Generalized continuous model theory and stability

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Logic S_∞-space

Let $L = (R_i^{n_i})_{i \in I}$ be a countable relational language and

$$\mathcal{X}_L = \Pi_{i \in I} 2^{\omega^{n_i}}$$

be the corresponding space under the product topology τ.

$\mathcal{X}_L$ is the space of all L-structures on ω:
$\mathrm{x} = (\ldots x_i \ldots) \in \mathcal{X}_L \Leftarrow$ structure $(\omega, R_i)_{i \in I}$,
R_i is the n_i-ary relation defined by $x_i : \omega^{n_i} \to 2$.

The **logic action** of S_∞ is defined on $\mathcal{X}_L$ by the rule:

$$g \circ \mathrm{x} = \mathrm{y} \Leftrightarrow \forall i \forall \bar{s} (y_i(\bar{s}) = x_i(g^{-1}(\bar{s}))).$$

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Other topologies

For any countable fragment F of $L_{\omega_1\omega}$, which is closed under quantifiers, all sets

$$Mod(\phi, \bar{s}) = \{M \in \mathcal{X}_L : M \models \phi(\bar{s})\} \text{ with } \bar{s} \subset \omega$$

form a basis defining another topology (denoted by t_F) of the S_∞-space $\mathcal{X}_L$.

The logic action of the group S_∞ on $\mathcal{X}_L$ is continuous with respect to t_F.

Space of expansions

Let $G \leq_{closed} S_\infty$.

When $M_0 = (\omega, \ldots)$ with $G = \mathrm{Aut}(M_0)$ then a topology similar to τ can be defined on the G-space of all L-expansions of M_0.

Having an appropriate fragment F of $L_{\omega_1\omega}$, a topology similar to t_F can be defined on this G-space.

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General case when $G \leq_{closed} S_\infty$

Fix $G \leq_{closed} S_\infty$ and
$(\langle \mathcal{X}, \tau \rangle, G)$ = Polish G-space with a countable basis.

Along with τ we shall consider another topology on $\mathcal{X}$.

Nice topology:

(below $\mathcal{N}^G$ = standard basis of the topology of G)

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Nice topology

Definition (H.Becker) A topology **t** on $\mathcal{X}$ is **nice** for the G-space $(\langle \mathcal{X}, \tau \rangle, G)$ if:

(A) **t** is a Polish, **t** is finer than τ and the G-action remains **t**-continuous.

(B) There exists a basis $\mathcal{B}$ for **t** (called **nice**) such that:

1. $\mathcal{B}$ consists of Borel sets and is countable;
2. for all $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2 \in \mathcal{B}$;
3. for all $B \in \mathcal{B}$, $\mathcal{X} \setminus B \in \mathcal{B}$;
4. for all $B \in \mathcal{B}$ and $u \in \mathcal{N}^G$, $B^{\Delta u}, B^{\star u} \in \mathcal{B}$;
5. for any $B \in \mathcal{B}$ there exists an open subgroup $H < G$ such that B is invariant under the corresponding H-action.

Logic space for Polish groups?

Question:
Is it possible to extend the generalised model theory of H.Becker to actions of Polish groups (without the assumption $G \leq S_\infty$) ?

Looking for terminology. Canonical structure for G

Let (G, d) be a Polish group with a left invariant metric ≤ 1.
If $(\mathcal{X}, d)$ is its completion, then $G \leq Iso(\mathcal{X})$.

J.Melleray: Any Polish *G is the automorphism group of the continuous structure on $\mathcal{X}$*, say M_G.

Let $S \subseteq_{cntble,dnse} \mathcal{X}$. Enumerate all orbits of G of finite tuples of S.

For the closure of such an *n*-orbit C define a predicate $R_{\overline{C}}$ on $(\mathcal{X}, d)$ (with continuity moduli $= id$) by

$$R_{\overline{C}}(y_1, ..., y_n) = d((y_1, ..., y_n), \overline{C}) \text{ (i.e. } inf\{d(\bar{y}, \bar{c}) : \bar{c} \in \overline{C}\}).$$

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The space of continuous structures

Fix a continuous signature L and Polish $(\mathcal{Y}, d)$;
S be a dense cntble $\subseteq \mathcal{Y}$.

- The **Polish space** $\mathcal{Y}_L$ *of continuous L-strctres on* $(\mathcal{Y}, d)$:

Metric: Enumerate all $(j, \bar{s})$, where $\bar{s} \in S$ and $|\bar{s}| = arity(R_j)$.
For L-structures M and N on $\mathcal{Y}$ let

$$\delta(M, N) = \sum_{i=1}^{\infty} \{2^{-i} |R_j^M(\bar{s}) - R_j^N(\bar{s})| : i \text{ is the number of } (j, \bar{s})\}.$$

Logic action: the Polish group $Iso(\mathcal{Y})$ acts on $\mathcal{Y}_L$ continuously

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Universality

Theorem ([CL], [IMI])
For any Polish group G there is Polish $(\mathcal{Y}, d)$ and a continuous relational signature L such that

- $G < Iso(\mathcal{Y})$
- for any Polish $(G, \mathcal{X})$ there is a Borel 1-1-map $\mathcal{M} : \mathcal{X} \to \mathcal{Y}_L$ s. t. for any $x, x' \in \mathcal{X}$ structures $\mathcal{M}(x)$ and $\mathcal{M}(x')$ are isomorphic if and only if x and x' are in the same G-orbit,

The map $\mathcal{M}$ is a Borel G-invariant 1-1-reduction of $(\mathcal{X}, E_G)$ to $(\mathcal{Y}_L, E_{Iso(\mathcal{Y})})$.

Looking for terminology

Find counterparts for $Mod(\phi, \bar{s})$ *and* $\bar{s}$*-stabilizers in* S_{∞},

for t_F and for nice topology.

Grey subsets and subgroups

A **grey subset** of $\mathcal{X}$, denoted $\phi \sqsubseteq \mathcal{X}$, is a function $\mathcal{X} \to [0,1]$.

It is **open (closed)**, $\phi \in \Sigma_1$ (resp. $\phi \in \Pi_1$), if the **cone** $\phi_{<r}$ (resp. $\phi_{>r}$) is open for all $r \in [0,1]$

(here $\phi_{<r} = \{z \in \mathcal{X} : \phi(z) < r\}$).

(We also define Borel classes Σ_α, Π_α).

When G is a Polish group, then $H \sqsubseteq G$ is called a **grey subgroup** if $H(1) = 0$, $\forall g \in G(H(g) = H(g^{-1}))$ and

$\forall g, g' \in G(H(gg') \leq H(g) + H(g'))$.

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Grey stabilizer

Basic example:

For $\bar{c}$ from $(\mathcal{Y}, d)$ and a linear δ with $\delta(0) = 0$

grey stabilizer $H_{\delta,\bar{c}} \sqsubseteq Iso(\mathcal{Y})$:

$$H_{\delta,\bar{c}}(g) = \delta((d(\bar{c}, g(\bar{c}))), \text{ where } g \in Iso(\mathcal{Y}).$$

Example: grey subsets of $\mathcal{Y}_L$

A **continuous formula** is an expression built from 0,1 and atomic formulas by applications of the following functions:

$$x/2\ ,\ x \dot{-} y = max(x-y,0)\ ,\ min(x,y)\ ,\ \ldots\ ,\ sup_x \text{ and } inf_x.$$

Any continuous sentence $\phi(\bar{c})$ defines a grey subset of $\mathcal{Y}_L$ which belongs to Σ_n for some n:

$$\phi(\bar{c}) \text{ takes } M \text{ to the value } \phi^M(\bar{c}).$$

Invariant grey subsets

Definition $\mathcal{X} = G$-space.
A grey $\phi \sqsubseteq \mathcal{X}$ is **invariant** with respect to $H \sqsubseteq G$ if for any $g \in G$ we have $\phi(g(x)) \leq \phi(x) \dot{+} H(g)$.

Example: Assuming that continuity moduli of L-symbols are id for any continuous $\phi(\bar{x})$ there is a linear function δ such that

$$H_{\delta,\bar{c}}(g) = \delta((d(\bar{c}, g(\bar{c})))\text{, where } g \in Iso(Y).$$

and the grey subset $\phi(\bar{c}) \sqsubseteq \mathcal{Y}_L$ satisfy

$$\phi^{g(M)}(\bar{c}) \leq \phi^M(\bar{c}) \dot{+} H_{\delta,\bar{c}}(g).$$

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Grey bases

1. Distinguish $\mathcal{R}$, a countable family of open grey $\sqsubseteq G$ so that
   - all $\rho_{<r}$ for $\rho \in \mathcal{R}$ and $r \in \mathbb{Q}$, form a basis of the topology of G.
   - $\mathcal{R}$ consists of **grey cosets**, i.e. for such $\rho \in \mathcal{R}$ there is a grey subgroup $H \in \mathcal{R}$ and an element $g_0 \in G$ so that for any $g \in G$, $\rho(g) = H(gg_0^{-1})$.

2. Considering a $(G, \mathcal{R})$-space $\mathcal{X}$ we distinguish a cntble family $\mathcal{U}$ of open grey sbsts of $\mathcal{X}$ generating the topol. τ.

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Nice basis

Definition. A family $\mathcal{B}$ of Borel grey subsets of the G-space $\mathcal{X}$ is a **nice basis** w.r.to $\mathcal{R}$ if:

- $\mathcal{B}$ is countable and generates the topol. finer than τ;
- for all $\phi_1, \phi_2 \in \mathcal{B}$, the functions $min(\phi_1, \phi_2)$, $max(\phi_1, \phi_2)$, $|\phi_1 - \phi_2|$, $\phi_1 \dot{-} \phi_2$ $\phi_1 \dot{+} \phi_2$ belong to $\mathcal{B}$;
- for all $\phi \in \mathcal{B}$ and rational $r \in [0,1]$, $r\phi$ and $1 - \phi \in \mathcal{B}$;
- for all $\phi \in \mathcal{B}$ and $\rho \in \mathcal{R}$, $\phi^{*\rho}, \phi^{\Delta\rho} \in \mathcal{B}$;
- any $\phi \in \mathcal{B}$ is invariant w.r.to some open grey subgrp $H \in \mathcal{R}$.

A topology $\mathbf{t}$ on $\mathcal{X}$ is $\mathcal{R}$-**nice** for the G-space $\langle \mathcal{X}, \tau \rangle$ if:
(a) $\mathbf{t}$ is Polish, $\mathbf{t}$ is finer than τ and $(G, \mathcal{X})$ is continuous w.r.to $\mathbf{t}$;
(b) there exists a nice basis $\mathcal{B}$ so that $\mathbf{t}$ is generated by all $\phi_{<q}$ with $\phi \in \mathcal{B}$ and $q \in \mathbb{Q} \cap (0,1]$.

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The Urysohn sphere

The Urysohn sphere $\mathfrak{U}$ is the unique Polish metric space of diameter 1 which is universal and ultrahomogeneous.

Approximating substructure: The **rational Urysohn sphere**, $\mathbb{Q}\mathfrak{U}$, is both ultrahomogeneous and universal for countable metric spaces with rational distances and diameter ≤ 1.

There is a nice embedding of $\mathbb{Q}\mathfrak{U}$ into $\mathfrak{U}$.
Let G_0 be a dense countable subgroup of $\mathrm{Iso}(\mathbb{Q}\mathfrak{U})$; we may view it as a subgroup of $\mathrm{Iso}(\mathfrak{U})$.

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The Urysohn sphere as a platform. $\mathcal{R}^{\mathfrak{U}}(G_0)$

Let $\mathcal{R}_0$ be the family of all clopen grey subgroups of $\mathrm{Aut}(\mathfrak{U})$ of the (truncated) form

$$H_{q,\bar{s}} : g \to q \cdot d(g(\bar{s}), \bar{s}), \text{ where } \bar{s} \subset \mathbb{Q}\mathfrak{U}, \text{ and } q \in \mathbb{Q}^+.$$

($\mathcal{R}_0$ is closed under conjugacy by elements of G_0)

Consider the closure of $\mathcal{R}_0$ under the function **max** and define $\mathcal{R}^{\mathfrak{U}}(G_0)$ to be the family of all G_0-cosets of grey subgroups from $\max(\mathcal{R}_0)$.

The Urysohn sphere as a platform. $\mathcal{R}^{\mathfrak{U}}(G_0)$

Let $\mathcal{R}_0$ be the family of all clopen grey subgroups of Aut($\mathfrak{U}$) of the (truncated) form

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The Urysohn space as a platform. $\mathcal{B}_{\mathcal{L}}$

Let $\mathcal{L}$ be a countable fragment of $L_{\omega_1 \omega}$ and

let $\mathcal{B}_{\mathcal{L}}$ be the family of all grey subsets of $\mathfrak{U}_{\mathcal{L}}$ defined by continuous $\mathcal{L}$-sentences (with parameters from $\mathbb{Q}\mathfrak{U}$) as above.

The space $\mathfrak{U}_L$

Theorem (IMI17)

The family $\mathcal{B}_{\mathcal{L}}$ is a $\mathcal{R}^{\mathfrak{U}}(G_0)$-nice basis.

Similar constructions (with weaker forms of this theorem, where **nice** *is replaced by* **good***):*

- The complex Hilbert space $l_2(\mathbb{N})$ (and the countable approximating substructure $\mathcal{Q}l_2$).
- The measure algebra on $[0, 1]$.

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Existence

Theorem ([IMI17]).
Let $(G, \mathcal{R})$ be a Polish group with $\mathcal{R}$ satisfying
(i) for every grey subgroup $H \in \mathcal{R}$ if $gH \in \mathcal{R}$, then $H^g \in \mathcal{R}$;
(ii) $\mathcal{R}$ is closed under **max** and multiplying by rationals.
Let $\langle \mathcal{X}, \tau \rangle$ be a G-space and
$\mathcal{U}$ be a countable family of Borel grey subsets of $\mathcal{X}$ generating a topology finer than τ, so that
each $\phi \in \mathcal{U}$ is invariant w.r.to some grey subgroup $H \in \mathcal{R}$.

Then there is an $\mathcal{R}$-nice topology for $(\langle \mathcal{X}, \tau \rangle, G)$ so that $\mathcal{U}$ consists of open grey subsets.

Ubiquity in the case of $\mathfrak{U}$

Theorem ([IMI17]). Let $G = \mathrm{Iso}(\mathfrak{U})$.
Consider the logic G-space $\mathfrak{U}_L$ under the standard topology τ.
Let $\mathcal{F}$ be a countable family of Borel grey subsets of $\mathfrak{U}_L$ generating a topology finer than τ such that any $\phi \in \mathcal{F}$ is invariant w.r.to a grey subgroup $H \in \mathcal{R}^{\mathfrak{U}}$.

Then there is an $\mathcal{R}^{\mathfrak{U}}$-nice topology $\mathbf{t}$ for the G-space $\langle \mathfrak{U}_L, \tau \rangle$ which is generated by some countable fragment of $L_{\omega_1 \omega}$ such that $\mathcal{F}$ consists of $\mathbf{t}$-open grey subsets.

Lindström

G is a Polish group with a grey basis $\mathcal{R}$ consisting of grey cosets, $\langle \mathcal{X}, \tau \rangle$ is a Polish G-space, ect.

Theorem

Let $\mathbf{t}$ be $\mathcal{R}$-good.
Let $Y = Gx_0$ for some (any) $x_0 \in Y$ and Y be a G_δ-subset of $\mathcal{X}$.
Then both topologies τ and $\mathbf{t}$ are equal on Y.

Companions

Let X_0 and X_1 be closed invariant subsets of $(\mathcal{X}, \mathbf{t})$.
X_1 is a **companion** of X_0 if τ-closures of X_0 and X_1 coincide and any element of $\mathcal{B}$ is τ-clopen on X_1.

Effros space

Given a Polish space $\mathcal{Y}$ the **Effros structure** on $\mathcal{F}(\mathcal{Y})$ is the Borel space with respect to the σ-algebra generated by

$$\mathcal{C}_U = \{D \in \mathcal{F}(\mathcal{Y}) : D \cap U \neq \emptyset\},$$

for open $U \subseteq \mathcal{Y}$.

Given a Polish group G and a continuous (or Borel) action $(G\ \mathcal{Y})$, grey basis $\mathcal{R}$ (for G) and $\mathcal{B}$ (for $\mathbf{t}$ on $\mathcal{Y}$) consider $\mathcal{F}((\mathcal{Y},\mathrm{t}) \times G)$.

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Complete theories

G is a Polish group with a grey basis $\mathcal{R}$ consisting of grey cosets, $(\mathcal{X}, \tau)$, $\mathbf{t}$, $\mathcal{B}$, ...

Observation. The set of indecomposable G-invariant members $X \in \mathcal{F}(\mathcal{X}, \mathbf{t})$ (i.e. "complete theories") is Borel.

Complexity of companions

Theorem. The set of pairs (X_0, X_1) of G-invariant members of $\mathcal{F}(\mathcal{X}, \mathbf{t})$ with the condition that X_1 is a companion of X_0 is Borel.

Universality of $\mathfrak{U}_L$

Coskey and Lupini:
For any Polish G and any standard Borel G-space $\mathcal{X}$ there is a continuous group monomorphism $\Phi : G \to \mathrm{Iso}(\mathfrak{U})$ and a Borel Φ-equivariant injection $f : \mathcal{X} \to \mathfrak{U}_L$.

All Polish groups can be considered as elements of $\mathcal{F}(\mathrm{Iso}(\mathfrak{U}))$,
Polish spaces are elements of $\mathcal{F}(\mathfrak{U}_L, \mathrm{t})$ and
Polish G-spaces are pairs from $\mathcal{F}((\mathfrak{U}_L, \mathrm{t}) \times \mathrm{Iso}(\mathfrak{U}))$.

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A substitute for stability?

Given a $(G, \mathcal{R})$-space $\mathcal{X}$, a nice/good basis $\mathcal{B}$,
a grey subset $\phi \in \mathcal{B}$ and a **t**-closed subset $Y \subseteq \mathcal{X}$

define the notion ϕ is **unstable** w.r. to Y in terms of generalised model theory.

Stable platform

- Let $\mathfrak{X}$ be a metric structure and N be a countable approximating substructure (denoted by $N = \mathbb{Q}\mathfrak{X}$).
- Let $\mathcal{R}^{\mathfrak{X}}(G_0)$, L, $\mathcal{B}_{\mathcal{L}}$ be as before ($G_0 \leq \mathrm{Iso}(\mathbb{Q}\mathfrak{X})$),
- Y be a **t**-closed subset of $\mathfrak{X}_L$ which corresponds to a complete first order theory of L-expansions of $\mathfrak{X}$,
- **Assume that $\mathfrak{X}$ is a stable continuous structure. Assume that $\phi(\bar{x}, \bar{x}')$ is unstable with respect to Y.**

Stable platforms

Then there exists $\bar{s}_1\bar{s}'_1 \ldots \bar{s}_n\bar{s}'_n \ldots$ such that there in some $M \in Y$ where

$$lim_i lim_j \phi(\bar{s}_i, \bar{s}'_j) \neq lim_j lim_i \phi(\bar{s}_i, \bar{s}'_j) \text{ and one of these limits exists,}$$

but for any formula $\hat{\theta}(\bar{x}, \bar{x}')$ of the language of $\mathfrak{X}$

$$lim_i lim_j \hat{\theta}(\bar{s}_i, \bar{s}'_j) = lim_j lim_i \hat{\theta}(\bar{s}_i, \bar{s}'_j).$$

The following platforms are stable:

- The complex Hilbert space $l_2(\mathbb{N})$.
- The Polish ultrametric Urysohn space for $\mathbb{Q} \cap [0, 1]$.

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A substitute

Let $\mathfrak{X}$, $\mathbb{Q}\mathfrak{X}$, $\mathcal{R}^{\mathfrak{X}}(G_0)$, L, $\mathcal{B}_{\mathcal{L}}$ ($G_0 \leq \mathrm{Iso}(\mathbb{Q}\mathfrak{X})$) and a **t**-closed subset $Y \subseteq \mathfrak{X}_L$ be as above.
Let $\bar{s}, \bar{s}' \in \mathbb{Q}\mathfrak{X}$ and $\phi(\bar{s}, \bar{s}') \in \mathcal{B}_{\mathcal{L}}$.

Definition. The grey set $\phi(\bar{s}, \bar{s}')$ is **unstable** with respect to Y if

- there are rational r_1 and $r_2 \in [0,1]$ such that $r_1 < r_2$ and
- for any n and any $\varepsilon \in \mathbb{Q} \cap [0,1]$ there exist $\bar{s}_1, \bar{s}'_1, \ldots, \bar{s}_n, \bar{s}'_n \in \mathbb{Q}\mathfrak{X}$ such that $d(tp^{\mathfrak{X}}(\bar{s}_i \bar{s}'_j), tp^{\mathfrak{X}}(\bar{s}\bar{s}')) \leq \varepsilon$ for all $i, j \leq n$ and

$$Y \cap \bigcap \{(\phi(\bar{s}_i, \bar{s}'_j) \dot{-} r_1)_{\leq \varepsilon} : i < j\} \cap \bigcap \{(r_2 \dot{-} \phi(\bar{s}_i, \bar{s}'_j))_{\leq \varepsilon} : j \leq i\} \neq \emptyset.$$

A Borel substitute

Given $(G,\mathcal{R})$-space $\mathcal{X}$, a nice basis $\mathcal{B}$, a grey subset $\phi \in \mathcal{B}$.
Definition. A t-closed $Y \subseteq \mathcal{X}$ is **unstable relatively to** $(\mathcal{X},\mathcal{B})$ if

- there is a grey set $\phi \in \mathcal{B}$ and grey subgroups $H, H' \in \mathcal{R}$ s. t. ϕ is invariant w. r. to $\max(H, H')$ and
- for any n and any $\varepsilon \in \mathbb{Q} \cap [0,1]$ there exist $g_{1,1}, g_{1,2}, \ldots, g_{1,n}, g_{2,1}, \ldots, g_{n-1,n}, g_{1,n}, \ldots, g_{n,n} \in G_0$ such that $H(g_{i,j}^{-1} g_{i,l}) \leq \varepsilon$ and $H'(g_{i,j}^{-1} g_{l,j}) \leq \varepsilon$ for all $i,j,l \in \{1,\ldots,n\}$ and

$$Y \cap \bigcap \{(g_{i,j}\phi)_{\leq\varepsilon} : i < j\} \cap \bigcap \{(g_{i,j}\phi)_{\geq 1-\varepsilon} : j \leq i\} \neq \emptyset.$$

Theorem. The family $\{Y \in \mathcal{F}(\mathcal{X},\mathrm{t}) \,|\, Y$ is G-invariant and stable relatively to $(\mathcal{X},\mathcal{B})\}$ is Borel.

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An example of a relatively stable piece over $\mathfrak{U}$

The platform $\mathfrak{U}$ is not stable.
Fix $\mathbb{Q}\mathfrak{U}$ (rational Urysohn space) , G_0 (dense cntable $\leq \mathrm{Iso}(\mathbb{Q}\mathfrak{U})$) , $\mathcal{R}^{\mathfrak{U}}(G_0)$, $\mathcal{B}_{\mathcal{L}}$. and let $\mathcal{X} = \mathfrak{U}_L$.

- *Continuous signature:* (d, P) where P is unary with continuity modulus id.
- *Age* $\mathcal{K}$: all finite metric spaces (A, d, P) such that there is a metric space $B \supseteq A$ and some $B_0 \subseteq B$ s.t. P extends to the function $d(x, B_0)$ on B.

$\mathcal{K}$ is a *Fraïssé class*, i.e. it has HP, JEP, NAP, PP (Polish Property), CP (Continuity Property).

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A relatively stable piece over $\mathfrak{U}$

Let M be the Faïssé limit of $\mathcal{K}$. Then M is isometric to a structure of the form $(\mathfrak{U}, d, P)$.

Theorem. Let $M = (\mathfrak{U}, d, P)$ be the Faïssé limit of $\mathcal{K}$.
Let Y be the t-closed set defined by $Th(M)$.
Then $Th(M)$ is separably categorical and any grey set $\phi(\bar{s}, \bar{s}')$ is stable with respect to Y.

(any formula $\phi(\bar{x}, \bar{y})$ is stable with respect to $Th(\mathfrak{U})$)

Corollary from the proof

Let M be as in the formulation of Theorem , $\phi(\bar{x},\bar{y})$ be any formula of the language of this structure and $p(\bar{x})$ and $q(\bar{y})$ be types of $Th(M)$.

(1) Then there is a 0-definable predicate $\theta(\bar{x},\bar{y})$ of $Th(\mathfrak{U})$ such that $\phi(\bar{x},\bar{y}) - \theta(\bar{x},\bar{y})$ (viewed in $[-1,1]$) is stable on $p(\bar{x}) \cup q(\bar{y})$.

(2) The condition that $\phi(\bar{x},\bar{y})$ is stable on $p(\bar{x}) \cup q(\bar{y})$ is equivalent to the condition that $\phi(\bar{x},\bar{y})$ is constant on $p(\bar{x}) \cup q(\bar{y})$.

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Appendix: Relative stability

Let M be a continuous metric structure and $\phi(\bar{x},\bar{y})$ and $\theta(\bar{x},\bar{y})$ be formulas with parameters from M.

Definition

We say that $\phi(\bar{x},\bar{y})$ is **stably equivalent to** $\theta(\bar{x},\bar{y})$ **with respect to** $Th(M)$ if the formula $(\phi-\theta)(\bar{x},\bar{y})$ (viewed in $[-1,1]$) is stable.

- Stable equivalence is an equivalence relation on the set of formulas.
- If a formula $\phi(\bar{x},\bar{y})$ is stably equivalent to $\theta(\bar{x},\bar{y})$ with respect to $Th(M)$ then if $\phi(\bar{x},\bar{y})$ is stable with respect to $\theta(\bar{x},\bar{y})$ and vice versa.

Relative stability of formulas

(without the assumption of satbility of $Th(\mathfrak{X})$)

$\phi(\bar{x},\bar{x}')$ is **unstable** w.r. to Y and $\hat{\theta}$ if there exists $\bar{s}_1\bar{s}'_1 \dots \bar{s}_n\bar{s}'_n \dots$ such that

$$lim_i lim_j \hat{\theta}(\bar{s}_i,\bar{s}'_j) = lim_j lim_i \hat{\theta}(\bar{s}_i,\bar{s}'_j)$$

and

$$lim_i lim_j \phi(\bar{s}_i,\bar{s}'_j) \neq lim_j lim_i \phi(\bar{s}_i,\bar{s}'_j)$$

in some $M \in Y$.

Appendix: Definability of types

Observation. Assume that $\phi(\bar{x}, \bar{y})$ is stably equivalent to $\theta(\bar{x}, \bar{y})$ with respect to $Th(M)$.
Let $p(\bar{x}) \in S_\phi(M)$ and $\bar{a}$ be its realization.
Let $q(\bar{x}) \in S_\theta(M)$ be defined by $\bar{a}$.
Let $\psi(\bar{y})$ be the definable predicate over M which defines the $(\phi - \theta)$-type of $\bar{a}$ over M

Then the type $p(\bar{x})$ is defined by the sum $\phi(p, \bar{y}) = \theta(q, \bar{y}) + \psi(\bar{y})$.

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Appendix: Existence of non-forking extensions

Let $A \subset M$ and $p(\bar{x}) \in S(A)$.
Let $\phi_i(\bar{x}, \bar{y})$ $(i \leq n)$ be formulas such that each of them is stably equivalent to $\theta(\bar{x}, \bar{y})$ with respect to $Th(M)$.
Let $\Delta = \{(\phi_i - \theta)(\bar{x}, \bar{y}) | 1 \leq i \leq n\}$.

Then there are definable predicates $\psi_i(\bar{y})$, $1 \leq i \leq n$, almost over A which define a Δ-type over M consistent with $p(\bar{x})$ such that $\phi_i(\bar{x}, \bar{y}) = \theta(\bar{x}, \bar{y}) + \psi_i(\bar{y})$.

(reformulation Lemmas 2.7 and 2.18 of [Pillay-Book-1996] and Lemma 2.2 and Corollary 2.4 of [Ben Yaacov-JSL-2008])

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