee) Assume T is NATP (antichain tree poperty), then T^G is NTAP.*

Depending on the choice of R we may (or not) preserve being superstable, strong, etc.

NTP2

Definition

A theory T has k-TP2 if there is a formula $\varphi(\vec{x},\vec{y})$ and tuples $\{\vec{a}_{i,j} : i,j<\omega\}$ s.t.

$$\begin{array}{llll} \varphi(\vec{x},\vec{a}_{00}) & \varphi(\vec{x},\vec{a}_{01}) & \varphi(\vec{x},\vec{a}_{02})\dots & \textit{k-inconsistent} \\ \varphi(\vec{x},\vec{a}_{10}) & \varphi(\vec{x},\vec{a}_{11}) & \varphi(\vec{x},\vec{a}_{12})\dots & \textit{k-inconsistent} \\ \varphi(\vec{x},\vec{a}_{20}) & \varphi(\vec{x},\vec{a}_{21}) & \varphi(\vec{x},\vec{a}_{22})\dots & \textit{k-inconsistent} \\ \vdots & \vdots & \vdots & \end{array}$$

Any path going down is consistent.

T has TP2 if it has 2-TP2 for some formula.

T has NTP2 if it does not have TP2.

Important: Consider formulas of the form $\varphi(x,\vec{y})$, i.e. when $|x|=1$ and the parameters in the matrix form an *indiscernible array*.

NTP2

Case 1: We can find a vertical solution with $x \in G$ and assume all $\vec{a}_{ij}$ are G-independent.

$$\begin{array}{llll} \varphi(x,\vec{a}_{00}) \wedge G(x) & \varphi(x,\vec{a}_{01}) \wedge G(x) & \varphi(x,\vec{a}_{02}) \wedge G(x) & \ldots \\ \varphi(x,\vec{a}_{10}) \wedge G(x) & \varphi(x,\vec{a}_{11}) \wedge G(x) & \varphi(x,\vec{a}_{12}) \wedge G(x) & \ldots \\ \varphi(x,\vec{a}_{20}) \wedge G(x) & \varphi(x,\vec{a}_{21}) \wedge G(x) & \varphi(x,\vec{a}_{22}) \wedge G(x) & \ldots \\ \vdots & \vdots & \vdots & \end{array}$$

And any path going down is consistent.
We may exchange $\varphi(x,\vec{a}_{00}) \wedge G(x)$ for a conjunction of a $\mathcal{L}$-formula and a R-module formula. R-module formulas are stable, if $T = Th(V,\ldots)$ is NTP_2, then there is no such array.

Case 2: We can find a vertical solution with $x \in \operatorname{acl}(G)$. Use subadditivity.

Case 3: We can find a vertical solution with $x \notin \operatorname{acl}(G)$. Approximate formulas by $\mathcal{L}$-formulas.

Thanks

Dziekuje Ci

NTAP

Let T be a first-order complete $\mathcal{L}$-theory. We say an $\mathcal{L}$-formula $\varphi(x, y)$ has k-antichain tree property ($k-$ATP) if for any monster model M, there is a tree-indexed set of parameters $(a_\eta)_{\eta \in 2^{<\omega}}$ such that

1. for any antichain X in $2^{<\omega}$, the set $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent;
2. for any pairwise comparable distinct elements $\eta_0, \ldots, \eta_{k-1} \in 2^{<\omega}$, $\{\varphi(x, a_{\eta_i}) : i < k\}$ is inconsistent

For $k = 2$ we write ATP.

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