Around Vaught's conjecture for weakly o-minimal theories

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Confirmed ordered cases

Vaught's conjecture holds for:

- theories of coloured linear orders (Rubin, 1973);
- theories of linear orders with Skolem functions (Shelah, 1978);
- o-minimal theories (Mayer, 1988).

Weakly o-minimal types

Definition

A type $p \in S_1(A)$ is *ordered* if there is an A-definable linear order $(D, <)$ s.t. $p(\mathfrak{C}) \subseteq D$. An ordered type is *weakly o-minimal* if every $\mathfrak{C}$-definable set D' has finitely many convex components on $p(\mathfrak{C})$ (i.e. $D' \cap p(\mathfrak{C})$ is a finite union of convex subsets of $p(\mathfrak{C})$).

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2. Every type in $S_1(\emptyset)$ is weakly o-minimal iff the theory is *weakly quasi-o-minimal* (every $\mathfrak{C}$-definable subset of $\mathfrak{C}$ is a Boolean combination of convex and $\emptyset$-definable sets).

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3. For weakly o-minimal p and any $B \supseteq A$, $S_p(B)$ is naturally linearly ordered (by $<$), and every $q \in S_p(B)$ is weakly o-minimal.

Global non-forking extensions

Let $p \in S_1(A)$ be weakly o-minimal witnessed by $(D, <)$.

Definition

There are only two A-invariant global extensions of p:

$\mathfrak{p}_l$ $[\mathfrak{p}_r] \coloneqq \{\varphi \in L(\mathfrak{C}) \mid \varphi(\mathfrak{C})$ contains an initial [final] part of $p(\mathfrak{C})\}$.

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These are also only two global non-forking extensions of p over A.

Proof. If $\mathfrak{p} \neq \mathfrak{p}_l, \mathfrak{p}_r$ extends p, then there is a bounded in $p(\mathfrak{C})$ set $\varphi(\mathfrak{C}, \bar{b})$, i.e. such that $a_0 < \varphi(\mathfrak{C}, \bar{b}) < a_1$ for $a_0 \models \mathfrak{p}_{l \restriction A\bar{b}_0}$ and $a_1 \models \mathfrak{p}_{r \restriction A\bar{b}}$. Take an automorphism f mapping a_0 to a_1, then $f^n[\varphi(\mathfrak{C}, \bar{b})]$ are 2-inconsistent.

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Remark

Independently on $<$, $\mathfrak{p}_l$ and $\mathfrak{p}_r$ are endpoints of $S_p(\mathfrak{C})$; all other *bounded* points are forking over A extensions of p.

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Denote $p^2(x, y) \coloneqq \mathfrak{p}^2_{r\restriction A}(x, y)$; $p^2(x, y)$ is equal to $\mathfrak{p}^2_{l\restriction A}(y, x)$.

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Theorem

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For weakly o-minimal $p, q \in S_1(A)$, and $a \models p$:

$$\mathscr{D}_q(a) := \left\{t \models q \mid t \underset{A}{\not\perp} a\right\}.$$

Since $\mathscr{D}_q(a) = \bigcup\{q'(\mathfrak{C}) \mid q' \in S_q(Aa) \smallsetminus \{\mathfrak{q}_{l\restriction Aa}, \mathfrak{q}_{r\restriction Aa}\}\}$, it is a convex, bounded (maybe empty) subset of $q(\mathfrak{C})$.

Simple types

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Weakly o-minimal type $p \in S_1(A)$ is *simple* if:

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Example

If $M = (\mathbb{Z}, <)$, p is the unique type over $\varnothing$ and $a \models p$, then $\mathscr{D}_p(a)$ is a $\mathbb{Z}$-copy around a, so p is non-simple.

If $M = (\mathbb{Q}, <)$, p is the unique type over $\varnothing$ and $a \models p$, then $\mathscr{D}_p(a) = \{a\}$, so p is simple.

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Let $M = (\mathbb{Q}, <, P_n)_{n<\omega}$ where $\{P_n(\mathbb{Q})\}_{n<\omega}$ is a dense partition of $\mathbb{Q}$ (by which I mean that between any two points there are points of every colour). Types over $\emptyset$ are $p_n(x)$ determined by $P_n(x)$, for each n, and $p_\infty(x)$ determined by $\{\neg P_n(x)\}_{n<\omega}$.

Types $p_n(x)$ are convex witnessed by $(P_n(\mathfrak{C}), <)$, but $p_\infty(x)$ is not convex.

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Remark

If p is simple and convex and $a \models p$, then $\mathscr{D}_p(a)$ is Aa-definable.

Vaught's conjecture – binary case (I)

Theorem

Let T be a binary weakly quasi-o-minimal theory. Then $I(T, \aleph_0) \leqslant \aleph_0$, except if:

(C_1) T is not small;

(C_2) there are infinitely many pairwise $\perp^w$ non-isolated types in $S_1(\emptyset)$;

(C_3) there is a non-simple type in $S_1(\emptyset)$;

(C_4) there is a non-convex type in $S_1(\emptyset)$;

(C_5) there is a non-isolated forking extension of some $p \in S_1(\emptyset)$ over 1-element domain.

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Type $p(x) \cup \bigcup_{i \in I} x \notin \mathscr{D}_p(a_i)$ can be omitted in a model $M_I \supseteq \bar{a}_I$.

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Thus non-isomorphic countable orders I and J give non-isomorphic models M_I and M_J.

Orthogonality

Definition

Let $p, q \in S_1(A)$ be weakly o-minimal and $p \not\perp^w q$, and $a \models p$. p and q are *forking orthogonal*, $p \perp^f q$, if $\mathscr{D}_q(a) = \emptyset$, and forking non-orthogonal, $p \not\perp^f q$, if $\mathscr{D}_q(a) \neq \emptyset$.

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Example

Let $M = (\mathbb{Q}, <, P)$, where $\{P(\mathbb{Q}), \neg P(\mathbb{Q})\}$ is a dense partition, and $p(x)$ and $q(x)$ types over $\emptyset$ determined by $P(x)$ and $\neg P(X)$, respectively. Then $p \not\perp^w q$ but $p \perp^f q$.

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Theorem

Both $\not\perp^w$ and $\not\perp^f$ are equivalence relations on $S_1(A)$.

Vaught's conjecture – binary case (II)

Theorem

Assume that $(C_1) - (C_5)$ don't hold; in particular, $\not\perp^w$ is finite equivalence relation on $S_1^{ni}(\emptyset)$.

- If $\not\perp^f$ is infinite on $S_1^{ni}(\emptyset)$, then $I(T, \aleph_0) = \aleph_0$;
- if $\not\perp^f$ is finite on $S_1^{ni}(\emptyset)$, then $I(T, \aleph_0) < \aleph_0$, Moreover, there is a (complicated) formula for $I(T, \aleph_0)$.

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Some parts of the proof (II)

A countable M is prime over maximal Morley sequences of $\not\perp^f$-representatives.

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A weakly o-minimal type p is *trivial* over A if for any n, $a_1 < \cdots < a_n \models p$ and $a_i \perp_A a_{i+1}$ imply $(a_1, \ldots, a_n) \models \mathfrak{p}^n_{r \restriction A}$ (equivalently, $(a_n, \ldots, a_1) \models \mathfrak{p}^n_{l \restriction A}$).

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Remark

If there is a non-simple, or simple and non-convex, trivial type over finite set, then $I(T, \aleph_0) = 2^{\aleph_0}$.

Shifts

Let $(D, <)$ be an A-definable order.

Definition

Let $S_t(x)$ be an L_A-formula (in variables t and x). It is a *(right) shift* (on D) if ($a, b \in D$):

- $S_a(\mathfrak{C})$ is a convex subset of D with minimum a;
- $a < b$ implies $\sup S_a(\mathfrak{C}) \leqslant \sup S_b(\mathfrak{C})$;
- $S_a^n(\mathfrak{C}) \subsetneq S_a^{n+1}(\mathfrak{C})$ for all n, where $S_t^{n+1}(x) \coloneqq (\exists t')(S_t^n(t') \wedge S_{t'}(x))$.

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Examples

- If $(D, <)$ is infinite and discrete, there is a shift.
- If f is an A-definable strictly increasing function on D s.t. $f(x) > x$ for all $x \in D$, there is a shift.

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Theorem (small)

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Corollary (few models)

There are no shifts over finite sets.

On the proof

Let S_t be a shift on $(D, <)$ (over A) and $a \in D$.

Let $S_a^\infty := \bigcup_{n<\omega} S_a^n$.

Consider any convex completion of

$$\{x \in D\} \cup \{S_a^n < x \mid n < \omega\} \cup \{\varphi \in L(Aa) \mid \varphi(S_a^\infty) = S_a^\infty\}.$$

If non-trivial, we find a dense set of isolated types over a finite extension of Aa, contradicting smallness.

How do we find a shift?

Consider the following situation $a, b \models p$, $a < b$ and $a \not\downarrow_A b$.

Then b belongs to some bounded Aa-definable set D', and consider the trace of $S_a := \{t \in D \mid (\exists d' \in D')\, a \leqslant t \leqslant d'\}$ on $p(\mathfrak{C})$.

If for some $b \models p$, $b \in S_a$, $\sup(S_a) < \sup(S_b)$, then S_a is a shift (maybe after some modification).

It is not possible to have $b \models p$, $b \in S_a$ and $\sup(S_a) > \sup(S_b)$, and this is by weakly o-minimality.

So if there are no shifts, $(x \leqslant y \wedge y \in S_x) \vee (y \leqslant x \wedge x \in S_y)$ relatively defines a convex equivalence relation on $p(\mathfrak{C})$.

If there are no shifts

Corollary

$\mathscr{D}_p(a)$ is the union of E-classes of a, for all (non-full) relatively A-definable convex equivalence relations on $p(\mathfrak{C})$.

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Theorem (few models)

Every type over a finite set is convex.