Periods, Power Series, and Integrated Algebraic Numbers

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1. Motivation

Periods play a fundamental role in transcendental number theory and arithmetic geometry.
For example they arise in pairing singular homology and algebraic de Rham cohomology: $(\sigma,\omega)\mapsto\int_\sigma\omega$.

Kontsevich and Zagier give the following definition in *Mathematics unlimited – 2001 and beyond, 771-808, Springer, Berlin, 2001*:

> *A period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.*

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In the following we consider periods that are real. These are precisely given by $\int_{\mathbb{R}^n} f(x)\,dx$ where $f:\mathbb{R}^n \to \mathbb{R}$ is semialgebraic over $\mathbb{Q}$ and absolutely integrable.

Examples for periods:

$$\pi = \mathrm{vol}\big(\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1\}\big),$$

$$\log(2) = \int_1^2 \frac{dx}{x},$$

$$\zeta(3) = \sum_{n=1}^{\infty}\frac{1}{n^3} = \int_{0<x<y<z<1}\frac{dx\,dy\,dz}{(1-x)yz}.$$

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- Describe periods in a more formalized way.
- Define new number systems in an analytic-geometric way capturing the important mathematical constants.

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Starting point:

In the following way one get periods: Take a real power series which is algebraic over the rational polynomial ring, take its formal antiderivative and evaluate this power series at a rational point contained in its domain of convergence.

Examples:

(1) Let

$$L(X)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}X^n$$

be the **logarithmic series**. We have

$$\log\left(\frac{1}{2}\right)=\int_1^{1/2}\frac{dt}{t}=L\left(\frac{1}{2}\right).$$

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$$A(X) = \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n n!)^2} \frac{X^{2n+1}}{2n+1}$$

be the **arcsine series**. We have

$$\frac{\pi}{6} = \arcsin\left(\frac{1}{2}\right) = \int_0^{1/2} \frac{dt}{\sqrt{1-t^2}} = A\left(\frac{1}{2}\right).$$

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2. Integrated Algebraic Series and Integrated Algebraic Numbers

We follow this approach.

We start with the power series which are algebraic over the rational polynomial ring. We call them simply **algebraic power series**.

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We consider the smallest class of rings of power series which contain the algebraic series and are closed under taking

(a) **Reciprocals**,

(b) **Rational Polynomial Substitutions**,

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(b) **Rational Polynomial Substitution:** We replace the coordinates by rational polynomials vanishing at the origin. For example $G(X_1+X_2)=\sum_{m,n=0}^{\infty}\binom{m+n}{m}X_1^mX_2^n$.

(c) **Rational Translation:** Given a convergent power series and a point in its domain of convergence with rational components we develop the series at this point. For example $G(X+1/2)=\sum_{n=0}^{\infty}(\sum_{k=n}^{\infty}\binom{k}{n}(1/2)^{k-n})X^n$.

(d) **Formal Antiderivative:** Given a power series we take the formal antiderivative with respect to an arbitrary variable. For example the formal antiderivative of $G(X)$ is given by $\sum_{n=0}^{\infty}X^{n+1}/(n+1)$.

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(d) **Formal Antiderivative:** Given a power series we take the formal antiderivative with respect to an arbitrary variable. For example the formal antiderivative of $G(X)$ is given by $\sum_{n=0}^{\infty} X^{n+1}/(n+1)$.

We call the series obtained in this way the **integrated algebraic power series**.

We obtain the following results:

Theorem A

The coefficients of the integrated algebraic power series form a countable real closed field.

We call this the **field of integrated algebraic numbers**.

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The periods are contained in the established number system:

Theorem B

A period is an integrated algebraic number.

So:

- Everything is encoded in power series.
- The analysis is reduced to evaluation of power series at rational points.

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3. Theorem A – Power Series Systems

Theorem:
The integrated algebraic power series form a **convergent Analysis system**.

We explain these terms.

System (of convergent power series):
Collection $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}_0}$ such that for each $n \in \mathbb{N}_0$:

- $\mathcal{S}_n$ is a subring of the ring of convergent power series system $\mathbb{R}\{X\} = \mathbb{R}\{X_1, \ldots, X_n\}$,
- closed under permutations of variables,
- $\Bbbk := \mathcal{S}_0$ is a subfield of $\mathbb{R}$ such that $\Bbbk[X] \subset \mathcal{S}_n \subset \Bbbk\{X\}$.

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Convergent Weierstraß System

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Convergent System
with Weierstraß division and preparation

Denef & $\Downarrow$ Lipshitz

Substitution, Inverse Series, Implicit Series

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New feature: Systems of convergent power series on subfields of the reals!

Examples:

- System of algebraic power series is a convergent Weierstraß system with coefficient field $\mathbb{A}$ of real algebraic numbers.
- System of integrated algebraic power series is a convergent Analysis system with coefficient field $\mathbb{IA}$ of integrated algebraic numbers.

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4. Theorem B – O-Minimality

Definition

A **restricted integrated algebraic function** is of the form

$$f(x) = \begin{cases} p(x) & \text{if} \quad x \in [-1,1]^n_{\mathbb{IA}}, \\ 0 & \phantom{\text{if}} \quad x \notin [-1,1]^n_{\mathbb{IA}}, \end{cases}$$

where $p(X)$ is an integrated algebraic power series converging on a neighbourhood of $[-1,1]^n_{\mathbb{IA}}$.

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Let $\mathcal{R}_{\mathbb{IA}}$ be the structure generated over the field $\mathbb{IA}$ by the restricted integrated algebraic functions.

We consider the following languages:

- $\mathcal{L}$: language of ordered rings
- $\mathcal{L}_{\mathbb{IA}}$: extension of $\mathcal{L}$ by symbols for every restricted integrated algebraic function
- $\mathcal{L}_{\mathbb{IA}}^{-1}$: extension of $\mathcal{L}_{\mathbb{IA}}$ by a symbol for the reciprocal $x \mapsto 1/x$
- $\mathcal{L}_{\mathbb{IA}}^{\mathbb{Q}}$: extension of $\mathcal{L}_{\mathbb{IA}}$ by symbols for every power function $x \mapsto x^q$ with rational q

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Lion-Rolin preparation:

$$f(x,y) = \underbrace{a(x)}_{\text{def.}} \cdot |y - \underbrace{\Theta(x)}_{\text{def.}}|^{p_0} \cdot U\Big(\big(\underbrace{b_k(x)}_{\text{defn.}}|y-\Theta(x)|^{q_k}\big)_{k=1}^{l}\Big)$$

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An **exponential integrated algebraic number** is the integral of a function definable in $\mathcal{R}_{\mathbb{IA}}(\exp)$.

Theorem:

The exponential integrated algebraic numbers are a countable ring containing the exponential periods and the Euler constant γ.

Exponential period: given by $\int \exp(f(x))g(x)\,dx$ where f and g are semialgebraic over $\mathbb{Q}$ with values in $\mathbb{C}$ and f has bounded imaginary part.

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Structures and Integration:

Structure	Universe	Integration	Number System
Semialg. over $\mathbb{Q}$ = alg. series	real algebraic numbers	$\longrightarrow$	periods
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6. Overviews:

Structures and Integration:

Structure	Universe	Integration	Number System
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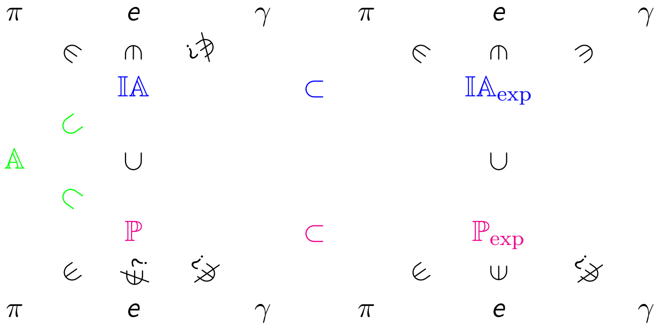
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