Newelski 60 meeting
Będlewo, December 2022
A stable analogue of Szemerédi regularity

Ehud Hrushovski

Independent systems.
A *simplicial complex* is a downward-closed set of subsets of $[n]$.
T any theory; $A(0) \leq M \models T$, $p_1, \ldots, p_n$ commuting invariant types over $acl(A(0))$.
An S *system:* $(A(u) : u \in S)$ substructures of M; $A(u) \subset A(v)$ if $u \subset v$.
Here: An *independent system:*

$$(a_1, \ldots, a_n) \models p_1 \otimes \cdots \otimes p_n | A_0;$$

$$\{a_i : i \in u\} \subset A(u) \subset acl(A_0, a_i : i \in u)$$

and for $|u| \geq 2$,

$$A(u) \perp^a_{\{A(v): v<u\}} \{A(v) : \neg(u \leq v))\}$$

(Galois independence.)

An S-system is *regular* if: for any $S' < S$, independent S'-system B, and embedding $A_{S'} \to B_{S'}$, the system $A + B$ remains independent.

Some points of reference:
Shelah, main gap; A_u elementary submodels. Independence: each $A(u)$ is stably independent from $\{A(v) : \neg(u \leq v)\}$ over $\{A(v) : v < u\}$.

The case:

$$A(u) = acl(\{a_i : i \in u\})$$

For T=ACF (e.g.), this is an independent system. This is key to proving simplicity of pseudo-finite fields.

Let $S \subset P([n])$ be a simplicial complex.
Define $S + S$ the *doubling* of S, a subcomplex of $\pm[n]$:
$|u| = \{|i| : i \in u\}$

$DS := \{u \subset \{\pm\}\times[n] : |u| \in S, \& \neg\exists i \in [n] \ (i \in u \wedge -i \in u).\}$

$D(A_S)$: described by a partial type: an independent system, with each $A_u \cong A_{|u|}$.

Theorem 1. *Let* $(a_1, \ldots, a_n) \models p_1 \otimes \cdots \otimes p_n$.

(a) *Let* $c \in acl(a_1, \ldots, a_n)$. *Then there exists a regular* $P(n)$*-system* $A(u) \subset acl(a_i : i \in u)$ *with* $A(u)$ *f.g. over* A_0, *and* $c \in A([n])$.

(b) *Let* A *be an independent* $P(n)$*-system. Then* A *is regular iff the doubling* $D(A)$ *is unique (a complete type).*

(a): a "finitized Löwenheim-Skolem", top down.
(b, 'only if'): using induction, reduce to $S = P(n)$, with result known for $P(n)^-$; form an S-system B_S, $B(u) = \cup A(u') : u' \in S + S, u' \cap [n] \subseteq u$; this extends A over $P(n)^-$.
We will concentrate on the 'if' direction in (b).

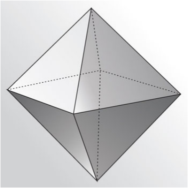
Proof for $n = 1$:
$tp(a)$ is stationary iff $tp(a, b)$ is uniquely determined, where a, b are independent, $tp(a) = tp(b)$.

Proof for $n = 2$

Towards $n \geq 3$

Note that $P(n)$ and $DP(n)$ are $n-1$-dimensional.

The geometric realization of $P(n)$ is an $n-1$-simplex. $DP(n)$ can be presented as the boundary of the convex hull of $\cup_{i\in[n]}\{\pm e_i\}$, where e_i is the i'th unit vector $(0,\cdots,0,1,0,\cdots 0) \in \mathbb{R}^n$. Thus $DP(n)$ is homeomrphic to an $n-1$-sphere.



Hence for any Abelian group C,

$$H^{n-1}(DP(n), C) \cong C$$

In particular if $C \neq 0$ then $H^{n-1}(DP(n), C) \neq 0$.

Since any $n-1$-face of $u \in DP(n)_{n-1}$ belongs to just two facets, The group $H^{n-1}(DP(n), C)$ can be defined as $C^{DP(n)_n}$ modulo the subgroup generated by all $\delta_u - \delta_{u'}$, where $u, u' \in DP(n)_n$ and $|u - u'| = n - 1$.

Proof for $n \geq 3$

Assume A is not regular; we have to show that $D(A)$ is not unique. We will define an invariant of $D(A)$ in $H^{n-1}(D(P(n))$.
We may assume the restriction of A to any horn $H_i = \{u : i \in u\}$ is unique. (Otherwise work over a_i and use induction to prove reducibility of DX.)
For $u \in P(n)^-$ have b_u, that we may take Galois over a_u; and $tp(A[n]/A_{<n})$ does not imply $tp(A[n]/B)$.
Let $G^i = Aut(b^i/A)$ where $b^i = B([n] - \{i\})$. Let $H = Aut(B/A) \leq \Pi_i G^i$. So $H \neq \Pi_i G^i$.

Lemma. *Let* $G^1, \dots, G^n$ *be groups,* $n \geq 3$*, and* $H \leq \Pi_{i\in[n]} G^i$ *Assume* $\pi_u(H) = \Pi_{i\in u} G^i$ *for* $u \in \binom{n}{n-1}$.
Then there exists a (unique!) abelian group C *with surjective homomorphisms* $j_i : G^i \to C$*, such that*

$$H = \{(x_1, \dots, x_n) \in G : \Sigma_i j_i(x_i) = 0\}$$

Proof. We may assume $N_1 =: \{x \in G_1 : (x, 1, \cdots, 1) \in H\} = 1$. To see that G^1 is abelian, let $x, y \in G^1$. Then $(x, 1, 1, \cdots, 1, x')$ and $(y, y', 1, \cdots, 1)$ are in H, for some $x' \in G_n$ and $y' \in G_2$. Taking commutators, we see that $([x, y], 1, \cdots, 1) \in H$ so $[x, y] \in N_1$.

□

Thus we may assume all $G^i = C$; for $v \in B_{n-1}$, b_v lies in an $A(v)$-definable C-torsor T_v; let $T_{[n]} = \Pi_v T_v / H$; then T is again a C-torsor, and by Galois theory, it has a point $t = t(A([n])$ defined over $A([n]) \cup B$.

Now let us move to $D(A)$. *Choose* elements b'_v of each T_v, $v \in D(A)_{n-1}$. Then for $a \in D(A)_n$, $t(a) - (b'_v)_{v \in \binom{a}{n-1}} \in C$. This gives a function from $D(P(n))_n$ to C. Changing the choice of b'_v matters only up to a coboundary. Hence we get a well-defined element of $H^{n-1}(D(P(n)) = C$.

But where this come from?

Regularity: partition version.

Consider formulas ϕ with a partition of their variables: $\phi(x_1, \ldots, x_n)$. The *boundary* of ϕ:

$$\phi^-(x_1, \ldots, x_n) = \bigwedge_{i=1}^{n} (\exists x_i)\phi$$

,

Definition. $\phi(x_1, \ldots, x_n)$ *is (ϵ-)* partition-regular *if it is statistically independent (relative to ϕ^-) from any stronger boundary formula. (up to ϵ.)*

Theorem (Szemeredi,Gowers, Rödl-Skokan). *1. Any $\phi(x_1, \ldots, x_n)$ can be partitioned into finitely any ϵ-regular formulas.*

2. Counting criterion for regularity: take new variables $x_{-1}, \ldots, x_{-n}$. ϕ is regular iff the 2^n formulas $\phi(x_{\pm 1}, \ldots, x_{\pm n})$ are statistically independent, relative to their boundaries.

Here is a precise and ϵ-free statement of (1).

Theorem. *Let μ be a definable measure, with Fubini. Let M be a model. Then*

$$B(u) := L_M(x_i : i \in u)$$

form an independent system of measure algebras.

(Independence = for the stable theory of atomless Boolean algebras with a finitely additive map to $[0, 1]$. (Ben-Yaacov))
Can be proved by methods of Tao, Towsner.
But (1) is apparently of relatively little use without the 'counting criterion' (2). (cf. Gowers norms.)

(2) is proved using analysis (Cauchy-Schwartz iterated $n-1$ times.)

Corollary (Expansions). *If* (T', μ') *expands* (T, μ), *and* ϕ *is partition-regular with respect to* (T, μ), *then* ϕ *is partition-regular with respect to* (T', μ').

An AG analogue of regularity

Basic observation:
o-minimality: A definable X has many subsets, or maps into a finite set (but very few finite covers.)
strong minimality: X has very few subsets, but many finite covers (maps from a definable set, with finite fibers.)
Serre, Grothendieck (implicitly Weil for curves) use this *topologically* to construct l-adic cohomology.
Here we work measure-theoretically.
'measure' $|Y/X| \longleftrightarrow$ degree of $Y \to X$.

Dualize the definition of regularity.

For a simplicial complex $S \subset P(n)$, an S-variety X is just a variety X_u for $u \in S$, and a dominant morphism $p_{u,v} : X_u \to X_v$ when $v \subset u$.
We assume $X_u \to \Pi_{i \in U} X_i$ is dominant and finite.
We say that X is irreducible if the fiber product

$$\widehat{X} := \{x \in \Pi_u X_u : p_{u,v}(x_u) = x_v\}$$

has a unique maximal-dimensional component.
Given a $P(n)$-variety X, let DX be the $DP(n)$-variety with $X_w := X_{|w|}$.

Theorem. *Let X be a $P(n)$-variety. If DX is irreducible, so is any $X \times_{X^-} Y$, Y an irreducible $P(n)^-$-variety over $X^- := X|P(n)^-$. (Special case for ACF of Theorem 1).*

Corollary (Expansions (dual)). *Assume* $T', p'_1, \ldots, p'_n$ *expands* $T, p_1, \ldots, p_n$, *and for* $a_1, \ldots, a_n \models p'_1 \otimes \cdots \otimes p'_n$, T' *adds no structure to* $acl_T(a_1, \ldots, a_n)$ *over* $a_1, \ldots, a_n$, *i.e.*

$$Aut(acl_{T'}(a_1, \ldots, a_n)) \twoheadrightarrow Aut(acl_T(a_1, \ldots, a_n)$$

If A_S *is regular for* T, *it is regular for* T'.

Pseudo-finite bridge

The AG (or stable) regularity was defined by *analogy* with combinatorial regularity; but in the setting of pseudo-finite fields they can be directly compared; a generic automorphism turns *covers* into *partitions*. This can be used to present definable regularity over pseudo-finite fields (Tao, Chevalier-Levi.)

$\pi : Y \to X$ a Galois cover; $X = Y/G$.
X a variety over a pseudo-finite field $F = Fix(\sigma)$.
$Y_h = \{y \in Y : \sigma(y) = h(y)\}$
$X_h = \pi(Y_h)$ partitions X according to conjugacy classes of G.
Geometric doubling $\to$ combinatorial doubling .

GVF's, Bedlewo 2017 (invitation, apology and update)

$$GVF_{k(t)} = Th_{\forall}(k(t)^{alg}, +, \cdot, ht_n)_{n=1,2,\cdots}$$

$$GVF_{\mathbb{Q}} \subseteq Th_{\forall}(\mathbb{Q}^{alg}, +, \cdot, ht_n)$$

Equality? See poster by Michał Szachniewicz.

theorem. *Any quantifier-free GVF type on a curve X over F is determined by:*

1. *The height (of the first nontrivial coordinate).*
2. *the F-adelic qf type. (values of F-adelic formulas.), and*
3. *the Néron character NW_p.*

An extension holds for any smooth projective variety X. It allows defining a *canonical extension* of a qf type over F to one over a GVF $K \geq F$. (1)-formulas over F. (2)-canonical extension of local types. (3) in Hilbert spaces. (4) mass zero to any exceptional divisor strictly over K.

https://people.maths.ox.ac.uk/hrushovski/GVFs/GVF3.pdf

Corollary. *Let $F = F \leq K$. Formulas on X over K are uniform limits of :*

- *algebraically bounded, finite formulas.*
- *formulas over F*
- *adelic formula $R_t(c, b, x)$ over K.*
- *formulas (x, c) giving values of the canonical bilinear map $A \times J$, with $c \in J(K^{alg})$*

To spell out the uniformity: for any such ϕ and $\epsilon > 0$ there exists a combination ψ of the three above forms, such that for any GVF structure L on $K(X)$ extending the given GVF structure on K, $|\phi - \psi|(K) < \epsilon$.

theorem. *The theory GVF is qf stable.*

I.e.: if (a_i, b_i) is a qf-indiscernible sequence and ϕ a formula, then $\phi(a_1, b_2) = \phi(b_1, a_2)$.
revived by: Karim Adiprasito, Itai Ben Yaacov, H.