Primitive pseudofinite permutation groups

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Context

Joint work with Martin Liebeck and Katrin Tent, part recent and in progress, part from
M.W. Liebeck, H.D. Macpherson, K. Tent, ‘Primitive permutation groups of bounded orbital diameter’, Proc. London Math Soc. (3) 100 (2010), 216–248.

Work with finite/pseudofinite permutation group (G,X), G faithful and transitive on X.

View as 2-sorted structure with definable action $G \times X \to X$ (or as pair (G, G_x)) where G_x is the stabiliser of some $x \in X$.
Definition. (G,X) is **primitive** if there is no proper non-trivial G-invariant equivalence relation on X.
Note: (G,X) is primitive iff G_x is a maximal subgroup of G.

Our goal: Understand primitive pseudofinite permutation groups (G,X).

Remarks.

1. Standard in finite permutation group theory to reduce to questions about primitive groups – e.g. imprimitive groups embed into wreath products.

2. ‘Primitive’ for permutation groups analogous to ‘simple’ for abstract groups. Analogues between our work on primitive pseudofinite permutation groups and John Wilson’s 1995 classification of simple pseudofinite groups.

3. One could consider ‘definably primitive’ permutation groups (no **definable** proper non-trivial G-invariant equivalence relation). Assuming infinite point stabiliser, definably primitive implies primitive for finite Morley rank (M, Pillay 1995) or in supersimple finite rank (Elwes, Ryten 2008).

4. Ultraproducts $\Pi_{n,k\to\mathcal{U}}(S_n, k\text{-subsets of } [n])$ definably primitive but not primitive pseudofinite permutation groups (with infinite point stabiliser).

5. Extensive work on primitive groups in finite Morley rank (M-Pillay 1995, Borovik-Cherlin 2008,...).

Definition. Orbital Ω of (G, X) – a G-orbit on $X^{[2]}$ (unordered 2-subsets of X). **Orbital graph** – vertex set X, edge set Ω.

Note: If Ω is an orbital of (G, X) then $G \leq \mathrm{Aut}(X, \Omega)$.

Fact: (D.G. Higman) If (G, X) is transitive then it is primitive if and only if all orbital graphs are connected.

Γ the random graph. Easy way to see $G = \mathrm{Aut}(\Gamma)$ is primitive: by homogeneity there are two orbital graphs, namely Γ and its complement, and both are connected by Alice Restaurant axioms.

Note: Orbital graphs are uniformly definable in (G, X) (cf. conjugacy classes in groups); connectedness not first-order.

Towards primitive pseudofinite permutation groups:
(a) Consider family of finite primitive permutation groups (G, X) with fixed bound on $|X^2/G|$. Well-understood setting, not total classification.

Here, bounded number of orbital graphs, each or bounded diameter, so connectedness first-order expressible, ultraproducts are primitive pseudofinite.

Special case with fixed bound on $|X^4/G|$ classified in Kantor-Liebeck-M 1989, towards (ω-categorical) smoothly approximable structures (Cherlin-Hrushovski 2003).

Bound on $|X^2/G|$ (but not $(|X^4/G|)$ picks up examples like

$$(\mathrm{PGL}_2(q), \mathrm{PG}_1(q)$$

$$(\mathrm{AGL}_1(q), \mathbb{F}_q)$$

– these are not ω-categorical in limit as $q \to \infty$.

(b) Family $\mathcal{F}_d$ of finite primitive permutation groups such that all orbital graphs have diameter at most d.

Essentially classified in [LMT 2010]. This yields
(i) classification of primitive pseudofinite permutation groups of bounded orbital diameter,
(ii) classification of ω-saturated primitive pseudofinite permutation groups.

Model-theoretically, the groups can be wild (e.g. theory of finite symmetric groups S_n is undecidable). In the pseudofinite case, the groups are very close to automorphism groups of NSOP_1-structures.
Conjecture: For (G, X) primitive pseudofinite of bounded orbital diameter, $N_{\mathrm{Sym}(X)}(G)$ is dense in the automorphism group of an NSOP_1-structure.

Care with subgroups of $\mathrm{P\Gamma L}_2(q)$ acting on $\mathrm{PG}_1(q)$ – subgroups of $\mathrm{Aut}(\mathbb{F}_q)$ encode subfields of $\mathbb{F}_q$, and the theory of pairs of finite fields is wild.

Methods for classification for bounded orbital diameter (case (b) above).

O'Nan-Scott Theorem + Aschbacher reduction + facts about maximal subgroups of finite simple groups + bound

Lemma: Let $(G, X) \in \mathcal{F}_d$, $x \in X$, $H = G_x$, Γ an orbit of H on X other than $\{x\}$. Then

(i) $1 + 2|\Gamma| + \ldots + (2|\Gamma|)^d \geq n$

(ii) $|H| \geq \Gamma$, so $|H| \geq n^{\frac{1}{d+1}}$ (for n large enough).

Proof of (i). Let Ω be the orbital including pairs $\{x, y\}$ where $y \in \Gamma$. The orbital graph (X, Ω) has degree $|\Gamma|$ or $2|\Gamma|$ (it is $2|\Gamma|$ if (x, y) and (y, x) are in different orbits for $\{x, y\} \in \Omega$).

The number of vertices at distance at most i from x is at most $1 + 2|\Gamma| + \ldots + (2|\Gamma|)^i$.

O'Nan-Scott Theorem

(applied in [LMT2010] in finite primitive permutation groups (G,X)).
Let $\mathrm{Soc}(G)$ be the direct product of the minimal normal subgroups of G. Then (by primitivity) $\mathrm{Soc}(G)$ is a direct product of isomorphic simple groups, and one of the following holds (over-simplified).

(1) $\mathrm{Soc}(G)$ non-abelian.
(a) $\mathrm{Soc}(G)$ simple non-abelian, that is, there is finite non-abelian simple group S with $S \leq G \leq \mathrm{Aut}(S)$.
(b) (Diagonal case) $\mathrm{Soc}(G) = S^k$, X identified with the right cosets of a diagonal subgroup of S^k isomorphic to the non-abelian simple group S.
(c) (Wreath product in product action) Suppose (H,Y) is primitive of type (1a), and $K \leq S_m$ is transitive. Then $G = H\mathrm{wr}K = H^m \rtimes K$ acts primitively on $X = Y^m$.
(Note: some small variants omitted.)

(2) S elementary abelian, so of form $C_p^n = V(n,p)$, acting regularly on X so identified with X. Here $G = V \rtimes H$ where $H = G_0 \leq \mathrm{GL}_n(p)$ is irreducible.

Description of bounded orbital diameter cases in O'Nan-Scott settings.

(1a) (non-abelian simple $S = \mathrm{Soc}(G)$). If Lie rank of S is unbounded, then have action on k-spaces for bounded k (includes case $S = \mathrm{Alt}_n$, when we get action on k-sets).

If Lie rank of S is bounded, then maximal subgroups of S are uniformly definable in S – except 'subfield subgroups' like $\mathrm{PSL}_d(q) < \mathrm{PSL}_d(q^r)$ for r prime. (Heavy use of description of maximal subgroups due to Liebeck and Seitz, via work on maximal subgroups of simple algebraic groups.) Limit theory is supersimple finite rank so definably primitive implies primitive. Get bounded orbital diameter, except in the subfield subgroup case (r unbounded).

(1b) (Diagonal case) Some cases of bounded orbital diameter (for k bounded).

(1c) Assuming (H, Y) has bounded orbital diameter, $H \mathrm{wr} K$ on Y^m has bounded orbital diameter iff m is bounded.

(2) (S elementary abelian, $G = V(n,p) \rtimes H$). If n is bounded, get bounded orbital diameter provided H contains scalars. If n unbounded, reduce to case when H is a classical group and $V(n,p)$ is its natural module.

Question. Does every primitive pseudofinite permutation group have bounded orbital diameter?

Note: Conceivably a counterexample (G, X) might arise from a sequence (G_n, X_n) of finite **imprimitive** permutation groups.

Still (without assuming primitivity of the (G_n, X_n)) get partial O'Nan-Scott description. After thinning out, one of the following holds:
(1) the G_n have elementary abelian socle, and G has elementary abelian or divisible torsion-free abelian socle V, with $G = V \rtimes H$ where H acts irreducibly on V, or

(2) $\mathrm{Soc}(G_n)$ is a direct product of isomorphic non-abelian simple groups.

Proof. Use Wilson's uniform definability of soluble radical in finite groups, + double centraliser argument.

Approach towards showing that **any** primitive pseudofinite (G, X) has bounded orbital diameter, so is captured by the [LMT2010] paper.

Suppose (G, X) is approximated by finite primitive (G_n, X_n). Suppose that in (G_n, X_n) there are uniformly 0-definable orbitals Γ_n and Δ_n (or sets of orbitals) and a function $f : \mathbb{N} \to \mathbb{N}$ such that if $k > f(n)$ then any pair $\{x, y\} \in \Delta_k$ lie at distance at least n in the Γ_k-orbital graph.

Then the orbital graph corresponding to Γ in (G, X) is disconnected, contradicting primitivity of (G, X).

(Compare argument to show there is no simple (as a group) infinite model of the theory of finite alternating groups.)

Proposition. Suppose the (G_n, X_n) appproximating (G, X) are primitive, and $\mathrm{Soc}(G_n)$ is alternating for each n. Then (G, X) has bounded orbital diameter (and is natural action on k-sets).

Sketch proof. We suppose G_n is symmetric, and write as Sym_n (fudging indices). Let H_n be a stabiliser $(G_n)_x$, for $x \in X_n$. So H_n is maximal in G_n.

Fact. Inside Sym_n can uniformly interpret $[n] = \{0, \ldots, n-1\}$, action of Sym_n on $[n]$, and $\mathcal{P}([n])$.

Case 1. H_n intransitive on $[n]$. Then by maximality of H_n in $G_n = \mathrm{Sym}_n$, H_n is the stabiliser of a subset of $[n]$ of size $k = k_n$. So (G_n, X_n) is action of Sym_n on k-subsets of $[n]$. If k is constant, then the (G_n, X_n) have orbital diameter k, as does (G, X), so suppose k is unbounded.

Let Γ_n be the orbital of pairs of k_n-sets of symmetric difference 2, and Δ_n the orbital of pairs of disjoint k_n-sets. These are uniformly 0-definable, and a pair $\{x, y\}$ in Δ_n are at distance k_n in the Γ_n-orbital graph, so (G, X) is imprimitive.

Case 2. H_n is transitive imprimitive on $[n]$. Then as H_n is maximal in Sym_n it is the full stabiliser of a partition of $[n]$ with all parts the same size, that is, $H_n = \mathrm{Sym}_r \,\mathrm{wr}\, \mathrm{Sym}_t$ for some $r, t > 1$ with $rt = n$. Similar argument to Case 1 contradicts primitivity of (G, X).

Case 3. H_n is primitive on $[n]$.

Key Fact: (Babai 1981) If H acts primitively on $[n]$ then either $H \geq \mathrm{Alt}_n$ or

$$\mu(H) \geq \frac{1}{2}(\sqrt{n} - 1),$$

where $\mu(H)$ is the smallest size of the support of a non-identity element of H.

Have a formula $\phi(g)$ expressing that for some r, g acts on $[n]$ as a product of r disjoint r-cycles (and fixed points).

Have formula $\psi(g,h)$ expressing $\phi(g)\wedge$ 'h is a cycle of g'.

$|\operatorname{supp}(g)|^2 \leq n$: $\quad \exists h,k(\operatorname{supp}(g) = \operatorname{supp}(h) \wedge \psi(k,h))$.

$|\operatorname{supp}(g)|^4 \leq n$: $\quad \exists h,k(\operatorname{supp}(g) = \operatorname{supp}(h) \wedge \psi(k,h) \wedge |\operatorname{supp}(k)|^2 \leq n)$.

$|\operatorname{supp}(g)|^8 \leq n$. $\quad$ Similar.

Hence can uniformly define (over $\emptyset$) in Sym_n the set C_3 of 3-cycles, and

$$C = \{g : n^{\frac{1}{8}} < |\operatorname{supp}(g)| < n^{\frac{1}{4}}\}.$$

Claim. An element $g \in C$ cannot be expressed as a word of bounded length of form $g = h_1 f^{\epsilon_1} h_2 \ldots f^{\epsilon_t} h_{t+1}$ (for $h_i \in H_n, f \in C_3$, $\epsilon_i \in \{1,-1\}$). (Otherwise we find $h_1 \ldots h_{t+1} \in H_n$ has support of size $|\operatorname{supp}(g)| \pm 3t$, contrary to Babai result.)

It follows that a point stabiliser H in (G,X) is not maximal, as $H < \langle H, f\rangle < G$ for $f \in C_3$. This contradicts primitivity of (G,X).