An Ax-Kochen/Ershov principle for deeply ramified fields

jt work with Konstantinos Kartas

Happy birthday Ludomir!

§ Introduction

Notation: (K,v) valued field, vK value group, Kv residue field.

$\mathcal{L}_{ring} = \{0,1,+,\cdot\}$ $\quad \mathcal{L}_{val} = \mathcal{L}_{ring} \cup \{\mathcal{O}\}$

$\mathcal{L}_{oag} = \{0,+,<\}$

Thm (Ax-Kochen/Ershov, '65): Let (K,v) and (L,w) be two henselian fields of residue characteristic 0. Then

$$\underbrace{(K,v) \equiv (L,w)}_{\text{in } \mathcal{L}_{val}} \iff \underbrace{Kv \equiv Lw}_{\text{in } \mathcal{L}_{ring}} \text{ and } \underbrace{vK \equiv wL}_{\text{in } \mathcal{L}_{oag}}$$

and, if $(K,v) \subseteq (L,w)$, we have

$$(K,v) \preccurlyeq (L,w) \iff Kv \preccurlyeq Lw \text{ and } vK \preccurlyeq wL$$

This has been generalized in several ways; e.g. replace "residue characteristic 0" by

- unramified with perfect residue field (AK/E)
- unramified with imperfect residue field (Anscombe-J.)
- alg. maximal Kaplansky (Ziegler/Ershov)

Definition: A valued field (K,v) with $\mathrm{char}(Kv) = p > 0$ is called **tame** if
- (K,v) is algebraically maximal
- vK is p-divisibel
- Kv is perfect.

Thm (F.V. Kuhlmann, '16)

Let (K,v) and (L,w) be two tame henselian fields of residue characteristic $p>0$. Then if $(K,v) \subseteq (L,w)$

$$(K,v) \preccurlyeq (L,w) \iff Kv \preccurlyeq Lw \text{ and } vK \preccurlyeq wL$$

If $\mathrm{char}(K) = \mathrm{char}(L) = p$, we also have

$$(K,v) \equiv (L,w) \iff Kv \equiv Lw \text{ and } vK \equiv wL.$$

Note: If L/K finite, $L \neq K$ such that $Kv = Lw$ and $vL = wL$, then $(K,v) \not\preccurlyeq (L,w)$

$\rightsquigarrow$ "henselian" is necessary, and so is "algebraically maximal"

We understand $\mathbb{Q}_p$, but not $\mathbb{F}_p((t))$ or $\mathbb{F}_p^{alg}((t))$.

TODAY: Generalization to some non-alg. max. henselian fields.
Replace "residue field" by "thickened residue field"

§ Perfectoid fields and their friends

Def: A valued field (K,v) of residue char. $p>0$ is called **perfectoid**, if
(1.) vK is archimedean but not discrete
(2.) (K,v) is complete
(3.) $\mathrm{Frob}: \mathcal{O}/p \to \mathcal{O}/p,\ x \mapsto x^p$ is surjective.

Remark: • If $\mathrm{char}(K) = p > 0$, (3.) just states that K is perfect.
• perfectoid fields do not form an elementary class
• If $v(p) \in vK$ is minimum positive, $\mathcal{O}/p = Kv$.
In a perfectoid field of $\mathrm{char}(K) = 0$, we have $(p) \subseteq \mathfrak{m}_v$, but $\mathfrak{m}_v \subsetneq (p)$
$\Rightarrow$ $\mathcal{O}/p$ is not an integral domain

Examples:
perfectoid $\begin{cases} \bullet\ (\widehat{\mathbb{Q}_p(1/p^\infty)}, v_p),\ (\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}, v_p) \\ \bullet\ (\widehat{\mathbb{F}_p((t))^{perf}}, v_t),\ (\widehat{\mathbb{F}_p^{alg}((t))^{perf}}, v_t) \end{cases}$

not perfectoid • $(\mathbb{Q}_p, v_p)$, $(\mathbb{F}_p((t)), v_t)$

Theorem (J.-Kartas, version 1)
Let $(K,v) \subseteq (L,w)$ be henselian fields of res. char $p>0$ with
(1.) value group archimedean and non-discrete
(2.) $\mathrm{Frob}: \mathcal{O}/p \to \mathcal{O}/p$ is surjective

Then

$$(K,v) \preccurlyeq (L,w) \iff \mathcal{O}_v/t \preccurlyeq \mathcal{O}_w/t$$

for some (equivalently: any) t with $0 < v(t) \leq v(p)$

Proof idea:

- mixed char $(0,p)$: $\qquad (t = p)$

  wlog (K,v) $\aleph_1$-sat, (L,w) saturated

$$K \xrightarrow[\text{divisible}]{(0,0)} Kv_0 \xrightarrow[\substack{rk-1\\ \mathbb{R}}]{(0,p)} Kv_p \xrightarrow{(p,p)} Kv$$

$$Kv_p = \mathrm{Frac}\left((\mathcal{O}_v/p)_{red}\right)$$

tame with divisible value group

$$\mathrm{Frac}\left((\mathcal{O}_v/p)_{red}\right) \preccurlyeq \mathrm{Frac}\left((\mathcal{O}_w/p)_{red}\right) = Lw_p$$

$$L \xrightarrow[\text{divisible}]{(0,0)} Lw_0 \xrightarrow[\substack{rk-1\\ \mathbb{R}}]{(0,p)} Lw_p \xrightarrow{(p,p)} Lw$$

Kuhlmann $\Rightarrow$ $\quad (K, v_p) \preccurlyeq (L, w_p)$

Kv_p, Lw_p stably embedded (by tameness)

$\Rightarrow (K,v) \preccurlyeq (L,w)$

In the general case: Need to show that

is still tame! $\boxtimes$

Open problems: • $(\mathbb{F}_p(t)^{hens}, v_t) \preccurlyeq (\mathbb{F}_p((t)), v_t)$?

• $(\mathbb{F}_p^{alg}(t)^{hens}, v_t) \preccurlyeq (\mathbb{F}_p^{alg}((t)), v_t)$?

Corollary: • $(\mathbb{F}_p(t)^{hens,perf}, v_t) \preccurlyeq (\mathbb{F}_p((t))^{perf}, v_t)$

• $(\mathbb{F}_p^{alg}(t)^{hens,perf}, v_t) \preccurlyeq (\mathbb{F}_p^{alg}((t))^{perf}, v_t)$

Why perfectoid fields?

Tilting construction (Scholze, based on Fontaine):

(K,v) perfectoid, $\operatorname{char}(K,Kv) = (0,p)$

$$\mathcal{O}^\flat = \varprojlim \left(\dots \xrightarrow{Frob} \mathcal{O}/p \xrightarrow{Frob} \mathcal{O}/p \right), \quad K^\flat = \operatorname{Frac}(\mathcal{O}^\flat)$$

$\Rightarrow (K^\flat, v^\flat)$ is called the tilt of (K,v), and there is $t \in \mathfrak{m}_{v^\flat}$ with

$$\mathcal{O}/p \cong \mathcal{O}^\flat/t$$

$\rightsquigarrow Kv \cong K^\flat v^\flat$ and $vK \cong v^\flat K^\flat$.

Quintessence: "K and $K^\flat$ are similar."

$\rightsquigarrow$ tilting allows to transfer arithmetic properties between K and $K^\flat$ and vice versa

Note: $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}^\flat = \mathbb{F}_p((t))^{perf} = \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}^\flat$

Corollary: (K,v), (L,w) perfectoid. Then

$$(K,v) \preccurlyeq (L,w) \iff (K^\flat, v^\flat) \preccurlyeq (L^\flat, w^\flat)$$

and

$$(K,v) \equiv (L,w) \Rightarrow (K^\flat, v^\flat) \equiv (L^\flat, w^\flat)$$

§ Leaving rank-1 behind: deeply ramified fields

Def: A valued field (K,v) of residue characteristic $p>0$ is called **deeply ramified** if

- $\mathrm{Frob}: \mathcal{O}_{\hat{v}}/p \to \mathcal{O}_{\hat{v}}/p$ is surjective and
- Δ_2/Δ_1 is non-discrete, for any convex subgroups $\Delta_1 \lneq \Delta_2 \leq vK$

Remark:
- If $\mathrm{char}(K) = p > 0$, then (K,v) is deeply ramified iff $(\hat{K}, \hat{v})$ is perfect.
- If $\mathrm{char}(K) = 0$, then
$$\mathcal{O}_{\hat{v}}/p = \mathcal{O}_v/p$$
- any perfectoid field is deeply ramified.

Theorem (J.- Kartas, version 2)
Let $(K,v) \subseteq (L,w)$ be perfect henselian deeply ramified fields of res. char $p>0$. Suppose there is $\varpi \in \mathfrak{m}_v$ s.th. $\mathcal{O}_v[1/\varpi]$ and $\mathcal{O}_w[1/\varpi]$ are defectless. Then:

$$(K,v) \preccurlyeq (L,w) \iff \mathcal{O}_v/\varpi \preccurlyeq \mathcal{O}_w/\varpi \text{ and } (vK, v\varpi) \preccurlyeq (wK, w\varpi)$$

Note: • The class of perfect henselian deeply ramified fields of residue characteristic $p>0$ with a distinguished element ϖ s.th. $\mathcal{O}_v[1/\varpi]$ is defectless is an elementary class.
$\rightsquigarrow$ suffices to check $\mathcal{O}_v[1/\varpi]$ is defectless in some model

• If vK is regular, the value group condition can be omitted.
$\rightsquigarrow$ version 2 generalizes version 1

• We only need Δ_2/Δ_1 to be non-discrete to make sure that (K,v) is not finitely ramified.

Open problem: $(\mathbb{F}_p((t)), v_t) \preccurlyeq (\mathbb{F}_p\underline{((t))((s^{\mathbb{Q}}))}, v_t)$?

Corollary: $(\mathbb{F}_p((t))^{perf}, v_t) \preccurlyeq (\underbrace{\mathbb{F}_p((t))^{perf}((s^{\mathbb{Q}}))}_{v_t}, v_t)$