Weak heir, coheirs and the Elis semigroups

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Basics

G is an infinite group
$\mathcal{A} \subseteq \mathcal{P}(G)$ is a G-algebra of sets
$S(\mathcal{A})$ is a G-flow
For $g \in G$, $\pi_g : S(\mathcal{A}) \xrightarrow{\approx} S(\mathcal{A})$
$\pi_g(p) = g \cdot p$.
$E(S(\mathcal{A})) := cl(\{\pi_g : g \in G\}) \subseteq S(\mathcal{A})^{S(\mathcal{A})}$,
the Ellis semigroup of $S(\mathcal{A})$. It is also a G-flow:
$g \cdot f = \pi_g \circ f$.
Minimal subflows of $E(S(\mathcal{A}))$ = minimal left ideals $I \lhd_m E(S\mathcal{A})$.
$J := \{u \in E(S(\mathcal{A})) : u^2 = u\}$. Then $J \cap I \neq \emptyset$ and for $u \in J \cap I$
uI is a group, an Ellis group of $S(\mathcal{A})$. Unique up to $\cong$.
$\beta G = S(\mathcal{P}(G)) \cong E(\beta G)$, why? Because $\mathcal{P}(G)$ is d-closed!

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Some definitions

Definition

- $p \in S(\mathcal{A}) \rightsquigarrow d_p : \mathcal{A} \to \mathcal{P}(G)$
  $d_p(U) = \{g \in G : g^{-1}U \in p\}$
  (Hrushovski's quantifier...)
  $d_p : \mathcal{A} \to \mathcal{P}(G)$ is a G-homomorphism
- $\mathcal{A}$ is d-closed if $\forall p \in S(\mathcal{A}) d_p[\mathcal{A}] \subseteq \mathcal{A}$.

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Properties of a d-closed $\mathcal{A}$

From now on we assume $\mathcal{A}$ is a d-closed G-algebra.

1. For $p \in S(\mathcal{A})$, $d_p \in End(\mathcal{A})$.
2. Let $d : S(\mathcal{A}) \to End(\mathcal{A})$, $p \mapsto d_p$. Then d is a bijection, induces $*$ in $S(\mathcal{A})$ such that $d : (S(\mathcal{A}), *) \xrightarrow{\cong} (End(\mathcal{A}), \circ)$
   $U \in p * q \iff d_q U \in p$
3. $(S(\mathcal{A}), *) \xrightarrow{\cong} (E(S(\mathcal{A})), \circ)$,
   $p \mapsto \ell_p$, $\ell_p(x) = p * x$

How does the topological dynamics of G-flows change when we change G?

How do the Ellis groups of $S(\mathcal{A})$ change when we change G?

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Combinatorial set-up

We work in the following **combinatorial set-up**:

- $G \prec H$ are group structures
  $Def(G)$ is the G-algebra of definable subsets of G (with parameters)
- $Def(G) \ni A = \varphi(G) \rightsquigarrow A^\# = \varphi(H) \subseteq H$
- $\mathcal{A} \subseteq Def(G)$ is a d-closed G-algebra
- $\mathcal{B} \subseteq \mathcal{P}(H)$ is a d-closed H-algebra such that $A^\# \in \mathcal{B}$ for every $A \in \mathcal{A}$.
- $\mathcal{B}|_G := \{B \cap G : B \in \mathcal{B}\} = \mathcal{A}$
- $^\# : \mathcal{A} \longrightarrow \mathcal{B}$ is a Boolean G-algebra monomorphism (respects G-translations)
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Model-theoretic set-up

Our combinatorial set-up is motivated by the following model-theoretic set-up:

- $G = G(M)$, we consider the G-flow $S_{ext,G}(M) = S(Def_{ext,G}(M))$ of external G-types over M
- $M \prec^* N$ and we want to compare algebraically $S_{ext,G}(M)$ and $S_{ext,G}(N)$ (including Ellis groups).

This reduces to the combinatorial set-up with $\mathcal{A} = Def_{ext,G}(M)$ and $\mathcal{B} = Def_{ext,G}(N)$.
$M \prec^* N$ means:
Consider M_{ext} in $L_{ext,M}$ and N_{ext} in $L_{ext,N}$. Then the symbols of $L_{ext,M}$ are identified with some symbols of $L_{ext,N}$ so that $M_{ext} \prec N_{ext}|_{L_{ext,M}}$.
So we can define $A^{\#}$ for $A \in Def_{ext,G}(M)$.
$\prec^*$-extensions exist... (standard)
Instead we shall compare algebraically $(S(\mathcal{A}), *)$ and $(S(\mathcal{B}), *)$ in the comnbinatorial set-up (including Ellis groups).

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Main obstacle: $r : (S(\mathcal{A}), *) \to (S(\mathcal{B}), *)$ is not a $*$-homomorphism. Additional assumptions needed...

Let $I \lhd_m S(\mathcal{B})$ and $u \in I \cap J$.

Let $K(uI) = Ker(d_p), p \in uI$ and $R(uI) = Im(d_p), p \in uI$.

Let $\mathcal{K} = \{Ker(d_p) : p \in I \lhd_m S(B)\}$ and $\mathcal{R} = \{Im(d_p) : p \in I \lhd_m S(\mathcal{B})\}$.

Proposition

1. $\{\text{Ellis groups of } (S(\mathcal{B}), *)\} \longleftrightarrow \mathcal{K} \times \mathcal{R}$
   $uI \mapsto (K(uI), R(uI))$
2. $K(uI) = K(u'I') \iff I = I'$
3. $\forall R \in \mathcal{R} \forall I \lhd_m S(\mathcal{B}) \exists! u \in I \cap J \; R = R(uI)$

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Main obstacle: $r : (S(\mathcal{A}), *) \to (S(\mathcal{B}), *)$ is not a $*$-homomorphism. Additional assumptions needed...
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1. $\{\text{Ellis groups of } (S(\mathcal{B}), *)\} \longleftrightarrow \mathcal{K} \times \mathcal{R}$
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The main results

Theorem 1

Assume $|\mathcal{R}| = 1$. Then the Ellis groups of $S(\mathcal{A})$ are isomorphic to some closed subgroups of the Ellis subgroups of $S(\mathcal{B})$.

Theorem 2

Assume $|\mathcal{K}| = 1$. Then the Ellis groups of $S(\mathcal{A})$ are homomorphic images of some subgroups of the Ellis groups of $S(\mathcal{B})$.

On the assumptions

1. $|\mathcal{R}| = 1$ means every (some) $I \triangleleft_m S(\mathcal{B})$ is a group equivalently: every (some) $I \triangleleft_m S(\mathcal{B})$ is distal.
2. $|\mathcal{K}| = 1$ means: there is exactly one $I \triangleleft_m S(\mathcal{B})$. equivalently: there is a generic type in $S(\mathcal{B})$.

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Weak heirs and (weak) coheirs

On the proofs

1. dual to each other
2. use weak heirs and (weak) coheirs

Definition. Let $q \in S(\mathcal{B})$.

1. q is a weak heir over $\mathcal{A}$ if $d_q A^{\#} = (d_{r(q)} A)^{\#}$ for every $A \in \mathcal{A}$.
2. q is a weak coheir over $\mathcal{A}$ if $\forall A, B \in \mathcal{A} \forall s \in S(\mathcal{B})$ if $d_s A^{\#} \cap B^{\#} \in q$, then $d_s A^{\#} \cap B^{\#} \cap G \neq \emptyset$.
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Lemma

Assume $q \in S(\mathcal{B})$.

1. q is a weak heir over $\mathcal{A}$ iff $r(s * q) = r(s) * r(q)$ for every $s \in S(\mathcal{B})$.
2. q is a weak coheir ober $\mathcal{A}$ iff $r(q * s) = r(q) * r(s)$ for every $s \in S(\mathcal{B})$.

$$CH(\mathcal{B}/\mathcal{A}) = \{q \in S(\mathcal{B}) : q \text{ is a coheir over } \mathcal{A}\}$$

Likewise $WCH(\mathcal{B}/\mathcal{A})$ (weak coheirs) and $WH(\mathcal{B}/\mathcal{A})$ (weak heirs).

Weak heirs and (weak) coheirs

Properties

1. $CH(\mathcal{B}/\mathcal{A}) \subseteq WCH(\mathcal{B}/\mathcal{A}) \subseteq S(\mathcal{B})$ are closed.
2. $WH(\mathcal{B}/\mathcal{A}), CH(\mathcal{B}/\mathcal{A}), WCH(\mathcal{B}/A)$ are non-empty sub-semigroups of $(S(\mathcal{B}), *)$
3. $r : WH(\mathcal{B}/\mathcal{A}) \to S(\mathcal{A})$ and $r : WCH(\mathcal{B}/\mathcal{A}) \to S(\mathcal{A})$ are $*$-epimorphisms.
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Proofs of the Theorems (idea):
Let $I \lhd_m S(\mathcal{A})$.
We find $I'' \lhd_m S(\mathcal{B})$, $u \in J \cap I$ and $u'' \in J \cap I''$ with $r[I''] = I$ (using weak heirs) and $u \in r[u'' I'']$.

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Proofs of the Theorems

Proof of Theorem 1

1. We define a group $\mathcal{G} \subseteq CH(\mathcal{B}/\mathcal{A})$ such that $r : \mathcal{G} \xrightarrow{\cong} uI$.
2. We define $\varphi : \mathcal{G} \to u''I'' = I''$ by $\varphi(x) = x * u''$, a $*$-homomorphism.
3. $\mathcal{G}'' := \varphi[\mathcal{G}]$ is a closed subgroup of I'' isomorphic to uI.

Proof of Theorem 2

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The choice of $\mathcal{H}$

Both groups $\mathcal{G}, \mathcal{H}$ are canonical (unique up to $\cong$). The choice of $\mathcal{H}$:

1. Let $I \lhd_m S(\mathcal{A})$ and $u \in J \cap I$.
2. Let $WH_u(\mathcal{B}) = r^{-1}(u) \cap WH(\mathcal{B}/\mathcal{A})$, a closed subsemigroup of $WH(\mathcal{B}/\mathcal{A})$.
3. Fix $I' \lhd_m WH_u(\mathcal{B})$. Let $K' = Ker(d_{q'}), q' \in I'$ (common kernel) and $\mathcal{R}'_u = \{Im(d_{q'}) : q' \in I'\}$.
4. I' is a disjoint union of isomorphic groups $u'I', u' \in J \cap I'$. Fix $u' \in J \cap I'$.
5. There is $I^+ \lhd S(\mathcal{B})$ with $I' = WH_u(\mathcal{B}/\mathcal{A}) \cap I^+$. Also, $r[I^+] = I$.
6. For every $q \in I$, we have that $I'_q := I^+ \cap WH(\mathcal{B}/\mathcal{A}) \cap r^{-1}(q) \neq \emptyset$ and for every $q' \in I'_q$, $Ker(d_{q'}) = K'$.
7. Let $R' = Im(d_{u'})$. Then $\mathcal{H} = \{q' \in \bigcup_{q \in uI} I'_q : Im(d_{q'}) = R'\}$.

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7. Let $R' = Im(d_{u'})$. Then $\mathcal{H} = \{q' \in \bigcup_{q \in uI} I'_q : Im(d_{q'}) = R'\}$.

The choice of $\mathcal{H}$

Both groups $\mathcal{G}, \mathcal{H}$ are canonical (unique up to $\cong$). The choice of $\mathcal{H}$:

1. Let $I \lhd_m S(\mathcal{A})$ and $u \in J \cap I$.
2. Let $WH_u(\mathcal{B}) = r^{-1}(u) \cap WH(\mathcal{B}/\mathcal{A})$, a closed subsemigroup of $WH(\mathcal{B}/\mathcal{A})$.
3. Fix $I' \lhd_m WH_u(\mathcal{B})$. Let $K' = Ker(d_{q'}), q' \in I'$ (common kernel) and $\mathcal{R}'_u = \{Im(d_{q'}) : q' \in I'\}$.
4. I' is a disjoint union of isomorphic groups $u'I', u' \in J \cap I'$. Fix $u' \in J \cap I'$.
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Final comments

But: the Ellis structural theorem holds for $WH(\mathcal{B}/\mathcal{A})$ and $\mathcal{H}$ is just an Ellis subgroup of $WH(\mathcal{B}/\mathcal{A})$.

Weak heirs and weak coheirs in the stable case

- $q \in WH_G(M/N)$ iff $q|_{\Delta_M}$ does not fork over M.
- $q \in WCH_G(M/N)$ iff $q|_{\Delta^*_M}$ does not fork over M.

Proposition

Assume T is stable, $a \in G(\mathfrak{C})$ and $M \prec N$ then $tp(a/N)$ is a weak heir over M iff $tp(a^{-1}/N)$ is a weak coheir over M.

Proposition

Assume T is stable, $M \prec N$ and $G = G(M)$ is abelian-by-finite. Then $WH_G(M/N) = WCH_G(M/N)$.