On large externally definable sets and NIP

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Joint work with Martin Bays, Omer Ben-Neria and Pierre Simon

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VC-classes and NIP

Definition

Let X be a set and $\mathcal{F} \subseteq \mathcal{P}(X)$. We say that $A \subseteq X$ is *shattered* by $\mathcal{F}$ if for every $S \subseteq A$ there is $F \in \mathcal{F}$ such that $F \cap A = S$. A family $\mathcal{F}$ is said to be a *VC-class* on X if there is some $n < \omega$ such that no subset of X of size n is shattered by $\mathcal{F}$. In this case the *VC-dimension* of $\mathcal{F}$ is the smallest integer n such that no subset of X of size $n + 1$ is shattered by $\mathcal{F}$.
Let T be a theory. A formula $\varphi(x, y)$ has the *independence property* or *IP* if the set-system $\{\varphi(M, b) \mid b \in M\}$ is not a VC-class for any (some) $M \models T$. The negation is *NIP*: a formula is NIP if this class is a VC-class (for any M).
T (or any $M \models T$) is *NIP* if every formula is NIP.

A question about cofinal subsets of $\mathbb{R}$

Question

Is there a cofinal family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{R})$ of finite subsets such that $\mathcal{F}$ is a VC-class?*

*Cofinal = *every finite set is contained in a set in $\mathcal{F}$.*

Note that there is a tension between two things: being a VC-class removes sets from $\mathcal{F}$ while being cofinal adds sets to $\mathcal{F}$.

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Note that there is a tension between two things: being a VC-class removes sets from $\mathcal{F}$ while being cofinal adds sets to $\mathcal{F}$.

If $\mathcal{F}$ is stable, then no such cofinal family exist, even for subsets of $\mathbb{N}$: Inductively choose $a_i \in \mathbb{N}, F_j \in \mathcal{F}$ such that $a_i \in F_j$ iff $i \leq j$. In stage j, choose $a_j \notin \bigcup \{F_j \mid j < i\}$ and F_j containing $\{a_i \mid i \leq j\}$.

Motivation

A set $X \subseteq M$ is *definable* if there is some formula $\psi(x)$ over M such that $X = \psi(M)$.

A set $X \subseteq M$ is *externally definable* if there is some elementary extension $N \succ M$ and some formula $\psi(x)$ over N such that $X = \psi(M)$.

Fact

T is stable iff every externally definable set over any model is definable.

Examples

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A cut C in a linear order is externally definable.

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Fact (Shelah)

Suppose that M *is a structure and* M^{Sh} *is an expansion given by adding predicates for all externally definable subsets in any number of variables. If* $\mathrm{Th}(M)$ *is NIP, then* $\mathrm{Th}\left(\mathrm{M}^{Sh}\right)$ *has quantifier elimination and is NIP.*

A question of Chernikov and Simon

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Due to the example of the random graph we cannot hope to get much information in case of IP.
In fact the situation is worse: for any cardinal κ there is a random graph N of size κ and an externally definable subset $X \subseteq N$ with no infinite definable subset.
Indeed, let N be the Skolem hull of an indiscernible sequence $I = \langle a_i \mid i < \kappa \rangle$. $\{a_i \mid i < \kappa \text{ even}\}$ is externally definable but every definable subset is finite.

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Example

Consider $M = (\mathbb{N} + \mathbb{Z}, <)$ whose theory is NIP. Then $\mathbb{N}$ is an externally definable subset with no infinite definable subset.

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Suppose that T is NIP. Is there a cardinal κ such that if $\mathcal{X}$ is externally definable of size $\geq \kappa$ then $\mathcal{X}$ contains an infinite definable set?

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Fact (Chernikov-Simon 2013)

Yes! One can take $\kappa = \beth_\omega$.

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Question (Chernikov-Simon 2013)

Can we choose κ to be $\aleph_1$?

Honest definitions

Definition

Suppose that $\varphi(x, y)$ is a formula, $N \succ M$ and $c \in N$. Say that a formula $\psi(x, z)$ (over $\emptyset$) is an *honest definition* of $\mathrm{tp}_{\varphi^{opp}}(c/M)$ if for every finite $A_0 \subseteq M$ there is some $b \in M^z$ such that

$$\varphi(A_0, c) \subseteq \psi(M, b) \subseteq \varphi(M, c).$$

Fact (Chernikov-Simon for NIP theories, Bays-K-Simon for NIP formulas)

If $\varphi(x, y)$ is NIP then there is a formula $\psi(x, z)$ that serves as an honest definition for any φ^{opp}-type (over any M).

Honest definitions

Suppose that M is NIP. Let $c \in N \succ M$ and let $X = \varphi(M, c)$ be externally definable.
Let $\psi(x, z)$ be an honest definition of $\varphi(M, c)$.
Then for every finite $X_0 \subseteq X$, there is some $b \in M^z$ such that

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$$X_0 \subseteq \psi(M, b) \subseteq X.$$

If we show that one of those $\psi(M, b)$'s is infinite, we found an infinite definable subset. So suppose none of them is infinite.
Let $\mathcal{F} = \{\psi(M, b) \mid b \in M^z, \psi(M, b) \subseteq X\}$.
We get that $\mathcal{F}$ is a *cofinal* family of finite subsets of X.
$\mathcal{F}$ is a VC-class since M is NIP.

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Chernikov-Simon's proof of the $\beth_\omega$ bound was through:

Fact (Chernikov-Simon, 2013)

There is no NIP cofinal family of finite subsets of $\beth_\omega$.
(The proof uses alternation rank and $\beth_\omega$ was used for the Erdös-Rado coloring theorem.)

Better bounds

Theorem (Bays, Ben-Neria, K., Simon)

Suppose that $\mathcal{F}$ is a cofinal family of finite subsets of $\aleph_\omega$. Then $\mathcal{F}$ has IP: it is not a VC-class.
More precisely, if $\mathcal{F}$ is a cofinal family of finite subsets of $\aleph_n$ then $\mathcal{F}$ has VC-dimension $> n$.

Corollary

Suppose that M is NIP. If X is an externally definable set of size $\geq \aleph_\omega$ then X contains a definable subset.

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Going back to the $\aleph_1$-question, we get:

Question

Suppose that $\mathcal{F}$ is a cofinal family of finite subsets of $\aleph_1$. Does it follow that $\mathcal{F}$ has IP?

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Theorem (Bays, Ben-Neria, K., Simon)

The answer is NO: there is a cofinal family $\mathcal{F}$ of finite subsets of $\aleph_1$ which is NIP (in fact of VC-dimension 2).

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The inductive condition on $<^\alpha$ is that for every finite set $A \subseteq \alpha$ there is some $A \subseteq B \subseteq \alpha$ such that B is closed under $\vdash$: if $\gamma, \beta \in B$ and $\gamma, \delta < \beta$ and $\delta <^\alpha \gamma$ then $\delta \in B$.
Let $\mathcal{F}$ be the set of finite subsets of ω_1 which are closed under $\vdash$.

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Let $\mathcal{F}$ be the set of finite subsets of ω_1 which are closed under $\vdash$.
Then $\mathcal{F}$ is NIP: for every $\alpha_0, \alpha_1, \alpha_2$, there is some permutation σ of $\{0, 1, 2\}$ such that $\alpha_{\sigma(0)}, \alpha_{\sigma(1)} \vdash \alpha_{\sigma(2)}$. This means that there can be no C $\in \mathcal{F}$ containing $\alpha_{\sigma(0)}, \alpha_{\sigma(1)}$ but not $\alpha_{\sigma(2)}$.

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Question

Is there a cofinal family of finite subset of $\aleph_2$ of VC-dimension 3?

A surprising undecidable statement

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Corollary

The following statement is independent of ZFC: there is a cofinal family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{R})$ of finite subsets such that $\mathcal{F}$ is a VC-class.

Proof.

By Gödel, ZFC is consistent with CH: $\aleph_1 = 2^{\aleph_0}$, so that it is consistent that there is such a family.
By Cohen, ZFC is consistent with $2^{\aleph_0} > \aleph_\omega$, implying that such a family does not exist. $\square$

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Theorem (Bays, Ben-Neria, K., Simon)

Let $\mathcal{X}$ be an uncountable set. If $\mathcal{F}$ is a cofinal family of finite subsets of $\mathcal{X}$, then the two-sorted structure $(\mathcal{X}, \mathcal{F}; \in)$ has IP.

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Theorem (Bays, Ben-Neria, K., Simon)

Let X be an uncountable set. If $\mathcal{F}$ is a cofinal family of finite subsets of X, then the two-sorted structure $(X, \mathcal{F}; \in)$ has IP.

Recall the setting:
M is NIP. $c \in N \succ M$ and $X = \varphi(M, c)$ is externally definable.
Let $\psi(x, z)$ be an honest definition of $\varphi(M, c)$.
Then for every finite $X_0 \subseteq X$, there is some $b \in M^z$ such that

$$X_0 \subseteq \psi(M, b) \subseteq X.$$

Let $\mathcal{F} = \{\psi(M, b) \mid b \in M^z, \psi(M, b) \subseteq X\}$.

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Let $\mathcal{F} = \{\psi(M, b) \mid b \in M^z, \psi(M, b) \subseteq X\}$.
Then $(X, \mathcal{F}; \in)$ is interpretable in M^{Sh},

Corollary

If M is NIP, every externally definable set of size $\geq \aleph_1$ contains an infinite definable subset.

General κ

In fact we get more:

Theorem

Let κ be any cardinal and let X have size $\geq \kappa^+$. If $\mathcal{F}$ is a family of subsets of X such that every finite subset of X is contained in a set from $\mathcal{F}$ (we call such families ω-cofinal) and each set in $\mathcal{F}$ has size $< \kappa$, then the two-sorted structure $(X, \mathcal{F}; \in)$ has IP.

Corollary

If M is NIP, every externally definable set of size $\geq \kappa^+$ contains a definable subset of size $\geq \kappa$.

A lemma

Lemma

Let κ be any infinite cardinal. Assume that:

1. $|X| \geq \kappa^+$.
2. $R \subseteq X^n$ *and* $1 \leq n$.
3. *For every* $a_1, \ldots, a_{n-1} \in X$, $|\{a_0 \in X \mid R(a_0, a_1, \ldots, a_{n-1})\}| < \kappa$.
4. *For every set* $A \subseteq X$ *of size* $|A| = n$, *for some* $a \in A$ *and some tuple* $\bar{a} \in (A \setminus a)^{n-1}$, $R(a, \bar{a})$ *holds.*

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4. *For every set* $A \subseteq \mathcal{X}$ *of size* $|A| = n$, *for some* $a \in A$ *and some tuple* $\bar{a} \in (A \setminus a)^{n-1}$, $R(a, \bar{a})$ *holds.*

Then, there is some partition of $\{1, \ldots, n-1\}$ *into nonempty disjoint sets* u, v *such that letting* $x := \langle x_i \mid i \in u \cup \{0\} \rangle$ *and* $y := \langle x_i \mid i \in v \rangle$, *the partitioned formula* $\phi(x, y) := R(x_0, x_1, \ldots, x_{n-1})$ *has IP.*

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Example

Choose, for each ordinal $\alpha < \omega_1$, an ω-order $<^\alpha$ on α. Let $R(\alpha, \beta, \gamma)$ hold iff $\alpha, \beta < \gamma$ and $\alpha <^\gamma \beta$.

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From the proof of the lemma, we get that $R(x, y; z)$ has IP.

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Question

Does $R(x; y, z)$ have IP? Note that letting

$$\mathcal{F} = \{\{\alpha \mid R(\alpha, \beta, \gamma)\} \mid \beta, \gamma \in \omega_1\},$$

$\mathcal{F}$ is a cofinal family of finite subsets of ω_1.

Idea of the proof, using the lemma

Suppose that $|X| \geq \kappa^+$ and that $\mathcal{F}$ is a cofinal family of subsets of X, each of size $< \kappa$. Suppose that $\mathrm{vc}(\mathcal{F}) = n$. For any $0 \leq k \leq n$ and any $m \leq k$, let $R_{m,k}(x_0, \ldots, x_k)$ be the relation defined by:

$$[\exists t \in \mathcal{F} \bigwedge_{1 \leq i \leq k} (x_i \in t)^{(i \leq m)}] \wedge [\forall t \in \mathcal{F} ((\bigwedge_{1 \leq i \leq k} (x_i \in t)^{(i \leq m)}) \rightarrow x_0 \in t)].$$

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Let $R(x_0, x_1, \ldots, x_n) = \bigvee_{m \leq k \leq n} R_{m,k}(x_0, \ldots, x_k)$. Then R satisfies the conditions of the lemma on X.

Some open questions

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Suppose that M *is a structure and* $X = \phi(M, c)$ *is externally definable of size* $\geq \aleph_1$. *Suppose that* ϕ *is NIP. Does it follow that* X *contains an infinite definable subset?*

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More generally,
we can define $\mathrm{ext}(T, \phi, \kappa)$ as the minimal λ (if exists) such that whenever $M \models T$ and $\mathcal{X} \subseteq M^k$ is externally definable by $\phi(x, c)$, then $\mathcal{X}$ contains a definable subset of size $\geq \kappa$. If T is NIP then $\mathrm{ext}(T, \phi, \kappa) \leq \kappa^+$. If the honest definition of ϕ is NIP and $\kappa = \aleph_\alpha$, $\mathrm{ext}(T, \phi, \kappa) \leq \aleph_{\alpha+\omega}$. If we assume only that ϕ is NIP, it is not even clear that $\mathrm{ext}(T, \phi, \aleph_0)$ exists.

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What is $\mathrm{ext}(T, \phi, \kappa)$ *when* ϕ *is NIP?*

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Question

Does $\operatorname{ext}(T, \aleph_0) = \infty$ *hold whenever* T *is IP?*

Thank you!