Convolution Semigroups of measures in NIP groups (joint w/ Kyle Gannon)

T – a complete L-theory

$\mathbb{M} \models T$ – a monster model of T, $M, N \prec \mathbb{M}$.

$S_x(A)$, $A \subseteq \mathbb{M}$

$S_x(\mathbb{M})$ – global types

Def A (Keisler) measure μ in var's x over $A \subseteq \mathbb{M}$ is a finitely additive prob. measure on the A-definable subsets of $\mathbb{M}_x$.

$\mathfrak{M}_x(A)$ – the compact Hausdorff space of Keisler measures over A in the variable x, equipped with the topology induced from $[0,1]^{L_x(A)}$ with the product topology.

Basis of open subsets: $\bigcap_{i<n} \{\mu \in \mathfrak{M}_x(A) : r_i < \mu(\varphi_i(x)) < s_i\}$

for $n \in \mathbb{N}$ and $\varphi_i(x) \in L(A)$, $r_i, s_i \in [0,1]$ for $i < n$.

Rem Every $\mu \in \mathfrak{M}_x(A)$ can be viewed as a measure on the clopen subsets of $S_x(A)$, then extends uniquely to a regular (countably additive) probability measure on Borel subsets of $S_x(A)$. Then the topology above corresponds to the weak*-topology.

Convolution products

the set of all global A-inv. types

Given $p \in S_x(\mathbb{M})$, $q \in S_y(\mathbb{M})$, $p \in \boxed{S^{inv}(\mathbb{M}, A)}$,

then $p \otimes q \in S_{xy}(\mathbb{M})$: for any small $M \preceq N \prec \mathbb{M}$,

we let $p \otimes q|_N = tp(a, b/N)$ for some/any $b \models q|_N$, $a \models p|_{Nb}$.

Assume that T expands a group, then given $p, q \in S_x(\mathbb{M}, A)$, we have an invariant type $p * q \in S_x(\mathbb{M}, A)$, via

$$\varphi(x) \in p * q \iff \varphi(x \cdot y) \in p_x \otimes q_y, \quad \text{for every } \varphi(x) \in L_x(\mathbb{M}).$$

Equivalently, $p * q = tp(a \cdot b / \mathbb{M})$, for some/any $(a, b) \models p \otimes q$.
(in some larger model).

Given $A \subseteq \mathbb{M}$, $S^{inv}_x(\mathbb{M}, A)$ – the (closed) set of global A-inv. types

$S^{fs}_x(\mathbb{M}, A)$ – (closed) set of global types finitely satisfiable in A

$S^{\dagger}_x(\mathbb{M}, A)$ for $\dagger \in \{inv, fs\}$.

$(S^{\dagger}_x(\mathbb{M}, M), *)$ – compact left-continuous semigroup.

i.e. for any $q \in S^{\dagger}_x(\mathbb{M}, M)$, $- * q : S^{\dagger}(\mathbb{M}, M) \circlearrowleft$ is continuous.

Motivation • By a theorem of Newelski, if T is stable, idempotent types $(S^{\dagger}_x(\mathbb{M}, M), *)$ correspond to type-definable subgroups of G.

• Classical case: If G is a locally compact topological grp, the space of regular Borel probability measure on G is equipped with the convolution product

$$\mu * \nu (A) = \boxed{\int_{y \in G} \int_{x \in G} \chi_A(x \cdot y)\, d\mu(x)\, d\nu(y)},$$ for $A \subseteq G$ Borel.

- If G is compact, then μ is idempotent iff the support of μ is a compact subgroup of G and μ restricted to it is the Haar measure. [Kawada, Itô '40], [Wendel '54].
- Same char. extends to locally compact abelian groups [Rudin '59], [Cohen '60], ...

Def $\mu \in M_x(\mathbb{M})$ is Borel definable (over $M \prec \mathbb{M}$) if:

1) for any $\varphi(x,y) \in L_{xy}$ and $b \in \mathbb{M}_y$, $\mu(\varphi(x,b))$ depends only on $tp(b/M)$ (i.e. μ is M-invariant)

2) $q \in S_y(M) \longmapsto \mu(\varphi(x,b)) \in [0,1]$ for some/any $b \models q$ is Borel.

Given $\mu \in \mathfrak{M}_x(\mathbb{M})$, $\nu \in \mathfrak{M}_y(\mathbb{M})$ with μ Borel def. / M,

$\mu \otimes \nu \in \mathfrak{M}_{xy}(\mathbb{M})$ via

$$\mu \otimes \nu(\varphi(x,y)) := \int_{S_y(M)} \mu(\varphi(x,q)) \, d\nu|_M(q)$$

Restrict to NIP groups, i.e. T is an expansion of a group and NIP.

If μ, ν are invariant, $\mu * \nu(\varphi(x)) := \mu_x \otimes \nu_y(\varphi(x \cdot y))$.

Let $\mathfrak{M}^{inv}_x(\mathbb{M}, M)$ — the set of global M-inv. measures

$\mathfrak{M}^{fs}_x(\mathbb{M}, M)$

$\mathfrak{M}^{\dagger}_x(\mathbb{M}, M)$ for $\dagger \in \{fs, inv\}$.

Fact [C, Gannon] If T is NIP, then $\mathfrak{M}^{\dagger}_x(\mathbb{M}, M)$ is compact, left-continuous semigroup.

Given $\mu \in M_x(A)$, $S(\mu) := \{p \in S_x(A) : \varphi(x) \in p \Rightarrow \mu(\varphi(x)) > 0\}$

the support of μ.

Not necessarily a group for an idempotent μ (i.e. $\mu = \mu * \mu$)

(e.g. $\mathcal{M} = (S^1, \cdot, C(x,y,z))$ — the circle group of rotations of the unit circle, C – circular ordering

, μ – the restriction of the Haar measure to definable subsets

$S(\mu)$ is not a group, $(S(\mu), *) \cong S^1 \times \{+,-\}$)

Fact [C., Gannon] Adapting Glicksberg, If $\mu \in M_x(\mathbb{M})$ is definable and idempotent, then $(S(\mu), *)$ is a compact, l.-c.t.cont. semigroup with no closed two-sided ideals.

Fact [C., Gannon] If T is stable, μ is any measure (or G is abelian, μ is g.s.), μ is idempotent iff μ is the unique left-invariant measure on the type-def. subgroup $St(\mu) = \{g \in \mathbb{M} : g \cdot \mu = \mu\}$.

Thm [C., Gannon]. Assume G is NIP, let I be a minimal left ideal of $\mathfrak{M}^+_x(\mathbb{M}, M)$. Then:

1) I is a closed convex subset of $\mathfrak{M}^+(\mathbb{M}, M)$.

2) For any $\mu \in I$, $\pi_*(\mu) = h$, where h is the normalized Haar measure on G/G^{00} and $\pi : G \to G/G^{00}$ is the quotient map.

3) For any idempotent $u \in I$, $u * I$ is trivial.

the ideal group

(In contrast to the case of types, where by the Ellis group conjecture of Newelski / Pillay, if G is def.-amenable, then $u * I \cong G/G^{00}$ — so often ~~non-~~trivial).

4) Assume G is $\boxed{\text{definably amenable}}$. In $\mathfrak{M}^{fs}_x(\mathbb{M}, M)$, minimal left ideals are of the form $I = \{\nu\}$, where $\nu \in \mathfrak{M}^{fs}_x(\mathbb{M}, M)$ is a $G(M)$-left-invariant.

5) In $\boxed{M_x^{inv}(\mathbb{M}, \mathcal{M})}$, there exists a unique minimal left (and two-sided) ideal $I = \{\mu \in M_x^{inv}(\mathbb{M}, \mathcal{M}) : \mu$ is $G(\mathbb{M})$-~~right~~-invariant$\}$.

The set $ex(I)$ is closed (hence I is a Bauer simplex) and equal to $\{\mu_p : p \in S^{inv}(\mathbb{M}, M)$ is right f-generic$\}$.

6) In the fsg, witnessed by a gen. stable G-inv. meas. μ, the $\{\mu\}$ is the unique minimal left ideal.

$\square$

Poulsen simplex

$\Updownarrow$

= extremal points are dense.